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Conditions for the existence of positive operator valued measures

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Abstract

Sufficient and necessary conditions are presented for the existence of (N, M)-positive operator valued measures ((N, M)-POVMs) valid for arbitrary-dimensional quantum systems. Firstly, a sufficient condition for the existence of (N, M)-POVMs is presented. It yields a simple relation determining an upper bound on the continuous parameter of an arbitrary (N, M)-POVM, below which all its POVM elements are guaranteed to be positive semidefinite. Secondly, for arbitrary optimal (N, M)-POVMs conditions on their existence are derived, which exhibit a close connection to the existence of sets of $(M - 1)N$ isospectral, traceless, orthonormal, and hermitian operators with particular common spectra. The specific form of the two possible types of spectra depends on whether M is smaller than the dimension of the quantum system under consideration or not. For the special case of $M = 2$ and dimensions, which are powers of two, these sets of operators always exist and can be expressed in terms of the Clifford algebra of tensor products of Pauli operators.

Key words: quantum information science, quantum correlations in quantum information science, quantum entanglement detection

1. Introduction

The development of efficient quantum measurement techniques is important for central tasks of quantum information processing [1, 2], such as quantum state reconstruction or the detection of characteristic quantum correlations. In this context (N, M)-POVMs [3] have been introduced recently as interesting one-parameter-continuous families of positive operator valued measures (POVMs). They describe numerous important quantum measurements in a unified way. These measurements include projective measurements with complete sets of mutually unbiased bases (MUBs) [4], mutually unbiased measurements (MUMs) [5], symmetric informationally complete POVMs (SIC-POVMs) [6, 7] and their generalizations (GSIC-POVMs) [8]. For purposes of quantum information processing informationally complete (N, M)-POVMs are particularly interesting, because they enable a complete reconstruction of quantum states.

Recent investigations exploring characteristic features of (N, M)-POVMs have concentrated on possible applications in quantum information processing and on basic theoretical questions concerning their existence. On the application side the potential of (N, M)-POVMs for the local detection of provable bipartite quantum entanglement [9] and quantum steering [10] has been investigated. As far as basic theoretical questions are concerned, it has been demonstrated that (N, M)-POVMs can always be constructed for sufficiently small values of their continuous parameter [3]. However, for larger or even maximal values of their continuous parameter, i.e., for optimal (N, M)-POVMs, it generally causes major theoretical problems to guarantee the positive semidefiniteness of all POVM elements involved. Despite considerable research efforts concentrating on the subclasses of SIC-POVM and MUBs, for example, open questions concerning their existence in arbitrary dimensions still remain [11, 12]. Thus, questions concerning the existence and construction of (N, M)-POVMs for large or even maximal values of their continuous parameter are still widely open.

In order to obtain a detailed theoretical understanding of (N, M)-POVMs and of their characteristic features there is a need to develop sufficient and necessary conditions, which guarantee their existence. It is a main intention of this paper to address this issue and to explore general features of the existence and construction of (N, M)-POVMs with the help of orthonormal hermitian operator bases. As a first main result we develop a sufficient condition for the existence of (N, M)-POVMs. This sufficient condition yields a simple upper bound on the continuous parameter, within which this existence is guaranteed. Thus, this result complements the already known property that (N, M)-POVMs can always be constructed for sufficiently small values of their continuous parameters [3]. As a second main result we present conditions on the existence of optimal (N, M)-POVMs. They exhibit a close connection to the existence of sets of $(M - 1)N$ isospectral, traceless, orthonormal, and hermitian operators with particular common spectra. It turns out that the particular form of the two possible types of common spectra depends on whether M is smaller than the dimension of the quantum system under consideration or not. This result generalizes an already known property of GSICs [8], i.e., optimal $(1, d^2)$-POVMs, to arbitrary optimal (N, M)-POVMs. Furthermore, for the special case of $M = 2$ and dimensions, which are powers of 2, these sets of operators always exist and can be expressed in terms of the Clifford algebra of tensor products of Pauli operators.

This paper is organized as follows. In Section 2 basic features of (N, M)-POVMs are summarized. In Section 2.1 their defining properties are recapitulated [3]. In Section 2.2 recent results [9] on their necessary relation to orthonormal hermitian operator bases are summarized. Furthermore, as a motivation for the subsequent discussion typical problems are discussed originating from the positive semidefiniteness of all its POVM elements. As one of our main results, in Section 3 a sufficient condition is presented, under which for any d-dimensional quantum system (N, M)-POVMs can be constructed. As a second main result in Section 4 conditions for the existence of optimal (N, M)-POVMs are presented, which exhibit a close connection to the existence of sets of $(M-1)N$ isospectral, traceless, orthonormal, and hermitian operators with particular common spectra. In Section 5 a necessary and sufficient condition for the existence of optimal $(N, 2)$-POVMs with $N \leq d^2 - 1$ is derived. It is shown that they can only exist in even-dimensional Hilbert spaces, and explicit constructions are presented for dimensions $d = 2^k$ with $k \in \mathbb{N}$.

2. Informationally complete positive operator valued measures

In this section elementary features of the recently developed (N, M)-POVMs [3] are summarized. In the first subsection their definition and resulting elementary properties are recapitulated. In the second subsection recently discussed basic relations between informationally complete (N, M)-POVMs and orthonormal hermitian operator bases are summarized [9]. Furthermore, some typical problems are discussed, which complicate the construction of (N, M)-POVMs by basis expansions in terms of orthonormal hermitian operator bases.

2.1. Basic properties

Let us consider a quantum system characterized by a d-dimensional Hilbert space. Ignoring the properties of a quantum state immediately after a measurement, the most general quantum measurement on this quantum system is described by a POVM [1, 2]. An M-element POVM is a set of M positive semidefinite operators, say $\Pi = \{\Pi_a \geqslant 0 | a = 1, \cdots, M\}$, which fulfill the completeness relation

$$\sum_{a=1}^{M} \Pi_a = \mathbb{1}_d \tag{1}$$

with the unit operator $\mathbb{1}_d$ of the quantum system's d-dimensional Hilbert space. Thereby, the indices $a \in \{1, \cdots, M\}$ coordinatize the different M possible real-valued measurement results, say $\mathcal{M}_a$. According to Born's rule the probability of measuring the result $\mathcal{M}_a$ is given by $p_a = Tr\{\varrho \Pi_a\}$, if the quantum system has been prepared in the quantum state $\varrho \geqslant 0$ immediately before the measurement. If the positive semidefinite operators Π are linearly independent for $a \in \{1, \cdots, M\}$ and $M = d^2$, a POVM is called informationally complete. In such a case an arbitrary quantum state ϱ can be reconstructed from all the d^2 measurement results. In the special case of orthogonal projection operators, i.e., $\Pi_a \Pi_{a'} = \delta_{aa'} \Pi_a$ for $a, a' \in \{1, \cdots, M\}$, a POVM describes a von Neumann measurement.

