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PAPER

Typical bipartite steerability and generalized local quantum measurements

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Abstract

Recently proposed correlation-matrix-based sufficient conditions for bipartite steerability from Alice to Bob for arbitrary measurements are applied to local informationally complete positive operator valued measures (POVMs) of the (N,M)-type. These POVMs include a large class of local generalized measurements of current interest [Siudzińska K 2022 ***Phys. Rev.*** A ***105*** 042209]. It is shown that the trace norm of correlation matrices with local (N,M)-POVMs is proportional to that of local orthonormal hermitian operator bases (LOOs). This implies that all types of informationally complete (N,M)-POVMs are equally powerful in detecting bipartite steerability from Alice to Bob and, in addition, they are as powerful as LOOs. In order to explore the typicality of steering numerical calculations of lower bounds on Euclidean volume ratios between steerable bipartite quantum states from Alice to Bob and all quantum states are determined with the help of a hit-and-run Monte-Carlo algorithm. These results demonstrate that with the single exception of two qubits this correlation-matrix-based sufficient condition significantly underestimates these volume ratios. These results are also compared to a recently proposed method that determines bipartite steerability from Alice's qubit to Bob's arbitrary dimensional quantum system by bipartite entanglement detection. It is demonstrated that in general this method is significantly more effective in detecting typical steerability provided non local entanglement detection methods are used which transcend local measurements.

1. Introduction

Basic assumptions underlying classical physics, particularly in general and special relativity, require consistency of all measurable statistical correlations with local realism [1]. As a consequence each measurement scenario imposes particular restrictions on measurable multipartite correlations. In general these classical correlations can be described within the framework of a local realistic theory (LRT) or local hidden variable (LHV) model [2]. These restrictions of possible classical correlations in local theories manifest themselves in Bell inequalities associated with classical probability polytopes [3, 4]. A striking phenomenon of quantum systems is their capability to violate these restrictions imposed by local realism, in particular in the context of measurement scenarios [5, 6]. This phenomenon of Bell-nonlocality has been demonstrated in a series of impressive experiments with increasing degrees of sophistication [7–9].

The possible existence of nonlocal quantum correlations has been pointed out as early as 1935 by Einstein, Podolsky and Rosen (EPR) [10] in their discussion of nonlocal properties of a certain class of pure quantum states. The peculiar properties of these quantum states have inspired numerous applications [11]. In his response to EPR Schrödinger termed these quantum states entangled. In addition, he coined the notion of steering for the characteristic quantum phenomenon underlying the physical discussion of EPR, namely the ability of one local observer, say Alice, to influence the measurements of another local observer, say Bob, by her measurements instantaneously [12, 13]. However, it was not until 2007 that Schrödinger's qualitative description of steering, frequently also termed EPR steering, was rigorously defined by Wiseman, Jones and Doherty [14, 15]. They based their notion of EPR steering on the existence of a local hidden state (LHS) model. These LHS models generalize LHV models by transcending the LHV-specific theoretical state concept with the help of quantum states thus allowing for a quantum theoretical description of the locally measurable probability distributions. This concept of EPR steerability captures some nonlocal aspects of quantum correlations and differs from Bell nonlocality and entanglement [16]. Although separability of quantum states is sufficient for EPR unsteerability, or equivalently, EPR steerability is sufficient for entanglement, the precise relation between these concepts of nonlocal quantum correlations is intricate, particularly for mixed quantum states. Contrary to entanglement, for example, EPR steerability is an asymmetric property with respect to the local observers involved because there are multipartite quantum states that allow one local observer to steer another observer but not vice versa. Furthermore, in contrast to entanglement EPR steerability is a measurement dependent notion. Whether a quantum state is EPR steerable or not depends on the local measurements which the observers can perform. Thus, increasing the measurement capabilities of a local observer should also increase the number of EPR steerable quantum states.

Apart from their significance for the foundations of quantum theory EPR steerable quantum states also have potentially promising applications in quantum information processing, in particular in the areas of quantum key distribution [17], quantum teleportation [18] and quantum secret sharing [19]. Because of this fundamental and practical significance of EPR steering, it is interesting to develop tests for EPR steerability from Alice to Bob. Although general necessary and sufficient conditions for bipartite EPR steerability of quantum states are known [16, 20, 21], these criteria involve intricate optimization procedures. In those, even the simplest case of two qubits requires considerable numerical efforts to determine typical statistical features of bipartite EPR steerability, such as volume ratios of EPR steerable versus all quantum states [20]. Up to now even in the simple case of two qubits numerically feasible necessary and sufficient conditions for EPR steerability from Alice to Bob are only known for cases in which Bob's local measurements are restricted to projective measurements. Although there is numerical evidence that these results also apply to general positive operator valued measurements (POVMs) [21] a general proof of this conjecture is still missing.

In view of these difficulties numerically less demanding conditions for EPR steerability, which are only sufficient and no longer necessary, offer a promising alternative for investigating typical statistical properties of EPR steerable bipartite quantum states from Alice to Bob for arbitrary dimensional bipartite quantum systems. Recently Lai and Luo [22] have developed such an approach based on the violation of an inequality which involves the trace norm of the correlation matrix of local measurements. As a violation of this inequality can be checked numerically in a rather straightforward manner, this approach may offer interesting perspectives for exploring the statistical properties of bipartite EPR steerability from Alice to Bob. So far, these authors have applied their sufficient condition to cases involving LOOs, mutually unbiased measurements (MUMs) [23] and general symmetric informationally complete positive operator-valued measures (GSIC) [24] for Alice and Bob. For these measurements they have explored bipartite EPR steerability for particular families of qubit-qubit, qubit-qutrit and qutrit-qutrit states and in addition for a particular one-parameter family of arbitrary dimensional bipartite isotropic quantum states. In view of these investigations the natural questions arise to which extent local measurements are capable of detecting arbitrary typical quantum states, which are not restricted to sets of measure zero, and how does the nature of these measurements influence their detectability.

