Detection of typical bipartite entanglement by local generalized measurements

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Motivated by the need for efficient local entanglement detection for applications in quantum information processing, sufficient conditions for arbitrary-dimensional local bipartite entanglement detection based on correlation matrices and joint probability distributions are investigated. In particular, their dependence on the nature of different classes of local measurements is explored for generalized measurements based on informationally complete (N, M) positive-operator-valued measures (POVMs) [Siudzinska, Phys. Rev. A **105**, 042209 (2022)]. It is shown that symmetry properties of (N, M) POVMs necessarily imply that these sufficient conditions for bipartite entanglement detection exhibit characteristic scaling properties relating equivalent sufficient conditions. Based on these general scaling properties, the efficiency of different classes of local quantum measurement detecting typical bipartite entanglement is investigated quantitatively. For this purpose Euclidean volume ratios between locally detectable bipartite entangled states and all bipartite quantum states are determined numerically with the help of a Monte Carlo algorithm. Our results demonstrate that physically realizable (N, M) POVMs are sufficient for optimal local entanglement detection. In particular, this implies that for this purpose the construction of optimal (N, M) POVMs is not necessary. As questions concerning the existence and construction of optimal (N, M) POVMs are still largely open, this may offer interesting perspectives for practical applications in quantum information processing.

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I. INTRODUCTION

Entanglement [1] is not only a characteristic quantum phenomenon of composite quantum systems distinguishing them significantly from classical physical systems, but also a valuable resource for quantum information science. It enables numerous applications, particularly in quantum key distribution and more generally in quantum communication [2]. For two-qubit and qubit-qutrit systems the Peres-Horodecki criterion [3,4] yields a simple necessary and sufficient condition for identifying bipartite entanglement based on a quantum state's negative partial transpose (NPT). However, the NPT property is only sufficient and no longer necessary for bipartite entanglement in higher-dimensional cases due to the intricate features of bound entanglement [1]. Nevertheless, establishing the NPT property of a bipartite quantum state is still an important (sufficient) method for ensuring its entanglement. As it is not known how to implement the partial transposition of an unknown bipartite quantum state, it is interesting to develop quantum measurements that yield sufficient conditions for efficiently detecting entanglement. Thereby, quantum measurements which can be performed locally, possibly even by far distant observers, are of particular interest for applications in quantum key distribution and quantum communication. In this context, the following natural question arises: How does the efficiency of entanglement detection by local measurements depend on the chosen quantum measurements and on the dimensionality of the systems involved for typical quantum states [5].

In this paper we address this question. The intention of our investigation is twofold. First, we want to compare sufficient conditions for bipartite entanglement detection resulting from different classes of local measurements. Second, we want to quantitatively explore the efficiencies of different classes of local measurements for entanglement detection of arbitrary bipartite quantum states. In this context local measurements involving generalized measurements based on informationally complete positive-operator-valued measures (POVMs) [6,7] are of particular current interest. We focus on (N, M) POVMs, which were recently introduced [8] in order to unify the theoretical description of numerous important classes of quantum measurements, including projective measurements involving mutually unbiased bases (MUBs) [9], mutually unbiased measurements (MUMs) [10], symmetric informationally complete measurements (SIC POVMs) [11,12], and their generalizations, so-called GSIC POVMs [13]. In principle, by measurements based on informationally complete (N, M) POVMs, quantum states can be reconstructed completely. Therefore, these generalized measurements are expected to be particularly well suited for detecting bipartite entangled states efficiently. For a quantitative exploration of this efficiency it is necessary to determine the fractions of bipartite entangled states which can be detected in the state space of all bipartite states. Therefore, a proper statistical exploration of the complete state space of all bipartite quantum states is required. In particular, restrictions to few-parameter families of quantum states, which have been frequently investigated in this context recently [5,14–17], form sets of measure zero in this complete state space and are incapable of exploring these typical statistical features. Furthermore, we want to investigate bipartite entanglement detection by arbitrary local (N, M) POVMs without restricting ourselves to (N, M) POVMs which can be constructed by a particular basis of Hermitian operators, such as the generalized Gell-Mann matrices [8,10,13].

As a first main result it will be shown that inherent symmetry properties of (N, M) POVMs necessarily imply that recently discussed sufficient conditions for bipartite entanglement [5,18–21] exhibit characteristic scaling properties which relate different equally efficient local entanglement detection scenarios. As a second main result, based on these sufficient conditions, we explore the efficiency of typical bipartite entanglement detection by local measurements numerically with a hit-and-run Monte Carlo algorithm. For this purpose, we numerically determine lower bounds on the Euclidean volume ratios between locally detectable bipartite entangled states and all bipartite quantum states and compare them with the corresponding volume ratios of NPT states. This is achieved with the help of a recently developed hit-and-run Monte Carlo algorithm [22,23].

This paper is organized as follows. In Sec. II A basic properties of (N, M) POVMs, as introduced originally by Siudzinska [8], are summarized. As a main result of Sec. II B, general relations between arbitrary orthonormal Hermitian operator bases and informationally complete (N, M) POVMs are derived. These relations have to be fulfilled necessarily and imply general scaling relations between different local entanglement detection scenarios of equal efficiency. These results constitute the basis for our subsequent investigations on the scaling relations between different local entanglement detection scenarios. In Sec. III correlation matrix-based and joint probability-based sufficient conditions for bipartite entanglement detection by local quantum measurements are discussed within a general framework. These results are applied to measurements based on arbitrary local orthonormal Hermitian operator (LOO) bases and on local (N, M) POVMs. As a main result, based on Sec. II, the resulting scaling relations are derived for LOOs and (N, M) POVMs. This result also unifies previously known sufficient conditions for bipartite entanglement detection which have been derived for some special cases. The general scaling relations of this section allow us to identify equivalent sufficient bipartite entanglement conditions which yield equally efficient entanglement detection scenarios. As a second main result, numerical Monte Carlo results on lower bounds of Euclidean volume ratios of detectable bipartite entangled states are presented in Sec. IV for local measurements involving (N, M) POVMs. These lower bounds are compared to the corresponding ratios of NPT bipartite quantum states. Based on these numerical results on locally detectable entanglement of typical bipartite quantum states, the efficiencies of different local entanglement detection procedures can be compared quantitatively.

II. INFORMATIONALLY COMPLETE GENERALIZED MEASUREMENTS

The recently introduced (N, M) POVMs describe numerous important quantum measurement procedures in a unified way [8]. In this section we are particularly interested in informationally complete (N, M) POVMs whose elements span the Hilbert space of all Hermitian operators of a quantum system. For the sake of completeness, defining general properties of (N, M) POVMs are summarized in Sec. II A. In Sec. II B conditions are derived which necessarily relate orthonormal Hermitian operator bases and informationally complete (N, M) POVMs irrespective of their positive semidefiniteness. For this purpose, explicit constructions of these (N, M) POVMs are not necessary. These necessarily valid relations are characterized by orthogonality preserving linear maps between linear Hermitian operator spaces. In the subspace of Hermitian operators, which are orthogonal to the unit operator of the quantum system's Hilbert space, these linear maps act conformally, thus scaling all elements of this subspace by the same positive amount. This characteristic scaling property is the main result of this section. In the subsequent section it is used to compare the 1-norms of correlation matrices and joint probability matrices for different measurement choices. This allows a comparison of different measurement settings for the sufficient bipartite entanglement conditions discussed in Sec. III.

A. General properties

Let us consider the d-dimensional Hilbert space $\mathcal{H}_d$ of a quantum system. An (N, M) POVM, described by the NM-tuple $\Pi = (\Pi_1, \dots, \Pi_{NM})$, is a set of NM positive-semidefinite operators $\Pi_i \geqslant 0$ $(i = 1, \dots, NM)$. They constitute N different POVMs formed by the N possible disjoint subsets of cardinality M. Therefore, the members Π_i of such an (N, M) POVM can be identified uniquely by ordered pairs (α, a), with α identifying the particular POVM and a identifying the possible classical measurement result of this particular POVM α. Such an identification can be obtained by the bijective map $i : \{(\alpha, a) \mid \alpha \in \{1, \dots, N\},\ a \in \{1, \dots, M\}\} \to \{1, \dots, NM\}$ with $i(\alpha, a) = (\alpha - 1)M + a$, for example. Therefore, each of the N POVMs fulfills the characteristic completeness relations for the possible distinguishable classical measurement results $a \in \{1, \dots, M\}$, i.e.,

$$\sum_{a=1}^{M} \Pi_{i(\alpha,a)} = \mathbb{1}_d \tag{1}$$

for each $\alpha \in \{1, \dots, N\}$. Thereby, $\mathbb{1}_d$ denotes the unit operator in the Hilbert space $\mathcal{H}_d$. In addition, the positive-semidefinite operators of an (N, M) POVM fulfill the characteristic additional relations [8]

$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\} = \frac{d}{M}, \tag{2}$$

$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\, \Pi_{i(\alpha,a')}\} = x\, \delta_{a,a'} + (1 - \delta_{a,a'}) \frac{d - Mx}{M(M-1)}, \tag{3}$$

$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\, \Pi_{i(\beta,b)}\} = \frac{d}{M^2} \tag{4}$$

for all $\beta \neq \alpha \in \{1, \dots, N\}$ and $a, a', b \in \{1, \dots, M\}$ if $N \geqslant 2$. In the degenerate case of a POVM with $N = 1$ the constraint (4) is not imposed. For given values of (d, N, M), the possible values of x are constrained by the relation $d/M^2 < x \leqslant \min(d^2/M^2, d/M)$. The (N, M) POVMs with maximal possible values of x are called optimal.