Recently (N, M)-POVMs [3] have been introduced as a unified way for describing numerous important quantum measurements, such as projective measurements with MUBs [4], MUMs [5], SIC-POVMs [6, 7], and their GSICs [8]. An (N, M)-POVM Π is a one-continuous-parameter family of N different M-element POVMs, i.e., $\Pi = \{\Pi_{i(\alpha, a)} | \alpha \in \{1, \cdots, N\}, a \in \{1, \cdots, M\}\}$, defined by the following relations

$$Tr\left\{\Pi_{i(\alpha,a)}\right\} = \frac{d}{M}, \tag{2}$$

$$Tr\left\{\Pi_{i(\alpha,a)}\ \Pi_{i(\alpha,a')}\right\} = x\ \delta_{a,a'} + (1 - \delta_{a,a'})\,\frac{d - Mx}{M\,(M-1)}, \tag{3}$$

$$Tr\left\{\Pi_{i(\alpha,a)}\ \Pi_{j(\beta,b)}\right\} = \frac{d}{M^2} \tag{4}$$

for all $\beta \neq \alpha \in \{1, \cdots, N\}$ and $a, a', b \in \{1, \cdots, M\}$. Thereby, for the sake of convenience we have introduced the coordinate function $I{:}(\alpha, a) \to i(\alpha, a)$. It maps the NM-tuples of the form (α, a), which for each value of α identify a particular POVM uniquely, bijectively onto the NM natural numbers $i, j \in \{1, \cdots, NM\}$. For given values of (d, N, M) the possible values of the continuous parameter x are constrained by the relation [3]

$$\frac{d}{M^2} < x \leq \min\left(\frac{d^2}{M^2}, \frac{d}{M}\right). \tag{5}$$

An (N, M)-POVM with maximal possible value of x is called optimal. Furthermore, an (N, M)-POVM Π is informationally complete, if it contains d^2 linearly independent positive semidefinite operators. As each of the N and M-element POVMs involved fulfills the completeness relation (1), this is equivalent to the requirement

$$(M-1)N + 1 = d^2\,. \tag{6}$$

For arbitrary dimensions four possible solutions of this relation are $(N, M) \in \{(1, d^2), (d+1, d), (d^2-1, 2), (d-1, d+2)\}$. The solution $(N, M) = (1, d^2)$ describes a one-parameter family of GSIC-POVMs [8] parameterized by the parameter x. SIC-POVMs are special cases of GSIC-POVMs with $x = 1/d^2$. The solution $(N, M) = (d+1, d)$ describes MUMs [5]. In the special case of $x = d^2/M^2 = d/M = 1$ these MUMs describe projective measurements of unit rank with maximal sets of $d+1$ MUBs.

2.2. Orthonormal hermitian operator bases and informationally complete (N, M)-POVMs

In this subsection we first recapitulate the general relations between informationally complete (N, M)-POVMs and orthonormal hermitian operator bases, which necessarily have to be fulfilled irrespectively of the positive semidefiniteness of the POVM elements involved [9]. They govern the construction of (N, M)-POVMs by basis expansions in terms of orthonormal hermitian operator bases. Subsequently, concentrating on the particular example of a MUM in dimension $d = 3$, it is exemplified that such a MUM can always be constructed for sufficiently small values of the continuous parameter x. This example demonstrates, why constructing (N, M)-POVMs for parameters x close to their lower bounds is always possible, while constructing optimal (N, M)-POVMs is rather difficult. These problems motivate the development of simple sufficient and necessary conditions for the existence of (N, M)-POVMs, which is pursued in the subsequent sections.

In a Hilbert space $\mathcal{H}_d$ of a d-dimensional quantum system an informationally complete (N, M)-POVM can always be expanded in a basis of d^2 linearly independent linear hermitian operators, say $G = (G_1, \cdots, G_{d^2})^T$. These hermitian operators can be chosen as orthonormal with respect to the Hilbert–Schmidt scalar product $\langle G_\mu | G_\nu \rangle_{HS} := Tr\left\{G_\mu^\dagger G_\nu\right\}$ with $G_\mu^\dagger = G_\mu$. They form a basis of the Hilbert space $\mathcal{H}_{d^2} = (\mathrm{Span}\,(G)\,, \langle\,\cdot\,|\,\cdot\,\rangle_{HS})$ of linear hermitian operators in $\mathcal{H}_d$ over the field of real numbers. This latter Hilbert space is a Euclidean vector space. Furthermore, such an orthonormal hermitian operator basis G can always be chosen so that

$$G_1 = \mathbb{1}_d/\sqrt{d}, \quad Tr\left\{G_\mu\right\} = 0 \tag{7}$$

for $\mu \in \{2, \cdots, d^2\}$. The resulting basis expansion of an arbitrary informationally complete (N, M)-POVM in such an orthonormal hermitian basis has the general form

$$\Pi = G^T S. \tag{8}$$

Thereby, S denotes the linear operator mapping $\mathcal{H}_{d^2}$ into the Hilbert space $\mathcal{H}_{NM}$ of hermitian operators, which contains all the elements of the (N, M)-POVM. Recently, it has been shown that for informationally complete (N, M)-POVMs the structure of this linear map S and of its corresponding real-valued $d^2 \times NM$ matrix $S_{\mu,i}$ is significantly constrained by the defining relations (3) and (4) [9]. In particular, these defining relations imply that the linear operator S^TS is symmetric, positive semidefinite and its matrix elemenets are real-valued. Therefore, its eigenvectors are orthogonal. The square root of this matrix S^TS defines the linear operator S up to an orthogonal $d^2 \times d^2$ matrix. Ignoring the positive semidefiniteness constraints of the POVM elements, it has been shown that the most general form of the linear operator $S : \mathcal{H}_{d^2} \to \mathcal{H}_{NM}$ is given by a $d^2 \times NM$ matrix of the form

$$S_{\mu,\mathrm{i}(\alpha,a)} = \sqrt{\Lambda_\mu} X^T_{\mu,\mathrm{i}(\alpha,a)}, \tag{9}$$

with the diagonal $d^2 \times d^2$ matrix Λ and the $NM \times d^2$ matrix $X_{i,\mu}$. The diagonal matrix Λ has only two different nonvanishing entries, which are the d^2 non-zero eigenvalues of S^TS. They are given by

$$\Lambda_1 = \frac{dN}{M}, \quad \Lambda_\mu = \Gamma = \frac{xM^2 - d}{M\,(M-1)} \tag{10}$$

for $\mu \in \{2, \cdots, d^2\}$. Thus, the eigenvalue Γ is ($d^2 - 1$)-fold degenerate, and the eigenvalue Λ_1 is non-degenerate. The real-valued $NM \times d^2$ matrix $X_{i,\mu}$ consists of d^2NM-dimensional orthonormal arrays, i.e.,

$$\sum_{i=1}^{NM} X_{i,\mu} X_{i,\nu} = \delta_{\mu\nu} \tag{11}$$

for $\mu, \nu \in \{1, \cdots, d^2\}$. They fulfill the relations

$$X_{i,1} = \frac{1}{\sqrt{NM}}, \quad \sum_{a=1}^{M} X_{\mathrm{i}(\alpha,a),\mu} = 0 \tag{12}$$

for $\mu \in \{2, \cdots, d^2\}$. As a consequence of the defining constraints (2) and (3) of (N, M)-POVMs these orthonormal NM-dimensional arrays also fulfill the relation

$$\sum_{\mu=2}^{d^2} \left(X_{i,\mu}\right)^2 = \frac{M-1}{M}\,. \tag{13}$$

It is apparent from (9) that all basis operators G_μ with $\mu \in \{2, \cdots, d^2\}$ are mapped conformally onto a ($d^2 - 1$)-dimensional subspace of $\mathcal{H}_{NM}$ by stretching the norms of all its elements by the factor $\sqrt{\Gamma}$. Only the basis operator G_1 is stretched by a different factor, namely $\sqrt{\Lambda_1}$. Therefore, ignoring the positive semidefiniteness constraints the defining properties of (N, M)-POVMs (3) and (4) imply the basis expansion

$$\Pi_{\mathrm{i}(\alpha,a)} = \frac{\mathbb{1}_d}{M} + \sqrt{\Gamma} \sum_{\mu=2}^{d^2} X_{\mathrm{i}(\alpha,a),\mu} G_\mu \tag{14}$$