Motivated by these developments in this paper we further explore basic properties of this recently proposed correlation-matrix-based sufficient condition for EPR steerability from Alice to Bob of Lai and Luo [22]. In particular, we focus on two main questions. Firstly, we want to explore how the capability of detecting EPR steerability from Alice to Bob in arbitrary dimensional bipartite quantum systems depends on the nature of the local quantum measurements for a broader class of measurements. In this respect informationally complete local measurements are of particular interest. For this purpose the recently introduced informationally complete (N,M)-POVMs [25] are well suited. These families of POVMs allow for a unified description of a large class of generalized measurements of current interest and include mutually unbiased bases (MUBs) [26], MUMs [23], symmetric informationally complete POVMs (SIC-POVMs) [27, 28] and their generalized analogs GSICs [24, 29]. As a main result it will be demonstrated that the trace norm of correlation matrices of local (N,M)-POVMs is proportional to that of LOOs. This scaling property implies that the sufficient condition for bipartite EPR steerability from Alice to Bob of Lai and Luo [22] becomes independent of the local informationally complete measurements performed within the class of (N,M)-POVMs. Furthermore, it will be shown that LOOs are equally powerful in determining EPR steerability from Alice to Bob as local informationally complete (N,M)-POVMs. Secondly, we want to explore how many EPR steerable bipartite quantum states from Alice to Bob can be detected by this correlation-matrix-based sufficient condition. For this purpose we sample random bipartite quantum states by a recently developed hit-and-run Monte-Carlo procedure [30]. In particular, we determine the resulting relative Euclidean volume ratios between EPR steerable states from Alice to Bob and all possible bipartite quantum states for different dimensions of Alice's and Bob's quantum systems. This way we are capable of exploring typical statistical features of EPR steerability from Alice to Bob.

This paper is organized as follows. For the sake of completeness in section 2 we summarize basic facts about the concepts of local realism and bipartite EPR steerability from Alice to Bob. Basic features of the recently introduced (N,M)-POVMs [25], which are the local measurements we will use for our subsequent investigation, are summarized in section 3. In section 4 the correlation-matrix-based sufficient condition for bipartite EPR steerability from Alice to Bob of Lai and Luo [22] is introduced and applied to local measurements based on (N,M)-POVMs in arbitrary dimensional bipartite quantum systems. In section 5 it is demonstrated that within the class of local informationally complete (N,M)-POVMs this sufficient EPR steerability condition exhibits a characteristic scaling property. This implies that this sufficient condition becomes independent of the particular informationally complete local measurements testing for EPR steerability. Furthermore, it becomes identical with the corresponding sufficient condition for LOOs. In section 6 numerical results of Monte-Carlo simulations are presented exploring Euclidean volume ratios of steerable bipartite quantum states from Alice to Bob and all bipartite quantum states for different dimensions of Alice's and Bob's quantum systems. Furthermore, these results are compared to the corresponding volume ratios based on a method recently proposed by Das *et al* [31], which reduces the determination of bipartite steerability from Alice to Bob to the determination of bipartite entanglement.

2. Local realism and bipartite steerability

Classical physical theories which are consistent with the physical laws of special relativity and its fundamental distinction between time-like and space-like events are governed by local realism [1]. As a consequence classical correlations between two space-like separated observers, say Alice and Bob, obey locality constraints. They can be expressed in terms of generalized Bell inequalities which describe probability polytopes [4]. Consider a simple bipartite scenario in which two space-like separated observers, Alice and Bob, perform random measurements of local observables, say $\alpha \in \mathcal{O}_A$ and $\beta \in \mathcal{O}_B$ with $\mathcal{O}_A$ and $\mathcal{O}_B$ describing sets of local observables of Alice and Bob. Each of these local observables has different possible measurement results, say $a \in \mathcal{M}_A$ and $b \in \mathcal{M}_B$. In a local realistic classical theory (LRT) the bipartite probability distribution $P(a, b|\alpha, \beta)$ of a possible joint local measurement, in which Alice and Bob randomly select observables α and β and obtain measurement results a and b, has the characteristic structure

$$P(a,\, b|\alpha,\, \beta) = \sum_{\lambda\in\Lambda} p(\lambda) P^A(a|\alpha,\, \lambda) P^B(b|\beta,\, \lambda),$$
$$\sum_{\lambda\in\Lambda} p(\lambda) = 1 \tag{1}$$

with the characteristic normalization $\sum_{a\in\mathcal{M}_A} P^A(a|\alpha,\, \lambda) = \sum_{b\in\mathcal{M}_B} P^B(b|\beta,\, \lambda) = 1$. Upper indices, such as A and B, refer to local observers, such as Alice (A) and Bob (B), small Greek letters symbolize measurements and small Latin letters symbolize measurement results. According to (1), for each random selection of local observables the joint probability distribution is a convex sum of local (conditional) probability distributions of Alice and Bob, i.e. $P^A(a|\alpha, \lambda) \geqslant 0$ and $P^B(b|\beta, \lambda) \geqslant 0$, which depend on a random variable $\lambda \in \Lambda$ with probability distribution $p(\lambda) \geqslant 0$. In this description this 'hidden variable' $\lambda \in \Lambda$ characterizes uniquely the classical state of the bipartite physical system in a classical state space Λ. According to Bells theorem [1], the constraints imposed on physical theories by local realism, as expressed by equation (1) in a bipartite scenario, can be violated by quantum correlations. In particular, the quantum correlations originating from entangled quantum states can violate local realism for particular choices of measurements. Recently violations of local realism by quantum systems have been demonstrated by impressive experiments with increasing degrees of sophistication [7–9].