By definition, an (N, M) POVM Π is informationally complete if it contains d^2 linearly independent positive operators. As each of the N POVMs involved fulfills the completeness relation (1), this is equivalent to the requirement

$$(M-1)N+1=d^2. \tag{5}$$

There are at least four classes of possible solutions of (5) [8], namely, $(N,M)\in\{(1,d^2),(d+1,d),(d^2-1,2),(d-1,d+2)\}$. The solution $(N,M)=(1,d^2)$ characterizes the special case of a one-parameter family of GSIC POVMs [13] parameterized by the parameter x. Symmetric informationally complete POVMs correspond to the special case of GSIC POVMs with $x=1/d^2$. The solution $(N,M)=(d+1,d)$ describes MUMs [10], which in the special case of $x=d^2/M^2=d/M=1$ further reduce to projective measurements of unit rank with maximal sets of $(d+1)$ MUBs. In the special case of a qubit, i.e., $d=2$, these four possible solutions of (5) reduce to two cases, namely, GSIC POVMs for $(N,M)=(1,4)$ and MUMs for $(N,M)=(3,2)$.

B. Informationally complete (N,M) POVMs and orthonormal Hermitian operator bases

To compare the sufficient entanglement conditions discussed in Sec. III, it is necessary to relate the correlation matrices and joint probability distributions of local informationally complete (N,M) POVMs to those of arbitrary orthonormal Hermitian operator bases. For this purpose, in this section we study the linear maps between Hermitian operators fulfilling the characteristic relations of (N,M) POVMs [cf. (3) and (4)] and arbitrary orthonormal Hermitian operator bases. In a Hilbert space $\mathcal{H}_d$ of a d-dimensional quantum system, an informationally complete (N,M) POVM can be expanded in a basis of d^2 linearly independent linear operators, say, $G=(G_1,\dots,G_{d^2})^T$, acting on $\mathcal{H}_d$. These operators can always be chosen as orthonormal Hermitian operators with respect to the Hilbert-Schmidt scalar product $\langle G_\mu|G_\nu\rangle_{\mathrm{HS}}:=\mathrm{Tr}\{G_\mu^\dagger G_\nu\}$ with $G_\mu^\dagger=G_\mu$. They form a basis of the Hilbert space $\mathcal{H}_{d^2}=(\mathrm{Span}(G),\langle\cdot|\cdot\rangle_{\mathrm{HS}})$ of linear operators in $\mathcal{H}_d$ over the field of real numbers. This latter space is a Euclidean vector space.

The resulting basis expansion of an arbitrary (N,M) POVM in such an arbitrary orthonormal Hermitian basis of linear operators has the general form

$$\Pi=G^TS, \tag{6}$$

with S denoting the linear operator which maps $\mathcal{H}_{d^2}$ into the possibly higher-dimensional Hilbert space $\mathcal{H}_{NM}$ of Hermitian operators. This is a consequence of the dimensional constraint $NM=d^2+N-1\geqslant d^2$ valid for informationally complete (N,M) POVMs. The structure of this linear map S and its corresponding $d^2\times NM$ matrix $S_{\mu,i}$ of real-valued coefficients is significantly constrained by the relations (3) and (4) characterizing essential features of (N,M) POVMs irrespective of their positive semidefiniteness in a basis independent way. This linear map is important for the subsequent discussion of entanglement detection because it allows us to relate equivalent sufficient entanglement conditions which result in equal entanglement detection efficiencies.

Let us determine the most general form of the linear operator $S:\mathcal{H}_{d^2}\to\mathcal{H}_{NM}$ mapping the Hilbert space $\mathcal{H}_{d^2}$ into the Hilbert space $\mathcal{H}_{NM}$. Thereby, we will ignore the constraints imposed by the positive semidefiniteness of the operators Π and only take into account the relations (3) and (4). This implies that the linear operator $S:\mathcal{H}_{d^2}\to\mathcal{H}_{NM}$ characterizes all relations which necessarily must be fulfilled between an orthonormal Hermitian basis G and any possible (N,M) POVM Π. In the Appendix it is demonstrated that one can always choose an orthonormal Hermitian operator basis $\tilde{G}$ in such a way that $\tilde{G}_1=\mathbb{1}_d/\sqrt{d}$ and $\mathrm{Tr}\{\tilde{G}_\nu\}=0$ for $\nu\in\{2,\dots,d^2\}$ [cf. (A11)] and that this basis implies the relation $\Pi=G^TS=\tilde{G}^T\tilde{S}$ with

$$\tilde{S}_{\nu,i}=\sqrt{\Lambda_\nu}X^T_{\nu,i}. \tag{7}$$

The orthonormal $NM\times d^2$ matrix $X_{i,\nu}$ is constructed by the $d^2=N(M-1)+1$ orthonormal eigenvectors of the $NM\times NM$ matrix $(S^TS)_{i,j}$ [cf. (A7)–(A9)] with nonzero eigenvalues

$$\Lambda_1=\frac{dN}{M},$$
$$\Lambda_\nu=\Gamma=\frac{xM^2-d}{M(M-1)} \tag{8}$$

for $i\in\{1,\dots,NM\}$ and $\nu\in\{2,\dots,d^2\}$. This latter matrix encodes the constraints (3) and (4) characterizing essential features of (N,M) POVMs without taking into account their positive semidefiniteness. From (7) it is apparent that all basis operators $\tilde{G}_\nu$ with $\nu\in\{2,\dots,d^2\}$ are mapped conformally onto a (d^2-1)-dimensional subspace of $\mathcal{H}_{NM}$ by scaling the norms of all these operators by a factor of $\sqrt{\Gamma}$. Only the basis operator $\tilde{G}_1$ is scaled by a different factor, namely, $\sqrt{\Lambda_1}$. In the subsequent section, it will be demonstrated that this special property, relating the orthonormal Hermitian operator basis $\tilde{G}$ to an existing arbitrary informationally complete (N,M) POVM Π, manifests in characteristic scaling relations for sufficient conditions of bipartite entanglement. We want to stress once again that, in view of the arguments leading to (7), for the derivation of these scaling relations the positive semidefiniteness of POVM operators is irrelevant. In this context it is worth mentioning that a recent construction of a family of entanglement witnesses also relies on a linear but different map between (N,M) POVMs and the witnesses for which the positive semidefiniteness of the POVM elements is irrelevant [24]. In the next section these scaling relations are used to compare the detection efficiencies of different sufficient bipartite entanglement conditions.

III. CORRELATIONS OF SEPARABLE BIPARTITE QUANTUM STATES

In this section recently discussed constraints imposed on local correlations of separable bipartite quantum states are summarized and generalized to (N,M) POVMs. Violating these constraints yields sufficient conditions for bipartite entanglement of arbitrary-dimensional quantum systems. In particular, we concentrate on inequalities for the 1-norms of local correlation matrices and joint local probability distributions, the strongest sufficient conditions for bipartite entanglement detection with (N,M) POVMs [5]. We explore the dependence of these sufficient bipartite entanglement conditions on the types of local measurements performed, including arbitrary (N,M) POVMs. As a main result it will be shown that, as far as correlation matrix-based sufficient entanglement conditions are concerned, LOOs are as powerful in detecting bipartite entanglement as any locally applied informationally complete (N, M) POVM. This is a consequence of the characteristic scaling properties presented in Sec. II B, which necessarily have to be fulfilled by informationally complete (N, M) POVMs. For joint bipartite probability distributions of informationally complete local (N, M) POVMs, these scaling properties manifest themselves more subtly. It will be shown that, for given dimensions of the local quantum systems, the resulting sufficient conditions for bipartite entanglement are identical for informationally complete local (N, M) POVMs characterized by the same rescaled x parameters [cf. (3) and (34)].