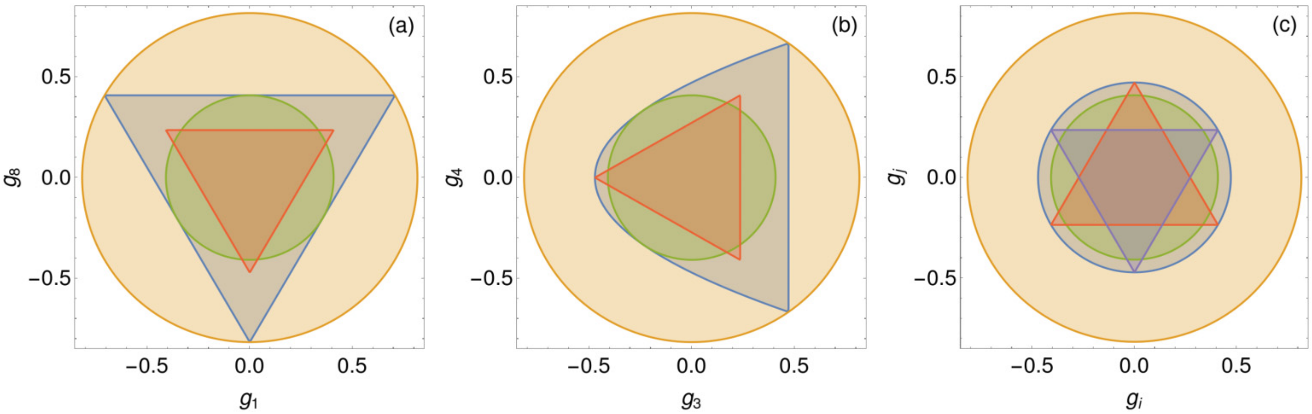
for each element of an informationally complete (N, M)-POVM with $i \in \{1, \cdots, NM\}$. In view of (7) the orthonormal hermitian operators G_μ for $\mu \in \{2, \cdots, d^2\}$ are only determined up to an orthogonal transformation of the orthogonal group $O(d^2 - 1)$. Furthermore, there is an additional freedom in choosing the $N \times d^2$ matrices $X_{\mathrm{i}(.,\,a)}$, within the constraints imposed by relation (12). From the basis expansion (14) it is apparent that possibilities for constructing positive semidefinite POVM elements may be severely constrained by not fully taking advantage of the freedom of choice of the orthonormal hermitian operator basis.

According to relation (6) an informationally complete (N, M)-POVM consists of $d^2 - 1 = N(M - 1)$ linear independent POVM elements. A strategy for its construction is to partition the orthonormal traceless hermitian operator basis $\{G_\mu\}$ with $\mu \in \{2, \cdots, d^2\}$ into N basis tuples B_α, each of which corresponds to a particular value of $\alpha \in \{1, \cdots, N\}$. This partitioning of the basis elements ensures that condition (4) is fulfilled. Accordingly, the basis expansion (14) is restricted to an ansatz of the form

$$\Pi_{\mathrm{i}(\alpha,a)} = \frac{\mathbb{1}_d}{M} + \sqrt{\Gamma} \sum_{G_\mu \in B_\alpha} X_{\mathrm{i}(\alpha,a),\mu} G_\mu. \tag{15}$$

Using this ansatz the allowed transformations are restricted to the orthogonal group $O\,(M-1)$ for each value of α, thus also restricting the achievable positive semidefinite operators for a given basis $\{G_\mu\}$. Therefore, the possible values of the continuous parameters x of such a construction depend on the chosen basis $\{G_\mu\}$ and its partitioning. In order to demonstrate this, let us consider the construction of a MUM for $d = 3$ as a special example of an informationally complete (4, 3)-POVM with $1/3 < x \le 1$. According to (15) the construction of this MUM can be interpreted geometrically. For this purpose let us identify the operator $\mathbb{1}_d/3$ with the origin of an 8-dimensional Euclidean space spanned by the hermitian operators G_μ for $\mu \in \{2, \ldots, 9\}$. Accordingly, we have to construct $N = 4$ equilateral triangles with this origin as their centroids

Fig. 1. Regions of positive semidefiniteness corresponding to four different partitions $B_\alpha, \alpha \in \{1, \cdots, 4\}$ of the traceless, hermitian operators for an informationally complete (4, 3)-positive operator valued measure (mutually unbiased measurements (MUM)) in dimension $d = 3$ with $\alpha = 1$ (*a*), $\alpha = 2$ (*b*), $\alpha = \{3, 4\}$ and $(i, j) \in \{(2, 5), (6, 7)\}$ (*c*): Geometrically each equilateral triangle with (0,0) as its centroid represents a set of three operators fulfilling (15). The restrictions imposed by positive semidefiniteness are visualized by the blue regions. The yellow regions represent the constraints (5). The green region is the circle of maximal radius $r_< = 1/\sqrt{6}$ with center (0, 0) located within the intersection of all blue regions of all four partitions. Within this circle equilateral triangles with centroid (0, 0) can be rotated by arbitrary angles. The two red equilateral triangles of Figs. 1*a* and 1*b* and the two equilateral triangles of Fig. 1*c* represent a possible maximal MUM with $x = 5/9$. Their vertices lie outside of the green circles.



to fulfill the characteristic completeness relation of POVMs (12) for each α. Furthermore, condition (5) implies that

$$\Gamma \sum_{G_\mu \in B_\alpha} \left(X_{i(\alpha,a),\mu}\right)^2 \leq r_>^2 := 2/3. \tag{16}$$

An (N, M)-POVM is optimal if the two sides of inequality (16) are equal.