Another characteristic quantum concept is steering [16]. It has been introduced originally by Schrödinger in 1935 [12, 13], in the same year in which Einstein, Podolsky and Rosen (EPR) have published their work questioning the completeness of quantum mechanical description and describing the well-known EPR paradoxon [10]. Steering, sometimes also called EPR steering, may be viewed as a generalization of the experimental scenario which forms the basic scenario discussed in the EPR paradoxon and which is described by (1) within the framework of a local realistic theory.

Bipartite scenarios are among the simplest ones for which the concept of EPR steering can be formulated. Accordingly, let us again consider two observers, Alice and Bob, performing general local measurements which can be described by local positive operator valued measures (POVMs) [32, 33], say $\{\Pi^A_{\alpha,a} \geqslant 0\}$ for Alice and $\{\Pi^B_{\beta,b} \geqslant 0\}$ for Bob. Thereby α and β denote the different measurements and a and b denote their corresponding measurement results. These POVMs obey the characteristic completeness relations

$$\sum_{a\in\mathcal{M}_A} \Pi^A_{\alpha,a} = \mathbb{1}_{d_A}, \qquad \sum_{b\in\mathcal{M}_B} \Pi^B_{\beta,b} = \mathbb{1}_{d_B} \tag{2}$$

with the unit operators $\mathbb{1}_{d_A}$ and $\mathbb{1}_{d_B}$ of the d_A- and d_B-dimensional local Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ of Alice's and Bob's quantum systems. The Hilbert space of the complete bipartite quantum system is given by $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$.

In a typical bipartite EPR steering from Alice to Bob, Alice prepares a bipartite quantum state, say $\rho \geqslant 0$, with corresponding reduced local density operators $\rho^A = \mathrm{Tr}_B\{\rho\}$ and $\rho^B = \mathrm{Tr}_A\{\rho\}$ for Alice and Bob. It is assumed that Alice can perform local measurements on the combined quantum state only with the help of her POVM $\{\Pi^A_{\alpha,a}\}$, and Bob can perform local measurements on this quantum state only with his POVM $\{\Pi^B_{\beta,b}\}$. According to quantum theory the bipartite probability distribution of a possible joint local measurement on this quantum state, in which Alice and Bob measure observables α and β with measurement results a and b, is given by

$$P(a,\, b|\alpha,\, \beta,\, \rho) = \mathrm{Tr}_{AB}\{\Pi^A_{\alpha,a} \otimes \Pi^B_{\beta,b}\rho\}. \tag{3}$$

A quantum state ρ is called EPR unsteerable from Alice to Bob with respect to measurements α and β if this joint probability distribution can be described within the framework of a local hidden state (LHS) model [14, 15]. Otherwise this quantum state is called EPR steerable from Alice to Bob concerning these measurements. More concretely, EPR unsteerability from Alice to Bob for local measurements described by the POVMs $\{\Pi^A_{\alpha,a}\}$ and $\{\Pi^B_{\beta,b}\}$ means, that there exists a statistical ensemble of reduced quantum states of Bob, say $\{(\lambda,\, \rho^B_\lambda)|\lambda \in \Lambda\}$ with the random variable $\{\lambda \in \Lambda\}$ being characterized by a probability distribution $\{p(\lambda) > 0|\lambda \in \Lambda\}$, which is consistent with the conditional joint probability (3), i.e.

$$\begin{aligned} P(a,\, b|\alpha,\, \beta,\, \rho) &= \sum_{\lambda\in\Lambda} p(\lambda) P^A(a|\alpha,\, \lambda) P^B(b|\beta,\, \lambda), \\ P^B(b|\beta,\, \lambda) &\coloneqq \mathrm{Tr}_B\{\Pi^B_{\beta,b}\rho^B_\lambda\}. \end{aligned} \tag{4}$$

It is apparent that this restriction on the joint probability distribution of Alice (A) and Bob (B) is weaker than the restriction (1) imposed by local realism as it involves a quantum mechanical evaluation of Bob's local conditional probability distribution $P^B(b|\beta,\, \lambda)$. Using (2) and the characteristic normalization for conditional probabilities this condition (4) implies the relations

$$\begin{aligned} \sum_{a\in\mathcal{M}_A} P(a,\, b|\alpha,\, \beta,\, \rho) &= \mathrm{Tr}_B\{\Pi^B_{\beta,b}\rho^B\} = \mathrm{Tr}_B\{\Pi^B_{\beta,b}\sum_{\lambda\in\Lambda} p(\lambda)\rho^B_\lambda\}, \\ \sum_{b\in\mathcal{M}_B} P(a,\, b|\alpha,\, \beta,\, \rho) &= \mathrm{Tr}_A\{\Pi^A_{\alpha,a}\rho^A\} = \sum_{\lambda\in\Lambda} p(\lambda) P^A(a|\alpha,\, \lambda). \end{aligned} \tag{5}$$

Thus, concerning Bob's measurement of the POVM $\{\Pi^B_{\beta,b}\}$, Bob's reduced quantum state ρ^B and the LHS state $\sum_{\lambda\in\Lambda} p(\lambda)\rho^B_\lambda$ are indistinguishable.

EPR steering from Bob to Alice is defined analogously. It is worth mentioning that the two concepts of EPR steerability or EPR unsteerability, namely from Alice to Bob on the one hand and from Bob to Alice on the other hand, are asymmetric. In particular, EPR steerability (or EPR unsteerability) from Alice to Bob does not necessarily imply EPR steerability (or EPR unsteerability) from Bob to Alice.