A. Correlation matrices of general local quantum measurements

Let us consider arbitrary sets of Hermitian operators, say, $\mathcal{A} = \{A_i; i = 1, \ldots, \overline{N}_A\}$ and $\mathcal{B} = \{B_j; j = 1, \ldots, \overline{N}_B\}$, of two local observers, say, Alice and Bob. These operators are supposed to describe local observables, i.e., Hermitian operators or local POVMs. The correlation matrix of a quantum state ϱ associated with these local measurements is defined as

$$[C(\mathcal{A}, \mathcal{B}|\varrho)]_{ij} = \mathrm{Tr}\{A_i \otimes B_j(\varrho - \varrho^A \otimes \varrho^B)\}, \tag{9}$$

with the reduced local quantum states of Alice and Bob, $\varrho^A = \mathrm{Tr}_B\{\varrho\}$ and $\varrho^B = \mathrm{Tr}_A\{\varrho\}$.

An arbitrary bipartite quantum state ϱ is separable if and only if it can be represented in the form of a convex combination of an ensemble of local quantum states of Alice and Bob, i.e.,

$$\varrho = \sum_m p_m \varrho_m^A \otimes \varrho_m^B, \tag{10}$$

with the probabilities $p_m \geqslant 0$ and $\sum_m p_m = 1$. The matrix elements of the correlation matrix of such a general separable quantum state ϱ are given by

$$[C(\mathcal{A}, \mathcal{B}|\varrho)]_{ij} = \sum_{nm} (V_{nm})_i (W_{nm})_j, \tag{11}$$

with the $\overline{N}_A$- and $\overline{N}_B$-dimensional correlation vectors V_{nm} and W_{nm}. The components of these correlation vectors are defined by

$$(V_{nm})_i = \sqrt{\frac{p_n p_m}{2}} \mathrm{Tr}\{A_i \varrho_n^A - A_i \varrho_m^A\} \tag{12}$$

for $i = 1, \ldots, \overline{N}_A$ and by an analogous expression for $(W_{nm})_j$ $(j = 1, \ldots, \overline{N}_B)$. Using the triangular and the Cauchy-Schwarz inequalities, the 1-norm of this correlation matrix, i.e., $\|C(\mathcal{A}, \mathcal{B}|\varrho)\|_1 = \mathrm{Tr}\{\sqrt{C^\dagger(\mathcal{A}, \mathcal{B}|\varrho) C(\mathcal{A}, \mathcal{B}|\varrho)}\}$, can be upper bounded by the relation [5,25]

$$\begin{aligned} \|C(\mathcal{A}, \mathcal{B}|\varrho)\|_1 &\leqslant \sum_{nm} \|V_{nm}\|_2 \|W_{nm}\|_2 \\ &\leqslant \sqrt{\sum_{nm} \|V_{nm}\|_2^2} \sqrt{\sum_{nm} \|W_{nm}\|_2^2} \leqslant \sqrt{\Sigma_A \Sigma_B}, \end{aligned} \tag{13}$$

with

$$\begin{aligned} \sum_{nm} \|V_{nm}\|_2^2 &= \sum_{i=1}^{\overline{N}_A} \sum_m p_m \big(\mathrm{Tr}\,\{A_i \varrho_m^A\}\big)^2 - \sum_{i=1}^{\overline{N}_A} (\mathrm{Tr}\{A_i \varrho^A\})^2, \\ \Sigma_A &= \max_{\sigma^A} \sum_{i=1}^{\overline{N}_A} [(\mathrm{Tr}\{A_i \sigma^A\})^2 - (\mathrm{Tr}\{A_i \varrho^A\})^2], \end{aligned} \tag{14}$$

and analogous expressions for $\sum_{nm} \|W_{nm}\|_2^2$ and Σ_B. Accordingly, the upper bounds Σ_A and Σ_B involve a generally complicated maximization over all local quantum states of Alice and Bob, σ^A and σ^B. In particular, they depend on the chosen local Hermitian operators $\mathcal{A}$ and $\mathcal{B}$. The inequality (13) is a general consequence of bipartite separability of the quantum state ϱ. It applies to correlation matrices of arbitrary local measurements performed on arbitrary-dimensional separable bipartite quantum states. For the special cases of MUMs and GSIC POVMs, these inequalities have already been derived previously [20]. A violation of the inequality (13) is a sufficient condition for bipartite entanglement.

B. Correlation matrices of local orthonormal Hermitian operators

Let us specialize the inequality (13) to the correlation matrix of two arbitrary LOOs, say, $G^A = (G_1^A, \ldots, G_{d_A^2}^A)^T$ and $G^B = (G_1^B, \ldots, G_{d_B^2}^B)^T$, with $d_A^2 = \overline{N}_A$ and $d_B^2 = \overline{N}_B$. For this purpose the quantities Σ_A and Σ_B on the right-hand side of (13) have to be evaluated explicitly by maximizing over all possible quantum states of Alice and Bob. For LOOs the maximizations involved in the evaluation of the right-hand sides of (14) can be performed easily and are given by

$$\begin{aligned} \Sigma_A &= \max_{\sigma^A} \mathrm{Tr}\{(\sigma^A)^2\} - \mathrm{Tr}\{(\varrho^A)^2\} = 1 - \mathrm{Tr}\{(\varrho^A)^2\}, \\ \Sigma_B &= \max_{\sigma^B} \mathrm{Tr}\{(\sigma^B)^2\} - \mathrm{Tr}\{(\varrho^B)^2\} = 1 - \mathrm{Tr}\{(\varrho^B)^2\}. \end{aligned} \tag{15}$$

Therefore, the inequality (13) reduces to the known form [18,19]

$$\|C(G^A, G^B|\varrho)\|_1^2 \leqslant [1 - \mathrm{Tr}\{(\varrho^A)^2\}][1 - \mathrm{Tr}\{(\varrho^B)^2\}]. \tag{16}$$

A violation of this inequality is a sufficient condition for entanglement of an arbitrary-dimensional bipartite quantum state ϱ. This sufficient condition involves the correlation matrix of LOOs and the purities of Alice's and Bob's reduced quantum states. It is identical to the enhanced realignment criterion of Zhang *et al.* [26].

C. Correlation matrices of local informationally complete (N, M) POVMs

Let us now specialize the inequality (13) to two local (N, M) POVMs, say, $\Pi^A = (\Pi_1^A, \ldots, \Pi_{N_A M_A}^A)$ and $\Pi^B = (\Pi_1^B, \ldots, \Pi_{N_B M_B}^B)$, performed by Alice and Bob. For the sake of convenience, we introduce the indexing of the POVM elements by the mappings $i(\alpha, a) = (\alpha - 1)M_A + a$ and $j(\beta, b) = (\beta - 1)M_B + b$. In the following, we concentrate on local (N, M) POVMs which are informationally complete so that relation (5) applies, i.e., $N_A(M_A - 1) + 1 = d_A^2$ and $N_B(M_B - 1) + 1 = d_B^2$.

With the help of the constraints (2)–(4) characterizing local (N, M) POVMs, also the maximizations determining the relevant upper bounds entering the inequality (13) can be worked out straightforwardly, yielding the result

$$\Sigma_A = \max_{\sigma^A} \sum_{i=1}^{N_A M_A} \left[\left(\mathrm{Tr}\{\Pi^A_{i(\alpha,a)} \sigma^A\} \right)^2 - \left(\mathrm{Tr}\{\Pi^A_{i(\alpha,a)} \varrho^A\} \right)^2 \right]$$
$$= \Gamma_A \Big[\max_{\sigma^A} \mathrm{Tr}\{(\sigma^A)^2\} - \mathrm{Tr}\{(\varrho^A)^2\} \Big] = \Gamma_A [1 - \mathrm{Tr}\{(\varrho^A)^2\}] \tag{17}$$

and an analogous expression for Σ_B. As a consequence, the correlation matrix of informationally complete local (N, M) POVMs obeys the inequality

$$\|C(\Pi^A, \Pi^B|\varrho)\|_1^2 \leqslant \Gamma_A \Gamma_B [1 - \mathrm{Tr}\{(\varrho^A)^2\}][1 - \mathrm{Tr}\{(\varrho^B)^2\}], \tag{18}$$

with

$$\Gamma_A = \frac{x_A M_A^2 - d_A}{M_A(M_A - 1)}, \quad \Gamma_B = \frac{x_B M_B^2 - d_B}{M_B(M_B - 1)} \tag{19}$$

for all separable bipartite quantum states [5]. Special instances of the inequality (18) involving MUMs and GSIC POVMs as local measurements have been discussed recently [20]. It is worth mentioning that both sides of this inequality depend on the parameters x_A, x_B, M_A, M_B, N_A, and N_B. The dependence of the correlation matrix $C(\Pi^A, \Pi^B|\varrho)$ on these parameters complicates a direct comparison of the bipartite entanglement detectable by different measurement settings based on violations of the inequality (18). However, this apparent complication can be circumvented by relating the correlation matrix of (N, M) POVMs to those of LOOs with the help of the linear map S discussed in Sec. II B.