In Fig. 1 the constraints imposed by the positive semidefiniteness of all POVM elements of this MUM are visualized graphically for the different partitions B_α. Thereby, the Gell-Mann matrices $\{g_1, \cdots, g_8\}$, as defined in appendix A, have been used as an orthonormal basis of traceless, hermitian operators. Accordingly, the four partitions have been chosen as $B_1 = \{g_1, g_8\}$, $B_2 = \{g_3, g_4\}$, $B_3 = \{g_2, g_5\}$, $B_4 = \{g_6, g_7\}$. For arbitrary partitionings the corresponding positive semidefinite regions have already been discussed recently [13]. In Fig. 1*a* the two-dimensional (2D) Euclidean subspace spanned by the unit vectors of partition B_1 is depicted. All points inside the blue triangle correspond to the convex set of positive semidefinite matrices according to (15). The vertices of any equilateral triangle within this blue triangle with the origin as its centroid constitute a triple of possible POVMs with $a \in \{1, 2, 3\}$ and $\alpha = 1$. The points inside the yellow circle correspond to all hermitian operators, which fulfill the necessary constraint (5). From inequality (16) it follows that an optimal POVM is an equilateral triangle, whose vertices are on the boundary of the yellow area so that they have maximal distance to the origin. The blue triangle itself constitutes a single optimal POVM, which can be constructed with the help of the partitioning B_1 of the Gell-Mann basis. The green circle is the maximal circle around the origin, which can be constructed inside the triangle of positive semidefinite elements. Its radius is given by $r_< = 1/\sqrt{6}$. The blue region of Fig. 1*b* shows the convex set of positive semidefinite hermitian matrices, which can be constructed in the 2D Euclidean subspace spanned by unit vectors of partition B_2. The vertices of any equilateral triangle constructed within this blue region with the origin as its centroid constitute a triple of possible POVM elements with $a \in \{1, \cdots, 3\}$ and $\alpha = 2$. Again, the points inside the yellow circle correspond to all hermitian matrices, which fulfill the necessary constraint (5). The two points of the blue area intersecting with the yellow area's boundary cannot be used for constructing an optimal POVM for $\alpha = 2$. The green circle is the maximal circle around the origin again with radius $r_< = 1/\sqrt{6}$, which can be constructed inside the blue region of positive semidefinite elements. The blue region of Fig. 1*c* shows the convex set of positive, semidefinite, hermitian matrices, which can be constructed in the 2D Euclidean subspace spanned by unit vectors of partitions B_3 or B_4. The vertices of any equilateral triangle constructed within this blue region with the origin as its centroid constitute a triple of possible POVMs with $a \in \{1, \cdots, 3\}$ and $\alpha = 3$ or $\alpha = 4$. Again the points inside the yellow circle correspond to all hermitian matrices fulfilling the necessary constraint (5). A MUM is given by four equilateral triangles of identical size inside the positive semidefinite area of each partition with the origin as its centroid. The red triangles in Figs. 1*a* and 1*b* and the two triangles in Fig. 1*c* show four equilateral triangles of maximal sizes, which can be constructed inside the blue regions of Figs. 1*a*–1*c*. These four triangles constitute an informationally complete MUM in dimension $d = 3$ with the maximal possible value of $x = 5/9$. The directions of some of these triangles with respect to the chosen partitioning of the Gell-Mann basis are not determined uniquely. In Fig. 1*a*, for example the red triangle can be rotated around its centroid continuously as long as it stays within the blue region of positive semidefiniteness. In particular, this implies that not all rotation angles are possible. However, in Fig. 1*c* rotations of the two triangles around the origin are possible for arbitrary angles. But in Fig. 1*b* the shape of the blue region implies that the position of the red triangle is fixed uniquely. From Figs. 1*a*–1*c* it is apparent that the vertices of the equilateral triangles representing the maximal informationally complete MUM are located outside of the green circle of radius $r_< = 1/\sqrt{6}$. In contrast, all POVM elements, whose equilateral triangles are constructed inside this green circle, can be rotated arbitrarily around the origin without affecting the positive semidefiniteness of their corresponding POVM elements.

In general it is a cumbersome task to determine criteria, under which all POVM elements of an (N, M)-POVM are positive semidefinite. This example demonstrates that, although for sufficiently small regions around the minimal possible value of $x = d/M^2$ (N, M)-POVMs can be constructed, difficulties increase with increasing values of x. Typically, the most complicated situations arise for constructions of optimal (N, M)-POVMs. In view of these problems it is of interest to develop conditions for the existence of optimal (N, M)-POVMs. Motivated by this need in the following such conditions will be developed.

3. A sufficient condition for the construction of (N, M)-POVMs

In this section a sufficient condition is derived, under which for a d-dimensional quantum system (N, M)-POVMs can always be constructed. This sufficient condition yields a simple upper bound on the continuous x-parameter (cf. inequality (21)), within which this can be achieved.

A general sufficient condition for positivity can be derived by using basic properties of positive semidefinite linear operators [14, 15]. For this purpose let us consider an arbitrary POVM element of an (N, M)-POVM in a d-dimensional Hilbert space. Its spectral representation is given by

$$\Pi_{i(\alpha,a)} = \sum_{\sigma=1}^{d} \lambda_\sigma P_\sigma \tag{17}$$

with its non-negative eigenvalues λ_σ and with the associated one-dimensional orthogonal projection operators P_σ fulfilling the completeness and orthogonality relations $\sum_{\sigma=1}^{d} P_\sigma = \mathbb{1}_d$ and $P_\sigma P_{\sigma'} = \delta_{\sigma,\sigma'} P_\sigma$. The constraint (2) yields the relation

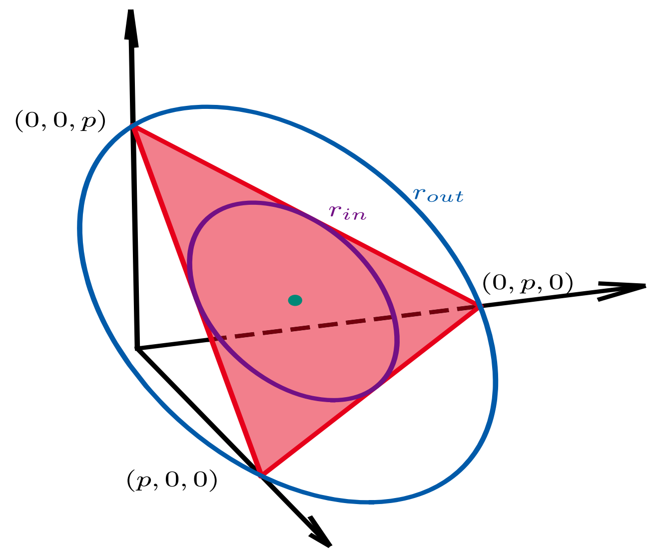
$$Tr\left\{\Pi_{i(\alpha,a)}\right\} = \frac{d}{M} = \sum_{\sigma=1}^{d} \lambda_\sigma. \tag{18}$$

Therefore, for given projection operators P_σ the set of all positive semidefinite POVM elements of this (N, M)-POVM constitute a $(d-1)$-dimensional simplex Δ_{d-1} in the d-dimensional Hilbert space (compare with Fig. 2). The centroid of this $(d-1)$-simplex is given by

$$C_{d-1} = \frac{Tr\left\{\Pi_{i(\alpha,a)}\right\}}{d} \sum_{\sigma=1}^{d} P_\sigma = \frac{1}{M}\mathbb{1}_d. \tag{19}$$

The boundary of this $(d-1)$-simplex, i.e., $\partial\Delta_{d-1}$, is the union of d different $(d-2)$-simplices Δ_{d-2}. It consists of all possible

Fig. 2. Visualization of the simpex Δ_{d-1} and its boundary $\partial\Delta_{d-1}$ and of their corresponding centroids in the elementary case $d = 3$ for $M \geqslant d$: The orthogonal axes are defined by the one-dimensional orthogonal projection operators $\{P_\sigma | \sigma \in \{1, 2, 3\}\}$ and their corresponding coordinates are the parameters λ_σ of (17) with $p = Tr\left\{\Pi_{i(\alpha,a)}\right\} = d/M$. The simplex Δ_2 is a triangle (red area). Its centroid C_2 is represented by the green point. The boundary of the red triangle $\partial\Delta_2$ consists of three 1-simplices Δ_1. The dark blue circle centered around C_2 with radius r_{in} is the maximal circle, which can be constructed inside Δ_2. It touches $\partial\Delta_2$ in the centroids of its three constituting simplices Δ_1. The light blue circle centered around C_2 with radius r_{out} represents the constraint (5).