3. Generalized measurements by positive operator valued (N,M)- measures

In this section we summarize basic results on informationally complete generalized measurements which can be described by (N,M)-POVMs [25]. They allow for a unified description of various more specialized measurement schemes, including MUBs [26], MUMs [23], SIC-POVMs [27, 28] and GSIC [24, 29].

In a d-dimensional Hilbert space a (N,M)-POVM, say Π, is a set of N POVMs such that each of them involves M positive semidefinite operators describing the different possible measurement results, i.e. $\Pi = \{\Pi_{\alpha,a} \geqslant 0|\ \alpha = 1, \cdots, N;\ a = 1, \cdots, M\}$. The parameter α identifies a particular measurement scheme associated with an experimental setup and the parameter a identifies the corresponding different possible measurement results. The operators $\Pi_{\alpha,a}$ of a (N,M)-POVMs satisfy the additional relations [25]

$$\mathrm{Tr}\{\Pi_{\alpha,a}\} = \frac{d}{M}, \tag{6}$$

$$\mathrm{Tr}\{\Pi_{\alpha,a}\,\Pi_{\alpha,a'}\} = x\,\delta_{aa'} + (1 - \delta_{aa'})\frac{d - Mx}{M(M-1)}, \tag{7}$$

$$\mathrm{Tr}\{\Pi_{\alpha,a}\,\Pi_{\alpha',a'}\} = \frac{d}{M^2} \tag{8}$$

for all $\alpha \neq \alpha'$. The possible values of x are constrained by the relation $d/M^2 < x \leqslant \min(d^2/M^2,\, d/M)$. A ($N$, M)-POVM, for which x assumes its largest possible value, is called optimal.

If a (N,M)-POVM Π contains d^2 linearly independent positive semidefinite operators it is called informationally complete. As each of the N POVMs fulfills the completeness relation (2), informational completeness constrains the parameters N, M and d by the relation [25]

$$(M-1)N+1=d^2. \tag{9}$$

Therefore, each possible solution of this equation yields a possible informationally complete (N,M)-POVM in a given d-dimensional Hilbert space. As a result at least four possible classes of informationally complete (N,M)-POVMs can always be constructed. They correspond to the possible solutions $(N, M) \in \{(1, d^2), (d+1, d), (d^2-1, 2), (d-1, d+2)\}$. In particular, the solution $(N, M) = (1, d^2)$ characterizes a one-parameter family of GSICs [24, 29]. The solution $(N, M) = (d+1, d)$ describes MUM [23], which in the special case $x = d^2/M^2 = d/M = 1$ reduces to projective measurements with maximal sets of $d+1$ MUBs. For a qubit, i.e. $d = 2$, these four possible solutions of (9) collapse to two cases, namely GSICs ($(N, M) = (1, 4)$) and MUMs ($(N, M) = 3, 2)$).

In a d-dimensional Hilbert space informationally complete (N,M)-POVMs can be represented by a basis of d^2 linearly independent hermitian operators. For this purpose it is advantageous to choose an orthonormal basis of hermitian operators, i.e. $\{G_i | G_i = G_i^\dagger,\ i = 1,\cdots,d^2\}$, with respect to the Hilbert-Schmidt (HS) scalar product $\langle G_i|G_{i'}\rangle := \mathrm{Tr}\{G_i^\dagger G_{i'}\}$. Such a basis spans the d^2-dimensional Hilbert space $\mathcal{H}_{d^2}$ of hermitian linear operators over the field of real numbers.

Starting from an arbitrary orthonormal basis $\mathcal{B} = \{|k\rangle;\ k = 1,\cdots,d\}$ in the d-dimensional Hilbert space $\mathcal{H}_d = (\mathrm{Span}(\mathcal{B}),\ \langle\cdot|\cdot\rangle)$ of a d-dimensional quantum system, an example of such a hermitian orthonormal operator basis is given by

$$\begin{aligned}
\tilde{G}_1 &= \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k\rangle\langle k|,\\
\tilde{G}_i &= \frac{1}{\sqrt{i(i-1)}}\left(\sum_{k=1}^{i-1}|k\rangle\langle k| - (i-1)|i\rangle\langle i|\right),\\
\tilde{G}_{md+n} &= \frac{1}{\sqrt{2}}(|m\rangle\langle n| + |n\rangle\langle m|),\quad 1\leqslant m < n \leqslant d,\\
\tilde{G}_{(m-1)d+n} &= \frac{i}{\sqrt{2}}(|m\rangle\langle n| - |n\rangle\langle m|),\quad 1\leqslant n < m \leqslant d
\end{aligned} \tag{10}$$

for $i = 2, \cdots, d$. This basis satisfies the additional convenient requirements $\tilde{G}_1\sqrt{d} = \mathbb{1}_d$, such that orthonormality implies $\mathrm{Tr}\{\tilde{G}_i\} = 0$. Any other orthonormal hermitian operator basis can be constructed from this one by applying an arbitrary real-valued orthogonal transformation, say $\tilde{O}$ with $\tilde{O}\tilde{O}^T = \mathbb{1}_{d^2}$, onto this operator basis. For our subsequent discussion it is convenient to arrange the basis operators of such an arbitrary hermitian operator basis in a d^2-dimensional column vector, say $G = (G_1,\cdots,G_{d^2})^T = \tilde{O}\tilde{G}$. The hermitian basis operators of any orthonormal basis fulfill the relation

$$\frac{1}{d}\sum_{i=1}^{d^2} G_i^2 = \mathbb{1}_d. \tag{11}$$

Analogously, an arbitrary (N,M)-POVM can be combined into a NM-dimensional row vector of positive semidefinite operators, i.e. $\Pi = (\Pi_1, \cdots, \Pi_{NM})$. Its expansion in terms of this basis is given by $\Pi = G^T S$ with the real-valued $d^2 \times NM$-matrix S, whose matrix elements are constrained by the defining properties (6), (7) and (8) of (N,M)-POVMs. In the subsequent sections we shall take advantage of the fact that the characteristic properties of the matrix S enable an efficient calculation of trace norms of correlation matrices for local (N,M)-POVMs.