For this purpose, let us expand each of these local (N, M) POVMs in arbitrary LOOs, say, G^A for Alice and G^B for Bob, i.e., $\Pi^A = (G^A)^T S^A$ and $\Pi^B = (G^B)^T S^B$. Thereby, S^A denotes the $d_A^2 \times N_A M_A$ matrix of real-valued expansion coefficients of Π^A for Alice and analogously for Bob. With the help of this basis expansion the correlation matrix of local informationally complete (N, M) POVMs can be related to the correlation matrix of these LOOs by

$$C(\Pi^A, \Pi^B|\varrho) = (S^A)^T C(G^A, G^B|\varrho) S^B. \tag{20}$$

In addition, using the results of Sec. II B and Appendix [cf. (A9)–(A11)], it is found that the 1-norms of both correlation matrices are related by

$$\|C(\Pi^A, \Pi^B|\varrho)\|_1 = \|\sqrt{\Lambda^A} C(\tilde{G}^A, \tilde{G}^B|\varrho) \sqrt{\Lambda^B}\|_1 \tag{21}$$

with the transformed LOOs $\tilde{G}^A = O^A G^A$ and $\tilde{G}^B = O^B G^B$ and with the diagonal matrices Λ^A and Λ^B of the nonzero eigenvalues of $(S^A)^T S^A$ and $(S^B)^T S^B$ [cf. (A7)]. For an arbitrary bipartite quantum state ϱ, the correlation matrix fulfills the relations

$$[C(\tilde{G}^A, \tilde{G}^B|\varrho)]_{1\nu} = [C(\tilde{G}^A, \tilde{G}^B|\varrho)]_{\mu 1} = 0 \tag{22}$$

for $\mu \in \{1, \dots, d_A^2\}$ and $\nu \in \{1, \dots, d_B^2\}$. As a result, the relation between the 1-norms of these correlation matrices obeys the simple scaling relation

$$\|C(\Pi^A, \Pi^B|\varrho)\|_1 = \sqrt{\Gamma_A \Gamma_B} \|C(G^A, G^B|\varrho)\|_1 \tag{23}$$

and the sufficient entanglement condition (18) simplifies to

$$\|C(G^A, G^B|\varrho)\|_1^2 = \frac{1}{\Gamma_A \Gamma_B} \|C(\Pi^A, \Pi^B|\varrho)\|_1^2$$
$$\leqslant [1 - \mathrm{Tr}\{(\varrho^A)^2\}][1 - \mathrm{Tr}\{(\varrho^B)^2\}]. \tag{24}$$

This inequality is the main result of this section. Its violation is a sufficient condition for bipartite entanglement detectable by local (N, M) POVMs. Thus, it is apparent that this sufficient entanglement condition is completely independent of the chosen local (N, M) POVMs and is identical to the one for LOOs [cf. the inequality (16)]. Therefore, concerning violations of the general inequality (13), LOOs are as powerful in detecting bipartite entanglement as local informationally complete (N, M) POVMs. This is a direct consequence of the symmetry of the relations defining (N, M) POVMs and the resulting scaling relation (23). Exploiting this scaling property offers interesting perspectives for practical applications. In particular, it may offer the possibility to circumvent possible problems concerning physical realizations of particular classes of local (N, M) POVMs by the use of known and easy-to-realize LOOs or local (N, M) POVMs without affecting the efficiency of bipartite entanglement detection.

D. Joint probability distributions

Based on the inequality (24) and its violation, sufficient conditions for bipartite entanglement can be derived, which involve the joint probability distribution of measurement results of local informationally complete (N, M) POVMs. For this purpose, let us consider the joint probability distribution

$$P(\Pi^A, \Pi^B|\varrho) = \mathrm{Tr}\{(\Pi^A)^T \otimes \Pi^B \varrho\} \tag{25}$$

resulting from local measurements of informationally complete (N_A, M_A) and (N_B, M_B) POVMs Π^A and Π^B. Applying the triangular inequality to the 1-norm of this joint probability distribution, we obtain the inequality

$$\|P(\Pi^A, \Pi^B|\varrho)\|_1 \leqslant \|C(\Pi^A, \Pi^B|\varrho)\|_1 + \|\mathrm{Tr}\{(\Pi^A)^T \otimes \Pi^B \varrho^A \otimes \varrho^B\}\|_1. \tag{26}$$

From the defining properties of informationally complete (N, M) POVMs (cf. Sec. II) we obtain the relation

$$\|\mathrm{Tr}\{(\Pi^A)^T \otimes \Pi^B \varrho^A \otimes \varrho^B\}\|_1 = \sqrt{U_A U_B},$$
$$U_A = \mathrm{Tr}\{(\varrho^A)^2\}\Gamma_A + \frac{d_A N_A/M_A - \Gamma_A}{d_A},$$
$$U_B = \mathrm{Tr}\{(\varrho^B)^2\}\Gamma_B + \frac{d_B N_B/M_B - \Gamma_B}{d_B}. \tag{27}$$

Thus, using the upper bounds of the correlation matrix (17), we find

$$\|P(\Pi^A, \Pi^B|\varrho)\|_1 \leqslant \sqrt{\Sigma_A \Sigma_B} + \sqrt{U_A U_B}. \tag{28}$$

For separable bipartite quantum states ϱ this inequality constrains the joint probabilities of the local measurement results and relates them to the purities of the reduced quantum states of Alice and Bob, i.e., $\mathrm{Tr}\{(\varrho^A)^2\}$ and $\mathrm{Tr}\{(\varrho^B)^2\}$. Violating this inequality by a bipartite quantum state ϱ yields a sufficient condition for its entanglement. With the help of the general inequality

$$\sqrt{\Sigma_A \Sigma_B} + \sqrt{U_A U_B} \leqslant \sqrt{\Sigma_A + U_A}\sqrt{\Sigma_B + U_B}, \tag{29}$$

a new upper bound can be derived on the right-hand side of (28), yielding the inequality

$$\|P(\Pi^A, \Pi^B|\varrho)\|_1 \leqslant \sqrt{\Sigma_A + U_A}\sqrt{\Sigma_B + U_B}, \tag{30}$$

with

$$\Sigma_A + U_A = \frac{d_A - 1}{d_A}\frac{d_A^2 + M_A^2 x_A}{M_A(M_A - 1)},$$
$$\Sigma_B + U_B = \frac{d_B - 1}{d_B}\frac{d_B^2 + M_B^2 x_B}{M_B(M_B - 1)}. \tag{31}$$

This sufficient entanglement condition has already been derived [8]. A violation of this inequality by a bipartite quantum state ϱ yields a sufficient condition for its entanglement. However, in view of (29), this sufficient condition for bipartite entanglement is generally weaker than the sufficient condition based on a violation of the inequality (28). It is apparent that the inequality (30) is independent of the purities of the reduced density operators of Alice and Bob. For special cases, namely, for SIC POVMs [21] and GSIC POVMs [14], the inequality (30) has already been derived. The joint probability distribution $P(\Pi^A, \Pi^B|\varrho)$ is a matrix of dimension $N_A M_A \times N_B M_B$. Its intricate dependence on all local (N, M) POVM parameters complicates a direct comparison of the bipartite entanglement detectable by different measurement settings. However, this apparent complication can be circumvented by relating the joint probability distribution $P(\Pi^A, \Pi^B|\varrho)$ of the local (N, M) POVMs to those of LOOs with the help of the linear map S discussed in Sec. II B.

The sufficient conditions for entanglement resulting from violations of (28) or (30) for the joint probability distributions of local (N, M) POVMs also exhibit scaling properties which allow us to relate them to the corresponding $(d_A^2 \times d_B^2)$-dimensional joint probability distribution $P(\tilde{G}^A, \tilde{G}^B|\varrho)$ of the local Hermitian bases $\tilde{G}^A$ and $\tilde{G}^B$. However, they differ from the scaling properties of the correlation matrix as discussed in Sec. III C. To derive these scaling relations, we first consider the common left-hand sides of the inequalities (28) and (30). Expanding the local (N, M) POVMs in the LOOs $\tilde{G}^A$ and $\tilde{G}^B$, we find with the help of the results of Sec. II B and the Appendix the relation

$$\|P(\Pi^A, \Pi^B|\varrho)\|_1 = \|\sqrt{\Lambda^A}P(\tilde{G}^A, \tilde{G}^B|\varrho)\sqrt{\Lambda^B}\|_1, \tag{32}$$

with the diagonal matrices Λ^A and Λ^B [cf. Eq. (A10)]. According to (A8), the common factors $\gamma_A = d_A(d_A - 1)/M_A(M_A - 1)$ and $\gamma_B = d_B(d_B - 1)/M_B(M_B - 1)$ can be extracted from the eigenvalues entering the diagonal matrices Λ^A and Λ^B, i.e.,

$$(\Lambda^A)_{11} = \gamma_A(d_A + 1),$$
$$(\Lambda^A)_{\nu\nu} = \Gamma_A = \gamma_A\frac{d_A\tilde{x}_A - 1}{d_A - 1},$$
$$(\Lambda^B)_{11} = \gamma_B(d_B + 1),$$
$$(\Lambda^B)_{\mu\mu} = \Gamma_B = \gamma_B\frac{d_B\tilde{x}_B - 1}{d_B - 1} \tag{33}$$

for $\nu \in \{2, \ldots, N_A(M_A - 1) + 1\}$ and $\mu \in \{2, \ldots, N_B(M_B - 1) + 1\}$ with the rescaled parameters

$$\tilde{x}_A = \frac{x_A M_A^2}{d_A^2}, \quad \tilde{x}_B = \frac{x_B M_B^2}{d_B^2}. \tag{34}$$