elements of the form (17) with at least one of the d eigenvalues vanishing. The centroid C_{d-1} has equal distances r_{in} to the centroids of all the d parts of $\partial\Delta_{d-1}$. This distance r_{in} defines the radius of the largest possible circle with center C_{d-1}, which lies within Δ_{d-1} and touches $\partial\Delta_{d-1}$ in one of its d centroids C_{d-2}. It is determined by the relation

$$\begin{aligned} r_{in}^2 &= Tr\left\{(C_{d-1} - C_{d-2})^2\right\} \\ &= \left(Tr\left\{\Pi_{i(\alpha,a)}\right\}\right)^2 \left[(d-1)\left(\frac{1}{d} - \frac{1}{d-1}\right)^2 + \frac{1}{d^2}\right] \\ &= \frac{\left(Tr\left\{\Pi_{i(\alpha,a)}\right\}\right)^2}{d(d-1)} = \frac{d}{M^2(d-1)}. \end{aligned} \tag{20}$$

Using (2) and (3) we arrive at the inequality

$$0 < Tr\left\{\left(\Pi_{i(\alpha,a)} - \mathbb{1}_d/M\right)^2\right\} = x - \frac{d}{M^2} \leq r_{in}^2 = \frac{d}{M^2(d-1)}. \tag{21}$$

According to (19) and (20) the centroid C_{d-1} as well as the radius r_{in} are independent of the choice of the orthonormal projection operators P_σ so that (21) applies to all POVM elements. Therefore, it can be concluded that fulfillment of inequality (21) is sufficient for the existence of an (N, M)-POVM. It guarantees the positive semidefiniteness of all its POVM elements. Note that in the special cases considered in Fig. 1*a*–1*c*, i.e., $M = 3$, $d = 3$, the distance r_{in} reduces to the value $r_{in} = 1/\sqrt{6} = r_<$, which is a basis and partition independent value.

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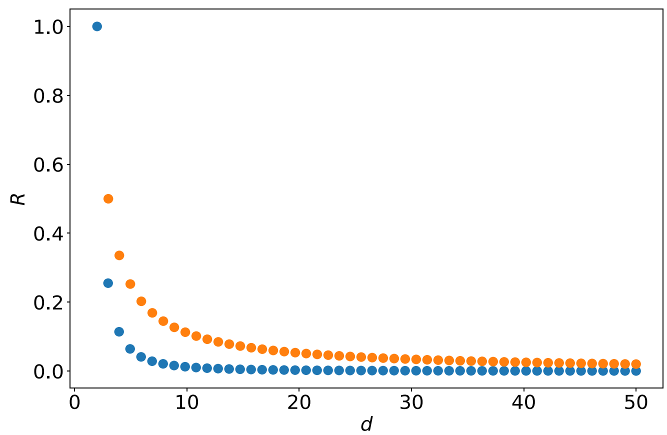
Fig. 3. Dimensional dependence of $R(d)$ according to (23) for $M \geqslant d$ (blue points) and for $2 = M < d$ (orange points): Cases with $2 < M < d$ are located between these two series of points and also rapidly converge to zero with increasing dimensions d.



One can also define a circle with center C_{d-1} and radius r_{out}, within which all possible (N, M)-POVMs are located according to the constraint (5). It is defined by

$$r_{\mathrm{out}}^2 = \min\left(\frac{d\,(M-1)}{M^2}, \frac{d\,(d-1)}{M^2}\right) \tag{22}$$

so that (5) reduces to the relation $0 < x - d/M^2 \leq r_{\mathrm{out}}^2$. An (N, M)-POVM with $x - d/M^2 = r_{\mathrm{out}}^2$ is called optimal. The ratio between the range of x-values around its minimal value of d/M^2, for which an (N, M)-POVM can always be constructed, i.e., r_{in}^2, and the corresponding maximal possible range, i.e., r_{out}^2, is given by

$$R\,(d) = \frac{r_{\mathrm{in}}^2}{r_{\mathrm{out}}^2} = \begin{cases} \dfrac{1}{(d-1)^2} & M \geq d, \\ \dfrac{1}{(M-1)\,(d-1)} & 2 \leq M < d. \end{cases} \tag{23}$$

In Fig. 3 the dependence of this ratio $R(d)$ is depicted for cases with $M \geqslant d$ (blue points) and for $2 = M < d$ (orange points). It is apparent that this ratio converges to zero rapidly with increasing dimensions of the quantum system's Hilbert space d. Correspondingly, the size of the interval of x-values, for which (N, M)-POVMs can be constructed with arbitrary choices of traceless, orthonormal, and hermitian operator bases, rapidly tends to zero. For cases with $2 < M < d$ the values of $R(d)$ are located inside the region between the two series of dotted points of Fig. 3. In the exceptional case of a qubit, i.e., $d = 2$, the inner and outer radius are identical, i.e., $r_{\mathrm{in}}^2 = r_{\mathrm{out}}^2 = 2/M^2$, and for $M = 4$ the set of positive semidefinite operators forms the Bloch sphere.

Fulfillment of the sufficient condition (21) allows the construction of (N, M)-POVMs for arbitrary choices of the $(d^2 - 1)$ traceless elements of the hermitian operator basis of $\mathcal{H}_{d^2}$ according to the ansatz (14). However, from the explicit expression of r_{in}^2 (cf. (20) and Fig. 3)) it is also apparent that the range of x-values, for which this sufficient condition can be fulfilled, decreases rapidly with increasing values of M. Thus, in general it is an intricate problem to construct (N, M)-POVMs, if the sufficient condition of (21) is not applicable. In particular, in these cases the choice of the traceless hermitian operator basis elements entering (14) can be crucial for the construction. Typically the most complicated situations arise for the construction of optimal (N, M)-POVMs. Motivated by these problems in the subsequent section we explore conditions for the construction of optimal (N, M)-POVMs.

4. Conditions for the existence of optimal (N, M)-POVMs

In this section conditions for the existence of optimal (N, M)-POVMs of a d-dimensional quantum system are presented. As a first main result it is shown that for $M \geqslant d$ the existence of $(M-1)N$ isospectral, traceless, orthonormal, and hermitian operators with a special common spectrum is necessary for the existence of an optimal (N, M)-POVM. As a second main result it is demonstrated that in cases with $2 < M < d$ the existence of an optimal (N, M)-POVM is even necessary and sufficient for the existence of $(M-1)N$ isospectral, traceless, orthonormal, and hermitian operators with a special but different form of their common spectrum.