4. Correlation-matrix-based sufficient conditions for bipartite EPR steerability from Alice to Bob

In this section the general form of the recently proposed sufficient condition for bipartite EPR steerability from Alice to Bob of Lai and Luo [22] is discussed and applied to local measurements of Alice and Bob which can be described by arbitrary (N,M)-POVMs or by LOOs. This sufficient condition is based on a violation of an inequality involving the trace- or 1-norm of the correlation matrix of these local measurements of Alice and Bob. It is valid for arbitrary dimensional quantum systems.

In order to investigate EPR steerability of bipartite quantum states with respect to two arbitrary local generalized measurements described by (N,M)-POVMs, say Π^A and Π^B, let us consider the associated correlation matrix $C(\Pi^A, \Pi^B|\rho)$ with matrix elements

$$(C(\Pi^A, \Pi^B|\rho))_{ij} = \mathrm{Tr}_{AB}\{\Pi^A(i)\otimes\Pi^B(j)(\rho - \rho^A\otimes\rho^B)\}. \tag{12}$$

Thereby, $\Pi^A(i)$ denotes the i-th component of Alice's row vector Π^A with the indexing $i(\alpha, a) := (\alpha-1)M_A + a \in \{1, \cdots, N_A M_A\}$. The analogous notation is used for Bob's POVM elements $\Pi^B(j)$.

If the bipartite quantum state ρ is EPR unsteerable from Alice to Bob with respect to these local measurements this correlation matrix can be rewritten in the form (cf (4))

$$\begin{aligned}(C(\Pi^A, \Pi^B|\rho))_{ij} &= \frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} p(\lambda)p(\lambda')V_i(\lambda, \lambda')W_j(\lambda, \lambda'),\\ V_{i(\alpha,a)}(\lambda, \lambda') &= P(a|\alpha, \lambda) - P(a|\alpha, \lambda'),\\ W_{j(\beta,b)}(\lambda, \lambda') &= \mathrm{Tr}_B\{\Pi^B_{\beta,b}\rho^B_{\lambda}\} - \mathrm{Tr}_B\{\Pi^B_{\beta,b}\rho^B_{\lambda'}\}\end{aligned} \tag{13}$$

with $i \in \{1, \cdots, N_AM_A\}$, $j \in \{1, \cdots, N_BM_B\}$. The trace- or 1-norm of this $N_AM_A \times N_BM_B$ correlation matrix can be restricted with the help of the triangular inequality and the Cauchy-Schwarz inequality yielding the result

$$\begin{aligned}\|C(\Pi^A, \Pi^B|\rho)\|_1 &\leqslant \frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} p(\lambda)p(\lambda')\|V(\lambda, \lambda')\|_2\|W(\lambda, \lambda')\|_2\\ &\leqslant \frac{1}{2}\sqrt{\sum_{\lambda,\lambda'\in\Lambda} p(\lambda)p(\lambda')\|V(\lambda, \lambda')\|_2^2}\sqrt{\sum_{\lambda,\lambda'\in\Lambda} p(\lambda)p(\lambda')\|W(\lambda, \lambda')\|_2^2}.\end{aligned} \tag{14}$$

Using the positivity of variances this upper bound can be further upper bounded with the help of the inequalities

$$\begin{aligned}\sum_{\lambda,\lambda'\in\Lambda}\sum_{i=1}^{N_AM_A} p(\lambda)p(\lambda')(V_i(\lambda, \lambda'))^2 &\leqslant 2\sum_{i=1}^{N_AM_A}(\mathrm{Tr}_A\{(\Pi^A)^2(i)\rho^A\} - (\mathrm{Tr}_A\{\Pi^A(i)\rho^A\})^2) :=2V_>,\\ \sum_{\lambda,\lambda'\in\Lambda}\sum_{j=1}^{N_BM_B} p(\lambda)p(\lambda')(W_j(\lambda, \lambda'))^2 &\leqslant 2\sum_{\lambda\in\Lambda}\sum_{j=1}^{N_BM_B}(\mathrm{Tr}_B\{\Pi^B(j)\rho^B_\lambda\})^2 - 2\sum_{j=1}^{N_BM_B}(\mathrm{Tr}_B\{\Pi^B(j)\rho^B\})^2\\ &\leqslant 2\left(\max_{\sigma^B}\sum_{j=1}^{N_BM_B}(\mathrm{Tr}_B\{\Pi^B(j)\sigma^B\})^2 - \sum_{j=1}^{N_BM_B}(\mathrm{Tr}_B\{\Pi^B(j)\rho^B\})^2\right) :=2W_>,\end{aligned} \tag{15}$$

which involve a maximization over all possible reduced quantum states of Bob σ^B.