The same factors γ_A and γ_B can also be extracted from the quantities U_A, U_B, Σ_A, and Σ_B of (17) and (27) entering the right-hand sides of the inequalities (28) and (30). Consequently defining the $(d_A^2 \times d_B^2)$-dimensional scaled joint probability distribution

$$\tilde{P}(\tilde{x}_A, \tilde{x}_B|\varrho) = \sqrt{\frac{\Lambda^A}{\gamma_A}}P(\tilde{G}^A, \tilde{G}^B|\varrho)\sqrt{\frac{\Lambda^B}{\gamma_B}}, \tag{35}$$

which only depends on the scaled parameters $\tilde{x}_A$ and $\tilde{x}_B$, the inequality (28) can be rewritten in the rescaled form

$$\|\tilde{P}(\tilde{x}_A, \tilde{x}_B|\varrho)\|_1 \leqslant \sqrt{\frac{\Sigma_A}{\gamma_A}}\sqrt{\frac{\Sigma_B}{\gamma_B}} + \sqrt{\frac{U_A}{\gamma_A}}\sqrt{\frac{U_B}{\gamma_B}} \tag{36}$$

and the inequality (30) becomes

$$\|\tilde{P}(\tilde{x}_A, \tilde{x}_B|\varrho)\|_1 \leqslant \sqrt{1 + \tilde{x}_A}\sqrt{1 + \tilde{x}_B}. \tag{37}$$

These inequalities are the main results of this section. It is apparent that for given dimensions d_A and d_B of Alice's and Bob's quantum systems these inequalities depend only on the scaled parameters $\tilde{x}_A$ and $\tilde{x}_B$ of (34) characterizing Alice's and Bob's local informationally complete (N, M) POVMs. For given dimensions d_A and d_B this implies that testing for violations of the inequalities (36) and (37) yields the same results for informationally complete (N, M) POVMs with the parameters M_A, x_A, M_B, and x_B and with the parameters $M'_A, x'_A = M_A^2 x_A/{M'_A}^2$ and $M'_B, x'_B = M_B^2 x_B/{M'_B}^2$. Therefore, according to (36) and (37), sufficient conditions for bipartite entanglement involving local MUMs with the parameters $M_A = d_A, x_A$ and $M_B = d_B, x_B$, for example, are identical to the sufficient conditions involving local GSIC POVMs with $M'_A = d_A^2$, $x'_A = x_A/d_A^2$, $M'_B = d_B^2$, and $x'_B = x_B/d_B^2$. The sufficient conditions (36) and (37) have been derived solely with the help of the eigenvalues of the linear operators S^A and S^B. Constraints on the local operators Π^A and Π^B due to positive semidefiniteness are not required for relating equivalent sufficient entanglement conditions and thus equally efficient entanglement detection scenarios. Thus, for this purpose an explicit construction of (N, M) POVMs is not required.

Let us conclude by comparing the sufficient bipartite entanglement conditions resulting from violations of our main results, i.e., the inequalities (24), (36), and (37), with other existing sufficient entanglement conditions for (N, M) POVMs.

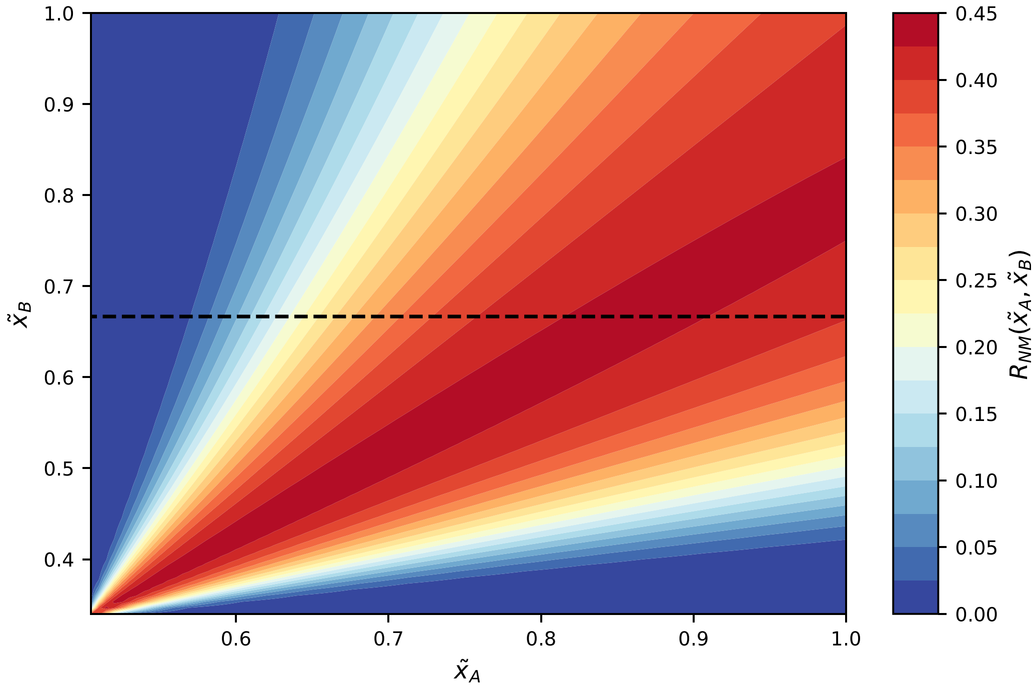

FIG. 1. Dependence of volume ratios $R_{NM}(\tilde{x}_A, \tilde{x}_B)$ between detected entangled and all bipartite quantum states on the scaled parameters $\tilde{x}_A$ and $\tilde{x}_B$ of Alice's and Bob's informationally complete local (N, M) POVMs as obtained from a violation of inequality (37) for $(d_A, d_B) = (2, 3)$. The horizontal dashed line indicates the upper bound for $\tilde{x}_B$ for (N, M) POVMs with $M_B = 2$. All (N, M) POVMs with these values of d_A and d_B yield the same results.

The sufficient bipartite entanglement conditions based on the inequalities (28) and (30) for joint probability distributions of local (N, M) POVMs have already been discussed in [8,16]. However, these investigations do not address the dependence of these inequalities on the parameters characterizing the local measurements chosen for entanglement detection. Thus, from these investigations it is not apparent which local (N, M) POVMs are more efficient in detecting bipartite entanglement than others. This aspect is addressed by our inequalities (36) and (37) analytically. The related aspects concerning efficiencies of bipartite entanglement detection are addressed by our numerical results presented in Figs. 1 and 2. The inequalities (36) and (37) explicitly show that, for given dimensions of the local quantum systems, local measurements involving informationally complete (N, M) POVMs with the same scaled parameters $\tilde{x}_A$ and $\tilde{x}_B$ [cf. (34)] are equally powerful in detecting bipartite entanglement. The correlation matrix-based sufficient condition derived by Lai and Luo [5] is identical to our Eq. (18). However, also this investigation does not address the issue of which local (N, M) POVMs are more efficient in detecting bipartite entanglement than others. Our inequality (24) addresses this issue by showing explicitly that informationally complete (N, M) POVMs are as powerful in detecting bipartite entanglement locally as LOOs. Finally, we want to mention that in some cases the inequalities (18) and (30) can also be replaced by weaker inequalities with the help of the relation $\sum_{i=1}^{d} |\mathcal{M}_{ii}| \leqslant \|\mathcal{M}\|_1$, valid for arbitrary $d \times d$ square matrices $\mathcal{M}$. Therefore, if we consider two informationally complete (N, M) POVMs with identical parameters, for example, the inequalities (18) and (30) imply the weaker inequalities

$$\sum_{i=1}^{NM} \mathrm{Tr}\left\{\Pi^A_{i(\alpha,a)} \otimes \Pi^B_{i(\alpha,a)} \varrho\right\} \leqslant \frac{d-1}{d} \frac{d^2 + M^2 x}{M(M-1)}, \tag{38}$$

$$\sum_{i=1}^{NM} \left|\mathrm{Tr}\left\{\Pi^A_{i(\alpha,a)} \otimes \Pi^B_{i(\alpha,a)} (\varrho - \varrho_A \otimes \varrho_B)\right\}\right| \leqslant \Gamma \sqrt{\left(1 - \mathrm{Tr}\left\{\varrho_A^2\right\}\right)\left(1 - \mathrm{Tr}\left\{\varrho_A^2\right\}\right)}, \tag{39}$$

with $d_A = d_B = d$. These sufficient conditions have already been discussed in [8,15,17]. However, sufficient conditions based on inequalities involving 1-norms, such as (18) and (30), can detect more entangled states than violations of the inequalities (38) and (39).