4.1. Optimal (N, M)-POVMs for $M \geqslant d$

Let us consider an optimal (N, M)-POVM Π of a d-dimensional quantum system with $M \geqslant d$ with $x = d^2/M^2$. According to the defining relations (2) and (3), each element $\Pi_{i(\alpha,a)}$ of this (N, M)-POVM fulfills the relations

$$Tr\left\{\Pi_{i(\alpha,a)}\right\} = \sum_{\sigma=1}^{d} \lambda_\sigma = \frac{d}{M}, \quad Tr\left\{\left(\Pi_{i(\alpha,a)}\right)^2\right\} = \sum_{\sigma=1}^{d} \lambda_\sigma^2 = \frac{d^2}{M^2} \tag{24}$$

with non-negative eigenvalues λ_σ for $\sigma \in \{1, \cdots, d\}$. Both relations constrain the possible values of these eigenvalues, because they imply $\sum_{1=\sigma<\sigma'}^{d} \lambda_\sigma \lambda_{\sigma'} = 0$. Therefore, the positive semidefiniteness of all eigenvalues λ_σ for $\sigma \in \{1, \cdots, d\}$ implies that there is only one non-zero eigenvalue of magnitude d/M. Correspondingly, an arbitrary POVM element of the optimal (N, M)-POVM Π has to be of the general form

$$\Pi_{i(\alpha,a)} = \frac{d}{M} |i\,(\alpha, a)\rangle \langle i\,(\alpha, a)| \tag{25}$$

The defining relations (2), (3), and (4) constrain the scalar products of the generally non-orthogonal but normalized eigenstates $|i(\alpha, a)\rangle$ by the relations

$$|\langle i\,(\alpha, a)\,|i\,(\alpha, a')\rangle| = \sqrt{\frac{M/d - 1}{M - 1}}, \quad |\langle i\,(\alpha, a)\,|i\,(\beta, b)\rangle| = \sqrt{\frac{1}{d}} \tag{26}$$

for $\alpha \neq \beta \in \{1, \cdots, N\}$, $a \neq a' \in \{1, \cdots, M\}$ and $a, a', b \in \{1, \cdots, M\}$.

For each $\alpha \in \{1, \cdots, N\}$ and $a \in \{1, \cdots, M-1\}$ one can construct the following traceless, hermitian operators

$$G_{i(\alpha,a)} = \frac{\sqrt{M-1}}{\left(\sqrt{M}+1\right)\sqrt{M^2 x - d}}\left(\mathbb{1}_d + A_{i(\alpha,a)}\right) \quad \text{with}$$
$$A_{i(\alpha,a)} = \sqrt{M}\,\Pi_{i(\alpha,M)} - \sqrt{M}\left(\sqrt{M}+1\right)\Pi_{i(\alpha,a)}. \tag{27}$$

It is straightforward to demonstrate that for an arbitrary (N, M)-POVM these $(M-1)N$ traceless hermitian operators

$G_{i(\alpha,a)}$ are orthonormal. For an optimal (N, M)-POVM we have $x = d^2/M^2$. In this special case, according to (25) and (27) each hermitian operator $A_{i(\alpha,a)}$ has a maximal rank of two. Therefore, the characteristic polynomials determining the eigenvalues Λ of $A_{i(\alpha,a)}$ have the general form

$$\Lambda^d + c_{d-1}\Lambda^{d-1} + c_{d-2}\Lambda^{d-2} = 0 \tag{28}$$

Using the Cayley-Hamilton theorem [16] it is easily found that

$$c_{d-1} = -Tr\left\{A_{i(\alpha,a)}\right\} = d,$$
$$c_{d-2} = \frac{1}{2}\left(\left(Tr\left\{A_{i(\alpha,a)}\right\}\right)^2 - Tr\left\{A^2_{i(\alpha,a)}\right\}\right) = \frac{d-d^2}{\sqrt{M}-1}. \tag{29}$$

Consequently, all $(M-1)N$ traceless, orthonormal, hermitian operators $\{G_{i(\alpha,a)}|\alpha = 1, \cdots, N, a = 1, \cdots, M-1\}$ have the same spectrum $\mathrm{Sp}(G_{i(\alpha,a)})$ determined by the solutions of (28), i.e.,

$$\mathrm{Sp}\left(G_{i(\alpha,a)}\right) = \left\{\left(\frac{\sqrt{M-1}}{\left(\sqrt{M}+1\right)\sqrt{d^2-d}}(1+\Lambda_+)\right)^{(1)}, \left(\frac{\sqrt{M-1}}{\left(\sqrt{M}+1\right)\sqrt{d^2-d}}(1+\Lambda_-)\right)^{(1)}, \left(\frac{\sqrt{M-1}}{\left(\sqrt{M}+1\right)\sqrt{d^2-d}}\right)^{(d-2)}\right\} \tag{30}$$

with

$$\Lambda_\pm = \frac{1}{2}\left(-d \pm \sqrt{d^2 + 4\frac{d^2-d}{\sqrt{M}-1}}\right). \tag{31}$$

The numbers in brackets in the exponents of (30) indicate the multiplicities of the corresponding eigenvalues.

Therefore, it can be concluded that for $M \geqslant d$ the existence of an optimal (N, M)-POVM Π implies the existence of a set of $(M-1)N$ isospectral, traceless, and orthonormal hermitian operators $G_{i(\alpha,a)}$ defined by (27), whose common spectrum is given by (30). Stated differently, the existence of a set of $(M-1)N$ isospectral, traceless, orthonormal, hermitian operators $G_{i(\alpha,a)}$, whose common spectrum is given by (30), is necessary for the existence of an optimal (N, M)-POVM. This result generalizes an already known property of GSICs [8], i.e., optimal informationally complete $(1, d^2)$-POVMs, to all optimal (N, M)-POVMs with $M \geqslant d$.

4.2. Optimal (N, M)-POVMs for $2 < M < d$

In this subsection we prove a necessary and sufficient condition for the existence of optimal (N, M)-POVMs. It is shown that the existence of $(M-1)N$ isospectral, traceless, orthonormal, hermitian operators with a special form of their common spectrum is equivalent to the existence of an optimal (N, M)-POVM. Although the discussion of this subsection also applies to cases with $M = 2$, these latter cases will be discussed separately in Section 5.

Let us consider an optimal (N, M)-POVM of a d-dimensional quantum system with $2 < M < d$. According to (5) it is characterized by the parameter $x = d/M$. Furthermore, the defining condition (3) implies that each pair of different elements of a single POVM α has to fulfill the relation (cf. (17))

$$0 = Tr\left\{\Pi_{i(\alpha,a)}\Pi_{i(\alpha,b)}\right\} = Tr\left\{\sqrt{\Pi_{i(\alpha,a)}}\sqrt{\Pi_{i(\alpha,b)}}\sqrt{\Pi_{i(\alpha,b)}}\sqrt{\Pi_{i(\alpha,a)}}\right\}$$
$$= \|\sqrt{\Pi_{i(\alpha,b)}}\sqrt{\Pi_{i(\alpha,a)}}\|^2_{\mathrm{HS}} \tag{32}$$

for all $\alpha \in \{1, \ldots, N\}$, $a, b \in \{1, \ldots, M\}$ with $a \neq b$. Thus, within a single POVM any two different elements are orthogonal, i.e., $\Pi_{i(\alpha,a)}\Pi_{i(\alpha,b)} = 0$ for $a \neq b$. Furthermore, as a consequence of the completeness relation (1), within a single POVM α all POVM elements are orthogonal projections of rank d/M (cf. (2)), i.e.,

$$\Pi_{i(\alpha,a)} = \Pi_{i(\alpha,a)}\sum_{b=1}^{M}\Pi_{i(\alpha,b)} = \Pi^2_{i(\alpha,a)} = \sum_{k=(a-1)d/M+1}^{ad/M}|\alpha,k\rangle\langle\alpha,k| \tag{33}$$