In order to simplify these expressions further we represent the local (N,M)-POVMs $\Pi^A = (G^A)^TS^A$ and $\Pi^B = (G^B)^TS^B$ by arbitrary LOOs. Analogously we represent Alice's and Bob's reduced states in the form $\rho^A = (G^A)^Tr^A$, $\rho^B = (G^B)^Tr^B$ and $\sigma^B = (G^B)^Ts^B$. Using (11), (A7) and (A8) we find

$$\begin{aligned}\sum_{i=1}^{N_AM_A}\mathrm{Tr}_A\{(\Pi^A)^2(i)\rho^A\} &= \mathrm{Tr}_A\{(G^A)^TS^A(S^A)^TG^A\rho^A\} = d_A\Gamma_A + \frac{N_Ad_A/M_A - \Gamma_A}{d_A},\\ \sum_{i=1}^{N_AM_A}(\mathrm{Tr}_A\{\Pi^A(i)\rho^A\})^2 &= (r^A)^TS^A(S^A)^Tr^A = \Gamma_A\mathrm{Tr}_A\{(\rho^A)^2\} + \frac{N_Ad_A/M_A - \Gamma_A}{d_A},\\ \sum_{j=1}^{N_BM_B}(\mathrm{Tr}_B\{\Pi^B(j)\rho^B\})^2 &= (r^B)^TS^B(S^B)^Tr^B = \Gamma_B\mathrm{Tr}_B\{(\rho^B)^2\} + \frac{N_Bd_B/M_B - \Gamma_B}{d_B},\\ \sum_{j=1}^{N_BM_B}\max_{\sigma^B}(\mathrm{Tr}_B\{\Pi^B(j)\sigma^B\})^2 &= \max_{s^B}(s^B)^TS^B(S^B)^Ts^B = \Gamma_B + \frac{N_Bd_B/M_B - \Gamma_B}{d_B}.\end{aligned} \tag{16}$$

Combining these upper bounds we finally obtain the inequality

$$\|C(\Pi^A, \Pi^B|\rho)\|_1 \leqslant \sqrt{\Gamma_A\Gamma_B}\sqrt{(d_A - \mathrm{Tr}_A\{(\rho^A)^2\})(1 - \mathrm{Tr}_B\{(\rho^B)^2\})} \tag{17}$$

with

$$\Gamma_A = \frac{x_AM_A^2 - d_A}{M_A(M_A - 1)}, \quad \Gamma_B = \frac{x_BM_B^2 - d_B}{M_B(M_B - 1)}. \tag{18}$$

Therefore, provided a bipartite quantum state ρ is EPR unsteerable from Alice to Bob with respect to the arbitrary (N,M)-POVMs Π^A and Π^B, the 1-norm of the associated correlation matrix, i.e. $\|C(\Pi^A, \Pi^B|\rho)\|_1$, fulfills inequality (17). Stated differently, a violation of inequality (17) is a sufficient condition for bipartite EPR steerability from Alice to Bob with respect to these local generalized measurements. Consequently, inequality (17) can be used to obtain upper bounds on measures of EPR unsteerable bipartite states from Alice to Bob and lower bounds on measures of bipartite EPR steerable bipartite states from Alice to Bob. For different (N,M)-POVMs and arbitrary quantum states the upper bound of the steering inequality (17) varies only by the scaling factor $\sqrt{\Gamma_A\Gamma_B}$. However, for different types of (N,M)-POVMs the dimensions of the correlation matrices determining the left hand side of (17) may change, thus complicating a direct comparison of detection efficiencies resulting from different values of N, M and x.

It is straightforward to derive an analogous inequality for an arbitrary set of local hermitian operators, say $\alpha = (\alpha_1, \cdots, \alpha_m)$ for Alice and $\beta = (\beta_1, \cdots, \beta_n)$ for Bob, describing local measurements. The resulting inequality is given by [22]

$$\|C(\alpha,\ \beta|\rho)\|_1 \leqslant \sqrt{V_>^A W_>^B} \tag{19}$$

with

$$V_>^A = \sum_{i=1}^{m} (\mathrm{Tr}_A\{(\alpha_i)^2\rho^A\} - (\mathrm{Tr}_A\{\alpha_i\rho^A\})^2),$$
$$W_>^B = \left(\max_{\sigma^B} \sum_{j=1}^{n} (\mathrm{Tr}_B\{\beta_j\sigma^B\})^2 - \sum_{j=1}^{n} (\mathrm{Tr}_B\{\beta_j\rho^B\})^2\right), \tag{20}$$

which also involves a maximization over all possible reduced quantum states of Bob σ^B.

5. Influence of local (*N*,*M*)-POVMs on violations of the EPR steering inequality (17)

The natural question arises how the sufficient condition for bipartite EPR steerability from Alice to Bob based on a violation of (17) depends on the type of local measurements performed by Alice and Bob. In this section it is demonstrated that inequality (17) exhibits a characteristic scaling property originating from a permutation symmetry inherent in the definition of (*N*,*M*)-POVMs. It implies that all informationally complete local measurements involving (*N*,*M*)-POVMS and all LOOs lead to one and the same inequality whose violation yields a sufficient condition for bipartite EPR steerability from Alice to Bob. The derivation of this scaling property is based on general relations between informationally complete (*N*,*M*)-POVMs and LOOs which have been obtained recently in an investigation on the detection of typical bipartite entanglement with the help of local measurements [34] and are summarized in the appendix.

Before we turn to the EPR steering detection with (*N*,*M*)-POVMs let us first investigate inequality (19) for the special case of LOOs for Alice and Bob as described by (10). Alice measures the hermitian basis operators $\tilde{G}^A = (\tilde{G}_1^A,\cdots,\tilde{G}_{d_A^2}^A)^T$ and Bob measures the hermitian basis operators $\tilde{G}^B = (\tilde{G}_1^B,\cdots,\tilde{G}_{d_B^2}^B)^T$. According to (20) the relevant upper bounds yield

$$V_>^A = \sum_{i=1}^{d_A^2} (\mathrm{Tr}_A\{(\tilde{G}_i^A)^2\rho^A\} - (\mathrm{Tr}_A\{\tilde{G}_i^A\rho^A\})^2) = d_A - \mathrm{Tr}_A\{(\rho^A)^2\},$$
$$W_>^B = \max_{\sigma^B} \sum_{j=1}^{d_B^2} (\mathrm{Tr}_B\{\tilde{G}_j^B\sigma^B\})^2 - \sum_{j=1}^{d_B^2} (\mathrm{Tr}_B\{\tilde{G}_j^B\rho^B\})^2 = 1 - \mathrm{Tr}_B\{(\rho^B)^2\}. \tag{21}$$