IV. NUMERICAL RESULTS

In this section, numerical results are presented exploring statistical features of the local detection of typical bipartite entanglement by violations of the inequalities (24), (36), and (37). Based on these sufficient conditions for bipartite entanglement detection, we determine lower bounds on the Euclidean volume ratios between entangled bipartite states and all quantum states for different dimensions d_A and d_B of Alice's and Bob's quantum systems. These volume ratios establish quantitative measures of the efficiencies with which different local entanglement detection procedures can detect unknown bipartite entangled states. For this purpose also the

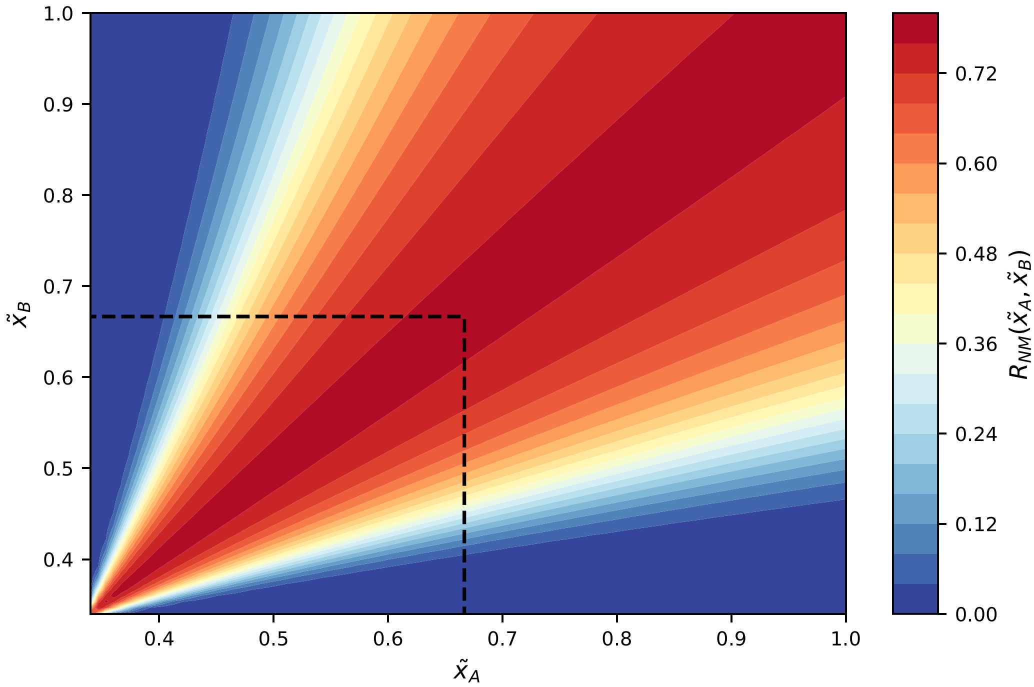

FIG. 2. Dependence of volume ratios $R_{NM}(\tilde{x}_A, \tilde{x}_B)$ between detected entangled and all bipartite quantum states on the scaled parameters $\tilde{x}_A$ and $\tilde{x}_B$ of Alice's and Bob's informationally complete local (N, M) POVMs as obtained from a violation of inequality (37) for $(d_A, d_B) = (3, 3)$. The vertical and horizontal dashed lines indicate the upper bounds for $\tilde{x}_A$ and $\tilde{x}_B$ for (N, M) POVMs with $M_A = 2$ and $M_B = 2$. All (N, M) POVMs with these values of d_A and d_B yield the same results.

dependence of the inequality (37) on the rescaled $\tilde{x}$ parameters [cf. (34)] is investigated.

Starting from the $d_A d_B$-dimensional Hilbert space $\mathcal{H}_{d_A d_B}$ describing Alice's and Bob's joint quantum system, one can construct the $(d_A d_B)^2$-dimensional Hilbert space $\mathcal{H}_{(d_A d_B)^2}$ of Hermitian linear operators acting on elements of $\mathcal{H}_{d_A d_B}$. This is a Hilbert space over the field of real numbers equipped with the Hilbert-Schmidt scalar product $\langle A|B\rangle_{\mathrm{HS}} := \mathrm{Tr}_{\mathrm{AB}}\{A^\dagger B\}$ for $A, B \in \mathcal{H}_{(d_A d_B)^2}$. Thus, it is a Euclidean vector space. On this Euclidean vector space volumes of convex sets of linear Hermitian operators, and thus quantum states $\varrho \geqslant 0$, can be defined naturally. The Euclidean volumes of quantum states can be estimated numerically with the help of Monte Carlo methods. We have developed a hit-and-run Monte Carlo algorithm [22] for estimating the volumes of convex sets of quantum states according to this Euclidean measure. Hit-and-run Monte Carlo methods were introduced originally by Smith [27]. They take advantage of a random walk inside a convex set to efficiently generate a uniform distribution over this convex set by iteration so that this distribution eventually becomes independent of the starting point, the completely mixed quantum state [28].

We randomly sampled $N = 10^8$ bipartite quantum states for different values of d_A and d_B with the help of this recently developed hit-and-run Monte Carlo algorithm [22,23] to determine lower bounds on Euclidean volumes of detected entangled states. In Table I the obtained lower bounds of the ratios R between the Euclidean volumes of entangled states and all bipartite quantum states are presented together with their estimated statistical errors. These errors have been estimated with the help of the procedure described in Ref. [22]. Thereby, for each pair of dimensions (d_A, d_B) four different

TABLE I. Lower bounds on volume ratios between entangled and all bipartite quantum states for different dimensions d_A and d_B of Alice's and Bob's quantum systems: bipartite NPT states (R_{NPT}) and bipartite states detectable by violations of (37) (R_{SIC1}), of (36) (R_{SIC2}), and of (24) (R_{cor}). The numbers in square brackets after the estimated statistical errors [22] (square roots of variances) indicate the relevant powers of 10^{-1}. For dimensions (3,4) and (4,4) the algorithm generated only NPT states. Therefore, the corresponding values of R_{NPT} do not involve any statistical uncertainties.

(d_A, d_B)	R_{NPT}	R_{SIC1}	R_{SIC2}	R_{cor}
(2,2)	$0.75784 \pm 1.7[4]$	$0.67060 \pm 2.2[4]$	$0.67947 \pm 2.1[4]$	$0.68860 \pm 2.1[4]$
(2,3)	$0.97303 \pm 7[5]$	$0.39732 \pm 5.6[4]$	$0.42998 \pm 5.5[4]$	$0.43853 \pm 5.5[4]$
(2,4)	$0.998696 \pm 1.6[5]$	$0.02710 \pm 2.7[4]$	$0.04361 \pm 3.4[4]$	$0.04504 \pm 3.5[4]$
(3,3)	$0.999895 \pm 4[6]$	$0.75680 \pm 8.2[4]$	$0.75754 \pm 8.2[4]$	$0.76364 \pm 8.1[4]$
(3,4)	1	$0.3605 \pm 1.8[3]$	$0.3742 \pm 1.8[3]$	$0.3795 \pm 1.8[3]$
(4,4)	1	$0.6378 \pm 7.7[3]$	$0.6380 \pm 7.7[3]$	$0.6419 \pm 7.7[3]$

lower bounds are presented, namely, the volume ratio resulting from bipartite NPT states ($R_{\rm NPT}$) and the volume ratios resulting from bipartite entangled states detectable by violations of inequalities (37) ($R_{\rm SIC1}$), (36) ($R_{\rm SIC2}$), and (24) ($R_{\rm cor}$). Because of the scaling relations discussed in the previous section, selecting a single class of (N, M) POVMs is sufficient for assessing the efficiency of local entanglement detection by different inequalities. For the results presented in Table I, we have chosen GSICs, i.e., $N_A = N_B = 1$, $M_A = d_A^2$, and $M_B = d_B^2$. The ratios $R_{\rm SIC1}$ and $R_{\rm SIC2}$ describe bipartite entanglement detection by local SIC POVMs performed by Alice and Bob with the maximal parameters $x_A = 1/d_A^2$ and $x_B = 1/d_B^2$ which corresponds to the scaled parameters $\tilde{x}_A = \tilde{x}_B = 1$. It is apparent from Table I that for $d_A = d_B$ the values of $R_{\rm SIC1}$ are consistent with the recently obtained results of Ref. [21]. The ratios $R_{\rm cor}$ describe bipartite entanglement detectable by correlation matrices of (N, M) POVMs, which, according to Sec. III C, are identical to the results of LOOs. These results show that all lower bounds on the ratios R based on local measurements are always smaller than the ratios $R_{\rm NPT}$ of bipartite NPT states. Furthermore, with increasing dimensions of Alice's and Bob's quantum systems, the ratios between bipartite entangled states, which are lower bounded by the ratios of NPT states, and all quantum states rapidly approach unity within our numerical accuracy. However, the lower bounds on these ratios detectable by local measurements do not reflect this tendency and increasingly underestimate the volume ratios of NPT states with increasing dimensions of the local quantum systems. Nevertheless, as expected from our discussion in Sec. III, the latter lower bounds always fulfill the relation $R_{\rm SIC1} \leqslant R_{\rm SIC2} \leqslant R_{\rm cor}$ consistent with our results (24), (26), (36), (29), and (37).