with a common orthonormal eigenbasis $\{|\alpha,k\rangle \mid k = 1, \cdots, d\}$ of the elements of a single POVM α. Necessarily, in dimension d an optimal (N, M)-POVM can only exist, if d is an integer multiple of M, i.e., $d/M \in \mathbb{N}$. Therefore, we conclude that for $2 < M < d$ condition (3) is equivalent to

$$\Pi_{i(\alpha,a)}\Pi_{i(\alpha,a')} = \Pi_{i(\alpha,a)}\delta_{aa'} \tag{34}$$

for optimal (N, M)-POVMs and $\alpha \in \{1, \ldots, N\}$ and $a \in \{1, \ldots, M\}$. According to this necessary condition the smallest dimension d, for example, for which an optimal informationally complete (N, M)-POVM can possibly be constructed, is given by $d = 8$. It is a $(21, 4)$-POVM with $x = 2$ and all its POVM elements are of rank two. According to (27) for $2 < M < d$ for each optimal (N, M)-POVM a set of $(M-1)N$ orthonormal, traceless, hermitian operators can be constructed. With the help of (33) its spectrum can be obtained easily yielding the result

$$\mathrm{Sp}\left(G_{i(\alpha,a)}\right) == \left\{\left(\frac{1}{\sqrt{d}}\right)^{(d/M)}, \left(\frac{1-\sqrt{M}\left(\sqrt{M}+1\right)}{\left(\sqrt{M}+1\right)\sqrt{d}}\right)^{(d/M)}, \left(\frac{1}{\left(\sqrt{M}+1\right)\sqrt{d}}\right)^{(d-2d/M)}\right\}. \tag{35}$$

Thereby, the relations

$$G_{i(\alpha,a)}|\alpha,k\rangle = \frac{1}{\sqrt{d}}|\alpha,k\rangle,$$
$$G_{i(\alpha,a)}|\alpha,l\rangle = \frac{\left(1-\sqrt{M}\left(\sqrt{M}+1\right)\right)}{\left(\sqrt{M}+1\right)\sqrt{d}}|\alpha,l\rangle,$$
$$\left[G_{i(\alpha,a)}, G_{i(\alpha,b)}\right] = 0 \tag{36}$$

with $a, b \in \{1, \ldots, M-1\}$, $\alpha \in \{1, \ldots, N\}$, $k \in \{(M-1)d/M + 1, \ldots, d\}$ and $l \in \{(a-1)d/M + 1, \ldots, ad/M\}$ have been used. Therefore, it can be concluded that for $2 < M < d$ the existence of an optimal (N, M)-POVM implies the existence of a set of $(M-1)N$ isospectral, traceless, orthonormal hermitian operators

$G_{i(\alpha,a)}$ defined by (27) with $x = d/M$, whose common spectrum is given by (35) and fulfills the relations (36).

However, for optimal (N, M)-POVMs with $2 < M < d$ this condition is also sufficient. In order to demonstrate this we start from rewriting (27) in the equivalent form

$$\Pi_{i(\alpha,M)} = \frac{\mathbf{1}_d}{M} + \frac{\sqrt{M^2x-d}}{M\sqrt{M-1}}\sum_{b'=1}^{M-1} G_{i(\alpha,b')},$$
$$\Pi_{i(\alpha,b)} = \frac{\mathbf{1}_d}{M} + \frac{\sqrt{M^2x-d}}{M\sqrt{M-1}\left(\sqrt{M}+1\right)}\sum_{b'=1}^{M-1} G_{i(\alpha,b')} - \frac{\sqrt{M^2x-d}}{\sqrt{M-1}\sqrt{M}}G_{i(\alpha,b)} \tag{37}$$

with $\alpha \in \{1, \cdots, N\}$ and $b \in \{1, \cdots, M-1\}$. Although these operators $\Pi_{i(\alpha,b)}$ fulfill the relations (2), (3), and (4), in general they are not positive semidefinite. However, in the case we are considering here, i.e., $x = d/M$ and $2 < M < d$, it is straightforward to demonstrate that for a set of $(M-1)N$ orthonormal, traceless, hermitian operators $\{G_{i(\alpha,b)} | \alpha = 1, \cdots, N, b = 1, \cdots, M-1\}$ of a d dimensional quantum system, which commute (cf. (36)) and whose common spectrum in given by (35), these operators $\Pi_{i(\alpha,b)}$ of (37) constitute an optimal (N, M)-POVM.

Therefore, for $2 < M < d$ the existence of a set of $(M-1)N$ isospectral, traceless, orthonormal, hermitian operators $G_{i(\alpha,a)}$, whose common spectrum is given by (35) and which fulfill the relations (36), is equivalent with the existence of an optimal (N, M)-POVM.

5. Optimal $(N, 2)$-POVMs

In this section optimal $(N, 2)$-POVMs of d-dimensional quantum systems are investigated. For them additional properties can be derived, which transcend the conditions discussed in Section 4.2.

Let us consider an arbitrary POVM element of an optimal $(N, 2)$-POVM as given by (17) with $x = d/2$ and positive semidefinite eigenvalues of the form $\lambda_\sigma = 1/2 + \eta_\sigma$ for $\sigma \in \{1, \cdots, d\}$. According to the arguments of Section 4.2 $\lambda_\sigma \in \{0, 1\}$ so that $|\eta_\sigma| = 1/2$. In addition, we have $d/2 \in \mathbb{N}$ so that the dimension d has to be even. Therefore, relation (3) can only be fulfilled, if the spectrum of the traceless, normalized and hermitian operators

$$K_{i(\alpha,a)} = \frac{\Pi_{i(\alpha,a)} - \mathbb{1}_d/2}{\sqrt{d/4}} \tag{38}$$

is given by

$$\mathrm{Sp}\left(K_{i(\alpha,a)}\right) = \left\{ +\frac{1}{\sqrt{d}}^{(d/2)}, -\frac{1}{\sqrt{d}}^{(d/2)} \right\} \tag{39}$$

for each $i(\alpha, a) \in \{1, \cdots, 2N\}$. The numbers in brackets in the exponents of (39) indicate the multiplicities of the corresponding eigenvalues. In view of relation (4) the operators $K_{i(\alpha,a)}$ and $K_{i(\beta,b)}$ are also orthogonal for $\alpha \neq \beta \in \{1, \cdots, N \leq d^2 - 1\}$ and $a, b \in \{1, 2\}$. Thereby, we have taken into account that for a d-dimensional quantum system the number of traceless, orthogonal, hermitian operators cannot exceed $d^2 - 1$. However, these operators $K_{i(\alpha,a)}$ and $K_{i(\beta,b)}$ are not orthogonal for $\alpha = \beta$ and $a \neq b$ and fulfill the relation

$$K_{i(\alpha,2)} = -K_{i(\alpha,1)}. \tag{40}$$

Therefore, for $N \leq d^2 - 1$ the existence of an optimal $(N, 2)$-POVM implies the existence of N isospectral, traceless, orthonormal, hermitian operators $\{K_{i(\alpha,1)} | \alpha \in \{1, \cdots, N\}\}$, whose common spectrum is given by (39). However, in view of (38) and (40) this conclusion can also be turned around. Thus, for $N \leq d^2 - 1$ the existence of an optimal $(N, 2)$-POVM is sufficient and necessary for the existence of N isospectral, traceless, orthonormal, hermitian operators $\{K_{i(\alpha,1)} | \alpha \in \{1, \cdots, N\}\}$, whose common spectrum is given by (39). Thereby, the case $N = d^2 - 1$ covers optimal informationally complete $(N, 2)$-POVMs. It should be mentioned that this existence criterion for optimal $(N, 2)$-POVMs with $N \leq d^2 - 1$ generalizes a recent weaker result [3], which was based on the weaker assumption $|\eta_\sigma| \leq 1/2$. This criterion is the special case of $M = 2$ of the results presented in Section 4.2.