Thus, inequality (19) reduces to

$$\|C(\tilde{G}^A,\ \tilde{G}^B|\rho)\|_1^2 \leqslant (d_A - \mathrm{Tr}_A\{(\rho^A)^2\})(1 - \mathrm{Tr}_B\{(\rho^B)^2\}). \tag{22}$$

As the 1-norm of a matrix is invariant under arbitrary local orthogonal transformations performed by Alice and Bob, say $\tilde{O}^A$ and $\tilde{O}^B$, inequality (22) also applies to any other LOOs for Alice and Bob, say $G^A = \tilde{O}^A\tilde{G}^A$ and $G^B = \tilde{O}^B\tilde{G}^B$, i.e.

$$\|C(G^A,\ G^B|\rho)\|_1 \leqslant \sqrt{(d_A - \mathrm{Tr}_A\{(\rho^A)^2\})(1 - \mathrm{Tr}_B\{(\rho^B)^2\})}. \tag{23}$$

The upper bound of the EPR steering inequality for (*N*,*M*)-POVMs (17) is the boundary of (23) scaled by the factor $\sqrt{\Gamma_A\Gamma_B}$. This scaling property enables us to compare the EPR steering inequalities of both measurement settings in a straight forward way despite the different dimensions of the correlation matrices of the (*N*,*M*) -POVMs and of the LOOs entering (17) and (23). For this purpose we have to relate the 1-norms of correlation matrices of local (*N*,*M*)-POVMs to the ones of LOOs. Let Alice and Bob use local informationally complete (*N*,*M*)-POVMs, say Π^A and Π^B, with basis expansions $\Pi^A = (G^A)^T S^A$ and $\Pi^B = (G^B)^T S^B$ involving arbitrary local orthonormal bases G^A and G^B. As outlined in the appendix the defining properties of informationally complete (*N*,*M*)-POVMs (2), (7), (8) and (9) imply that the 1-norms of the correlation matrices exhibit the simple scaling property

$$\|C(\Pi^A,\ \Pi^B|\rho)\|_1 = \sqrt{\Gamma_A\Gamma_B}\,\|C(G^A,\ G^B|\rho)\|_1. \tag{24}$$

Thus, the trace 1-norm of the correlations matrices of local (*N*,*M*)-POVM is also proportional to that of LOOs with the scaling factor $\sqrt{\Gamma_A\Gamma_B}$. As a result inequality (17) simplifies to the inequality for LOOs (23), i.e.

$$\|C(\Pi^A,\ \Pi^B|\rho)\|_1 = \sqrt{\Gamma_A\Gamma_B}\,\|C(G^A,\ G^B|\rho)\|_1 \leqslant \sqrt{\Gamma_A\Gamma_B}\sqrt{(d_A - \mathrm{Tr}_A\{(\rho^A)^2\})(1 - \mathrm{Tr}_B\{(\rho^B)^2\})}. \tag{25}$$

Therefore, a violation of inequality (23) yields the sufficient condition for bipartite EPR steerability from Alice to Bob not only for arbitrary LOOs but also for arbitrary informationally complete local (*N*,*M*)-POVMs. In general, the property of EPR steerability depends not only on the quantum state but also on the class of measurements involved. Since the EPR steering inequality (25) is identical for all (*N*,*M*)-POVMs and LOOs, the quantum states

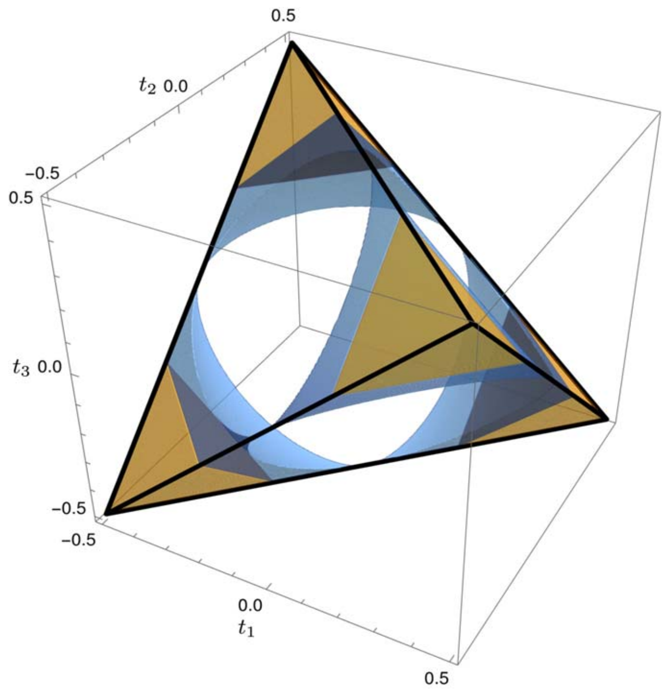

Figure 1. Bell diagonal two-qubit quantum states which are steerable with respect to different measurements: Steerable states with respect to local measurements involving (N,M)-POVMs due to violations of inequality (25) or equivalently (23)(yellow regions); steerable states with respect to projective measurements due to the criteria (27) and (29) (yellow and blue regions); unsteerable states with respect to projective measurements due to the criteria (27) and (29) (uncolored central convex region).

violating this inequality are steerable with respect to all these measurements. In view of this result the recent observations of Lai and Luo [22], which apply to particular families of quantum states and particular SIC-POVMs, MUBs and LOOs only, appear as special cases of this general consequence of characteristic scaling properties of (N,M)-POVMs.