So far our numerical results have concentrated on optimal (N, M) POVMs with $\tilde{x}_A = \tilde{x}_B = 1$ for cases with $d \leqslant M$. Now we investigate characteristic features of arbitrary values of these parameters. In Figs. 1 and 2 the volume ratios of bipartite entangled states detectable by violations of the inequality (37) and their dependence on the scaled parameters $\tilde{x}_A$ and $\tilde{x}_B$ of Alice's and Bob's local informationally complete (N, M) POVM measurements [cf. (34)] are depicted for a qubit-qutrit and a qutrit-qutrit system. These results are based on a statistical ensemble of 10^7 randomly sampled bipartite quantum states. According to the scaling properties discussed in Sec. III D and the discussion of Sec. II, these results describe the volume ratios of all local (N, M) POVMs of Alice and Bob with scaled parameters in the maximally allowed ranges $1/d_A < \tilde{x}_A \leqslant \min(1, M_A/d_A)$ and $1/d_B \leqslant \tilde{x}_B \leqslant \min(1, M_B/d_B)$. For $d_A = 2$ the possible informationally complete (N, M) POVMs are characterized by the values $(N_A, M_A) \in \{(1, 4), (3, 2)\}$ so that the corresponding possible range of scaled x parameters is given by $\frac{1}{2} < \tilde{x}_A \leqslant 1$ for all these possible (N, M) POVMs. For $d_B = 3$ the possible informationally complete (N, M) POVMs are characterized by the values $(N_B, M_B) \in \{(1, 9), (2, 5), (4, 3), (8, 2)\}$ so that the corresponding possible range of scaled x parameters is given by $\frac{1}{3} < \tilde{x}_B \leqslant 1$ for $M_B \geqslant 3$ and $\frac{1}{3} < \tilde{x}_B \leqslant \frac{2}{3}$ for $M_B = 2$. The black dashed straight horizontal line in Fig. 1 marks this upper bound of $\frac{2}{3}$ for $\tilde{x}_B$ for the case of $M_B = 2$ and $d_B = 3$. The values above this black dashed horizontal line describe cases in which $d_B \leqslant M_B$ so that the upper limit of the scaled parameter $\tilde{x}_B$ is given by unity. In the case depicted in Fig. 2 ($d_A = d_B = 3$) the corresponding upper bounds for (N, M) POVMS with $M_A = 2$ and $M_B = 2$ are also indicated by black dashed straight vertical and horizontal lines. The values outside these black dashed lines describe cases in which either $d_A < M_A$ or $d_B < M_B$ so that the upper limit of the scaled parameters $\tilde{x}_A$ or $\tilde{x}_B$ is given by unity.

Symmetric informationally complete POVMs are characterized by the parameter values $\tilde{x}_A = 1$ and $\tilde{x}_B = 1$ [21]. It is apparent from Fig. 1 that the volume ratios detectable by SIC POVMs are not close to the maximal possible amounts of entanglement which can be detected by local measurements. This is consistent with the gaps between the values of $R_{\rm SIC1}$ and $R_{\rm cor}$ for unequal dimensions of the subsystems in Table I. Every pair of local (N, M) POVMs with rescaled parameters $(\tilde{x}_A, \tilde{x}_B)$ belonging to the dark red area of Fig. 1 detects more entangled states than local SIC POVMs. For subsystems with identical dimensions and local (N, M) POVMs with identical rescaled parameters $\tilde{x}$ the volume ratios increase slightly with decreasing values of $\tilde{x}$. For $d = 3$ and $\tilde{x} = 0.35$, for example, the volume ratio of the entangled states is given by $R(\tilde{x}) = 0.76296 \pm 8.2 \times 10^{-4} > R_{\rm SIC1}$ (cf. Table I). This demonstrates that (N, M) POVMs with well-chosen parameters $\tilde{x}_A$ and $\tilde{x}_B$ can detect more entangled states than SIC POVMs. Furthermore, there are always parameter values of $\tilde{x}_A$ and $\tilde{x}_B$ close to the lower bounds, i.e., the lower left corners of Figs. 1 and 2, for which the volume ratios are close to their maximal values. Thus, (N, M) POVMs with x values close to their lower bounds can always be used for close to maximal local bipartite entanglement detection. The upper bounds on the scaled $\tilde{x}$ parameters, i.e., $\min(M/d, 1)$, are irrelevant for achieving this goal. In particular, optimal (N, M) POVMs are not necessary for this purpose. This observation offers interesting perspectives for physical realizations of efficient local bipartite entanglement detection as (N, M) POVMs with sufficiently small x values close to their lower bounds can always be constructed [8,10,13].

V. CONCLUSION

Basic properties of sufficient conditions for arbitrary-dimensional bipartite entanglement detection based on correlation matrices and joint probability distributions of local measurements have been investigated. On the one hand, the dependence of these sufficient conditions on the nature of the local measurements was explored for generalized measurements based on informationally complete (N, M) POVMs. On the other hand, the efficiency of these classes of local measurements for typical bipartite entanglement detection was investigated quantitatively by numerically determining volumes of detectable entangled states in the state space of all possible bipartite quantum states with the help of a hit-and-run Monte Carlo algorithm.

As a first main result it was shown in Sec. III that inherent symmetry properties of informationally complete (N, M) POVMs imply that sufficient conditions for bipartite entanglement exhibit characteristic scaling properties. These necessarily valid conditions relate equivalent sufficient entanglement conditions and thus also equally efficient entanglement detection scenarios. Their derivation is solely based on the validity of the constraints (3) and (4) characterizing (N, M) POVMs. Therefore, explicit constructions of (N, M) POVMs are not required to establish these scaling properties. They imply, for example, that for given dimensions of the quantum systems involved local entanglement detection by joint probability distributions of such (N, M) POVMs with a given efficiency can always be achieved by GSICs with $N_A = N_B = 1$, $M_A = d_A^2$, and $M_B = d_B^2$. This general aspect is of interest for realizations of local entanglement measurements as even the construction of optimal GSICs, i.e., SICs, is relatively well understood for quantum systems of dimensions of at least up to 151 [29], whereas questions concerning the explicit construction of optimal (N, M) POVMs for arbitrary values of N and M are still largely open. Furthermore, as a result of these characteristic scaling properties, local entanglement detection by correlation matrices of informationally complete (N, M) POVMs is entirely independent of all their parameters and is identical to the corresponding results of LOOs [cf. (24)]. Thus, concerning correlation matrices, local bipartite entanglement detection by (N, M) POVMs is as powerful as its detection by LOOs.

Based on a hit-and-run Monte Carlo algorithm, as a second main result the potential of informationally complete (N, M) POVMs for local bipartite entanglement detection was explored quantitatively. Our numerical results demonstrate that with increasing dimensions of the local quantum systems, detection of entangled bipartite states by such (N, M) POVMs becomes less and less efficient. However, the total number of bipartite entangled states increases (cf. Table I). With our Monte Carlo algorithm also the x dependence of joint probability distributions was explored numerically for low-dimensional local quantum systems. These results (cf. Figs. 1 and 2) demonstrate that for bipartite entanglement, detection maximal efficiency can always be achieved by (N, M) POVMs whose x values are close to their lower possible bounds of d_A/M_A^2 and d_B/M_B^2 for x_A and x_B. Furthermore, for well-chosen x values (N, M) POVMs can detect more entangled states than SIC POVMs or other optimal (N, M) POVMs. Contrary to optimal (N, M) POVMs, for which questions concerning their existence and construction in arbitrary dimensions are still largely open, (N, M) POVMs with x values close to their lower possible bounds can always be constructed [8,10,13]. This observation offers interesting perspectives for practical applications of efficient local entanglement detection by (N, M) POVMs, particularly in the areas of quantum communication and quantum key distribution.

Although these results already shed light on characteristic features concerning the potential of local measurements for bipartite entanglement detection and its dependence on different classes of local quantum measurements, they also trigger questions of interest for subsequent research. For example, our numerical results have established that in bipartite scenarios with increasing dimensions of the quantum systems, local measurements significantly underestimate even the amount of NPT entangled states. It would be interesting to explore whether a similar tendency is also observable in multipartite scenarios and how locally detectable multipartite entanglement depends on the local quantum systems' dimensions and on the quantum measurements' character. Nevertheless, open questions even remain in the significantly simpler context of bipartite quantum systems. For practical applications it would be interesting to find more efficient methods for bipartite entanglement detection, which are at least capable of detecting a significant fraction of all NPT entangled states. In this context the additional use of local operations and classical communication may be useful. From the theoretical point of view, investigations concerning the explicit construction of arbitrary optimal (N, M) POVMs or proofs of their nonexistence in particular dimensions constitute natural next steps for future research.