In the special cases of even dimensions of the form $d = 2^k$ with $k \in \mathbb{N}$, $N \leq d^2 - 1$ isospectral, traceless, orthonormal, and hermitian operators can easily be constructed with the help of the Clifford algebra generated by tensor products of Pauli operators. Accordingly, these operators are given by

$$\frac{1}{\sqrt{2^k}}\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \sigma_{i_k} \tag{41}$$

with $(i_1, \cdots, i_k) \neq (0, \cdots, 0)$ and with the Pauli operators

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{42}$$

Therefore, for $N \leq d^2 - 1$ optimal $(N, 2)$-POVMs in dimension $d = 2^k$ can easily be constructed. Their $2N$ elements are given by

$$\Pi_{i((i_1,\cdots,i_k),1)} = \frac{\mathbb{1}_d}{2} + \frac{1}{2}\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \sigma_{i_k},$$
$$\Pi_{i((i_1,\cdots,i_k),2)} = \frac{\mathbb{1}_d}{2} - \frac{1}{2}\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \sigma_{i_k} \tag{43}$$

with $(i_1, \cdots, i_k) \neq (0, \cdots, 0)$. Nevertheless, this explicit basis construction still leaves the question open, to which extent such optimal informationally complete $(N, 2)$-POVMs also exist in other even dimensions $d \neq 2^k$.

6. Summary and conclusions

Motivated by the recent interest in basic theoretical properties of (N, M)-POVMs, we have explored general features of their existence and construction with the help of orthonormal, hermitian operator bases for arbitrary d-dimensional quantum systems. A sufficient condition has been derived for the existence of an arbitrary (N, M)-POVM. It generalizes the already known property, that (N, M)-POVMs can always be constructed for sufficiently small values of their continuous x-parameter [3]. In particular, it yields an explicit expression for an upper bound on this continuous x-parameter, below which all POVM elements are guaranteed to be positive semidefinite. Furthermore, conditions on the existence of optimal (N, M)-POVMs have been presented. They exhibit a close connection to the existence of sets of $(M-1)N$ isospectral, traceless, orthonormal, hermitian operators with particular forms of their common spectra. The particular form of the two possible types of common spectra depends on whether M is smaller than the dimension of the quantum system under consideration or not. These conditions generalize a property, recently found for the special case of GSICs [8], to arbitrary optimal (N, M)-POVMs. This connection motivates further research on the construction of such sets of isospectral, traceless, orthonormal hermitian operators in order to shed new light on the construction of optimal (N, M)-POVMs. For the special cases with $M = 2$ a necessary and sufficient condition has been derived for the existence of optimal $(N, 2)$-POVMs with $N \leq d^2 - 1$. Thereby, also a relation to the existence of a set of N isospectral, traceless, orthonormal, hermitian operators has been established. Such operators with the required common spectrum can only exist in even dimensions. For dimensions $d = 2^k,\ k \in \mathbb{N}$ these operators can easily be constructed with the help of the Clifford algebra generated by the k-fold tensor products of the Pauli operators.

The recently introduced (N, M)-POVMs [3] are potentially interesting for numerous tasks of quantum information processing, such as the exploration of provable entanglement in quantum communication or quantum state tomography. Our presented sufficient and necessary conditions do not only shed new light on currently open questions concerning their existence and construction but also concerning their application for practical purposes. Our presented sufficient condition for their existence, for example, has established an explicit upper bound on the continuous x-parameters guaranteeing their existence. Combining this result with the recent observation [9], that typical bipartite entanglement can be detected locally in an optimal way by local (N, M)-POVMs fulfilling this sufficient condition, suggests interesting applications of (N, M)-POVMs for the detection of provable entanglement in quantum communication protocols. In view of these promising aspects also for applications we are confident that (N, M)-POVMs will play an interesting and practically useful role in future work exploring the intricacies of quantum correlations.

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Data availability

Data generated or analyzed during this study are provided in full within the published article.

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References

1. A.S. Holevo, Statistical structure of quantum theory. Springer, Berlin. 2001. doi:10.1007/3-540-44998-1.
2. J.A. Bergou, M.S. Hillery, and M. Saffman, Quantum information processing: theory and implementation. Springer, Cham. 2021. doi:10.1007/978-3-030-75436-5.
3. K. Siudzinska, Phys. Rev. A, **105**, 042209 (2022). doi:10.1103/PhysRevA.105.042209.
4. W.K. Wootters, and B.D. Fields, Ann. Phys. **191**, 363 (1989). doi:10.1016/0003-4916(89)90322-9.
5. A. Kalev, and G. Gour, New J. Phys. **16**, 053038 (2014). doi:10.1088/1367-2630/16/5/053038.
6. J.M. Renes, R. Blume-Kohout, A.J. Scott, and C.M. Caves, J. Math. Phys. **45**, 2171 (2004). doi:10.1063/1.1737053.
7. A.E. Rastegin, Phys. Scr. **89**, 085101 (2014). doi:10.1088/0031-8949/89/8/085101.
8. G. Gour, and A. Kalev, J. Phys. A: Math. Theor. **47**, 335302 (2014). doi:10.1088/1751-8113/47/33/335302.
9. M. Schumacher, and G. Alber, Phys. Rev. A, **108**, 042424 (2023). doi:10.1103/PhysRevA.108.042424.
10. M. Schumacher, and G. Alber, Phys. Scr. **98**, 115234 (2023). doi:10.1088/1402-4896/acfc7c.
11. C.A. Fuchs, M.C. Hoang, and B.C. Stacey, Axioms, **6**, 21 (2017). doi:10.3390/axioms6030021.
12. P. Horodecki, L. Rudnicki, and K. Zyczkowski, PRX Quantum, **3**, 010101 (2022). doi:10.1103/PRXQuantum.3.010101.
13. G. Kimura, Phys. Lett. A, **314**(5-6), 339 (2003). doi:10.1016/S0375-9601(03)00941-1.
14. I. Bengtsson, and K. Zyczkowski, Geometry of quantum states. Cambridge University Press, Cambridge. 2006. doi:10.1017/CBO9780511535048.
15. G. Kimura, and A. Kossakowski, Open Sys. Inf. Dyn. **12**, 207 (2005). doi:10.1007/s11080-005-0919-y.
16. G. Frobenius, J. Reine Angew. Math. **84**, 1 (1878).

Appendix A. Gell-Mann basis for $d = 3$

The Gell-Mann basis, which has been used in obtaining Figs. 1*a*–1*c*, is defined by the matrices

$$g_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$g_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$g_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad g_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$g_7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad g_8 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A1}$$

These $(d^2 - 1) = 8$ hermitian matrices have vanishing traces and are orthogonal with respect to the Hilbert–Schmidt scalar product. Together with the properly normalized unit matrix they form an orthonormal basis of the Hilbert space $\mathcal{H}_{d^2}$ for $d = 3$.