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APPENDIX: SPECTRAL REPRESENTATION OF *S*

In this Appendix, the most general form of the linear operator $S : \mathcal{H}_{d^2} \to \mathcal{H}_{NM}$ is determined, which relates NM Hermitian linear operators Π, not necessarily positive semidefinite, to an arbitrary orthonormal Hermitian operator basis, say, G, i.e., $\Pi = G^T S$, so that the basic properties (3) and (4), also characterizing (N, M) POVMs, are fulfilled. Thus, the linear operator $S : \mathcal{H}_{d^2} \to \mathcal{H}_{NM}$ characterizes all relations which necessarily have to be fulfilled between an arbitrary orthonormal Hermitian basis G and any possible (N, M) POVM Π irrespective of the positive semidefiniteness. In order to determine this linear operator $S : \mathcal{H}_{d^2} \to \mathcal{H}_{NM}$, we start from (6) and from the constraints (3) and (4). For $N \geqslant 2$ these constraints can be rewritten in the form

$$(S^T S)_{i(\alpha,a),j(\alpha',a')} = \Gamma \delta_{i(\alpha,a),j(\alpha',a')} - \frac{\Gamma}{M}\left(\bigoplus_{\alpha=1}^{N} J_\alpha\right)_{i(\alpha,a),j(\alpha',a')} + \frac{d}{M^2} J_{i(\alpha,a),j(\alpha',a')}, \tag{A1}$$

with

$$\Gamma = \frac{xM^2 - d}{M(M-1)}, \tag{A2}$$

the $NM \times NM$ matrix J of all ones, i.e., $J_{i(\alpha,a),j(\alpha',a')} = 1$, and the $M \times M$ matrices J_α of all ones, i.e., $(J_\alpha)_{i(\alpha,a),j(\alpha',a')} = \delta_{\alpha,\alpha'}$. In the degenerate case of $N = 1$ Eq. (A1) has to be replaced by

$$(S^T S)_{i(1,a),j(1,a')} = \Gamma \delta_{i(1,a),j(1,a')} + \frac{d - Mx}{M(M-1)} (J_1)_{i(1,a),j(1,a')}. \tag{A3}$$

Let us first consider cases with $N \geqslant 2$. It is apparent from (A1) that for $N \geqslant 2$,

$$S^T S J = \frac{dN}{M} J. \tag{A4}$$

Therefore, $X_{i,1} = 1/\sqrt{NM}$ with components $i \in \{1, \ldots, NM\}$ is a normalized eigenvector with eigenvalue $\Lambda_1 = dN/M$. As

$(S^T S)_{i,j}$ is a symmetric real-valued matrix it can be diagonalized and all its other eigenvectors have to be orthogonal to this particular eigenvector. From the form of (A1) it is also apparent that all orthonormal vectors $X_{i,\nu}$ with $\sum_{a=1}^{M} X_{i(\alpha,a),\nu} = 0$ have the same eigenvalue Γ [cf. (A2)] because $(J_\alpha X)_{i,\nu} = (JX)_{i,\nu} = 0$ for $\alpha \in \{1,\ldots,N\}$. For each α, constructing $M - 1$ such vectors is possible. Therefore, the eigenvalue Γ is $N(M-1)$-fold degenerate. There are $N - 1$ additional linear independent orthonormal vectors $X_{i(\alpha,a),\nu}$ with the properties $(\oplus_{\alpha=1}^{N} J_\alpha X)_{i,\nu} = MX_{i,\nu}$ and $(JX)_{i,\nu} = 0$. They all have the same eigenvalue 0. All in all, there are $1 + N(M-1) + (N-1) = NM$ orthogonal eigenvectors and we obtain the spectrum of the symmetric linear operator $S^T S$,

$$\mathrm{Sp}(S^T S) = \left\{\Gamma^{[N(M-1)]}, \frac{dN}{M}^{(1)}, 0^{(N-1)}\right\}, \tag{A5}$$

with the exponents indicating the multiplicities of the eigenvalues. In the degenerate case with $N = 1$, the zero eigenvalue disappears. Thus, provided the relation (5) characterizing informationally complete (N, M) POVMs is fulfilled, the dimension D of the eigenspace of the nonzero eigenvalues of $S^T S$ is given by

$$D = N(M-1) + 1 = d^2. \tag{A6}$$

As a result, the spectral representation of the symmetric linear operator $S^T S$ is given by

$$(S^T S)_{i,j} = \sum_{\mu=1}^{d^2} X_{i,\mu} \Lambda_\mu X^T_{\mu,j}, \tag{A7}$$

with

$$\Lambda_1 = \frac{dNM}{M^2}, \quad X_{i,1} = \frac{1}{\sqrt{NM}},$$
$$\Lambda_\nu = \Gamma, \quad \sum_{a=1}^{M} X_{i(\alpha,a),\nu} = 0 \tag{A8}$$

for each $\alpha \in \{1,\ldots,N\}$, $i \in \{1,\ldots,NM\}$, and $\nu \in \{2,\ldots,d^2\}$. The $NM \times d^2$ matrix $X_{i,\mu}$ fulfills the orthogonality condition

$$\sum_{i=1}^{NM} (X^T)_{\mu,i} X_{i,\nu} = \delta_{\mu,\nu} \tag{A9}$$

for $\mu, \nu \in \{1,\ldots,d^2\}$. As a consequence of (5), the most general form of the $d^2 \times NM$ matrix $S_{\mu,i}$, which is consistent with (3) and (4), is given by

$$S_{\mu,i} = \sum_{\mu'=1}^{d^2} O^T_{\mu,\mu'} \sqrt{\Lambda_{\mu'}} X^T_{\mu',i}, \tag{A10}$$

with the arbitrary real-valued orthogonal $d^2 \times d^2$ matrix O, i.e., $OO^T = O^T O = P_{d^2}$. Thereby, P_{d^2} denotes the projection operator onto the d^2-dimensional eigenspace of nonzero eigenvalues of the linear operator $S^T S$ acting in the Hilbert space $\mathcal{H}_{NM}$. The additional constraint (1) characterizing any POVM implies the additional relation

$$\mathbb{1}_d = \sum_{a=1}^{M} \Pi_{i(\alpha,a)} = \sqrt{d} \sum_{\mu=1}^{d^2} G_\mu O^T_{\mu,1}, \tag{A11}$$

where we have taken into account the property (A8) of the eigenvectors of $S^T S$. This property together with (A11) implies that also condition (2) is fulfilled. All these considerations concerning the spectral representation of $S^T S$ also apply to the degenerate case of $N = 1$. However, in this latter case, all eigenvalues are nonzero. Therefore, the map $S : \mathcal{H}_{d^2} \to \mathcal{H}_{NM}$ as defined by (A10) fulfills both relations (3) and (4) and the additional dimensional constraint (5). In a new basis of orthonormal Hermitian operators defined by $\tilde{G} = OG$ [cf. (A10)], we obtain the relation $\Pi = G^T S = \tilde{G}^T \tilde{S}$ with

$$\tilde{S}_{\nu,i} = \sqrt{\Lambda_\nu} X^T_{\nu,i} \tag{A12}$$

and $\tilde{G}_1 = \mathbb{1}_d/\sqrt{d}$ and $\mathrm{Tr}\{\tilde{G}_\nu\} = 0$ for $\nu \in \{2,\ldots,d^2\}$ [cf. (A11)]. In this new basis $\tilde{G}$, it is apparent that, according to (A5), the map S maps $\mathcal{H}_{d^2}$ injectively onto a d^2-dimensional subspace of $\mathcal{H}_{NM}$ by preserving orthogonality in such a way that all basis operators $\tilde{G}_\nu$ with $\nu \in \{2,\ldots,d^2\}$ are mapped conformally onto a $(d^2 - 1)$-dimensional subspace of $\mathcal{H}_{NM}$ by scaling the norms of all operators by a factor of $\sqrt{\Gamma}$. As these basis operators are orthogonal to $\tilde{G}_1$, they are characterized by the basis-independent property $\mathrm{Tr}\{\tilde{G}_\nu\} = 0$ for $\nu \in \{2,\ldots,d^2\}$. It is only the basis operator $\tilde{G}_1$ which is scaled by a different factor, namely, $\sqrt{\Lambda_1} = \sqrt{dN/M}$. In Secs. III C and III D it was demonstrated that this special property relating an arbitrary basis of orthonormal Hermitian operators, G or $\tilde{G}$, to an arbitrary informationally complete (N, M) POVM Π manifests in general scaling relations for sufficient conditions of bipartite entanglement.

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