For the simple case of Bell diagonal states of the form

$$\rho_{Bell} = \frac{1}{4}\mathbb{1}_2 \otimes \mathbb{1}_2 + \sum_{i=1}^{3} \frac{t_i}{2}\sigma_i^A \otimes \sigma_i^B \tag{26}$$

with the Pauli spin operators σ_i^A and σ_i^B for Alice and Bob it is straightforward to determine the quantum states which are steerable from Alice to Bob and which can be detected by a violation of inequality (25) or equivalently (23) with the help of LOOs or local informationally complete (N,M)-POVMs. For Bell diagonal states EPR steering is symmetric. Therefore, a steerable quantum state from Alice to Bob is also steerable from Bob to Alice. For Bell diagonal states and projective measurements a necessary and sufficient condition for EPR steerability has recently been derived by Nguyen *et al* [21]. Accordingly, the Bell diagonal states on the border of EPR steerability are solutions of the equation

$$1 = 2\pi N_T |\det T| \tag{27}$$

with the diagonal matrix $T = \mathrm{diag}(t_1, t_2, t_3)$. The normalization constant N_T is determined by the relation

$$N_T^{-1} = \int_{S_2} \mathrm{d}S(\vec{n})[\vec{n} \cdot T^{-2}\vec{n}]^{-2} \tag{28}$$

with the unit vectors $\vec{n}$ being integrated over the surface of the unit sphere S_2. Equivalently these states can be defined by the relation [35]

$$2\pi = \int_{S_2} \mathrm{d}S(\vec{n})\sqrt{\vec{n} \cdot T^2\vec{n}}\,. \tag{29}$$

In figure 1 the yellow regions within the tetrahedron of all Bell diagonal quantum states represent the steerable states detected by (25). The yellow and blue regions in this figure indicate the steerable quantum states with respect to projective measurements according to the criteria of (27) and (29). The uncolored central region represents the Bell diagonal two-qubit states which are unsteerable with respect to projective measurements according to (27) and (29).

The sufficient condition based on a violation of inequality (25) may be improved further by changing the LOOs of Alice and Bob to another set of local hermitian operators, say $\tilde{\alpha}^A$ and $\tilde{\alpha}^B$, thereby relaxing the orthonormality constraints. This may be achieved by a simple rescaling of Alice's local observables of the form

$$\tilde{\alpha}_i^A = h_i G_i^A \tag{30}$$

with the real-valued parameters h_i [22]. With these new local measurements inequality (19) assumes the form

$$\|C(\tilde{\alpha}^A, G^B|\rho)\|_1 \leqslant \sqrt{\tilde{V}_>^A(1 - \mathrm{Tr}_B\{(\rho^B)^2\})} \tag{31}$$

with

$$\tilde{V}_>^A = \sum_{i=1}^{d_A^2} h_i^2(\mathrm{Tr}_A\{(G_i^A)^2\rho^A\} - (\mathrm{Tr}_A\{G_i^A \rho^A\})^2). \tag{32}$$

For a given state ρ it can be tested whether the parameters h_i can be chosen so that inequality (31) can be violated. Thus, for a given state ρ it may be possible to find an optimal set of parameters h_i, such that inequality (31) is violated even if inequality (23) is still fulfilled. Such an optimization of the sufficient condition for EPR steerability from Alice to Bob by variations of the parameters h_i of Alice's LOO or local informationally complete (N,M)-POVM breaks the scaling property. Thus, a violation of inequality (31) may detect more EPR steerable states from Alice to Bob than a violation of inequality (23).

6. Numerical results

Numerous investigations have already concentrated on criteria and sufficient conditions for steerability and have applied them to restricted classes of quantum states which typically form zero-measure sets within the convex set of all quantum states [16]. However, so far questions concerning the statistical typicality of steerability and its detectability by local measurements are largely unexplored. In this section we concentrate on these latter issues and explore Euclidean volume ratios between bipartite steerable and all bipartite quantum states. In particular, we concentrate on detecting EPR steerability from Alice to Bob by local quantum measurements based on violations of inequality (31). The required volume ratios between EPR steerable quantum states from Alice to Bob and all quantum states for different dimensions of Alice's and Bob's quantum systems will be determined with the help of a recently developed hit-and-run Monte-Carlo method [30, 36]. This way the efficiency can be explored with which typical bipartite steerability can be detected with the help of local measurements.

Starting from a $d_A d_B$-dimensional Hilbert space $\mathcal{H}_{d_A d_B}$ describing a bipartite quantum system the corresponding $(d_A d_B)^2$-dimensional Hilbert space $\mathcal{H}_{(d_A d_B)^2}$ of hermitian linear operators can be constructed. With respect to the Hilbert-Schmidt (HS) scalar product $\langle A|B\rangle_{HS} := \mathrm{Tr}_{AB}\{A^\dagger B\}$ for $A,\ B \in \mathcal{H}_{(d_A d_B)^2}$ this Hilbert space $\mathcal{H}_{(d_A d_B)^2}$ is a Euclidean vector space on which volumes of sets of hermitian linear operators and of convex sets of quantum states $\rho \geqslant 0$ can be defined naturally. Numerically the Euclidean volumes of quantum states can be determined efficiently with the help of a recently developed hit-and-run Monte-Carlo algorithm [30] which has already been applied successfully to the determination of Euclidean volumes of bipartite quantum states. This efficient Monte-Carlo method, which has been introduced originally by Smith [37], relies on the realization of a random walk inside a convex set that converges efficiently to a uniform distribution over this convex set and, moreover, is independent of the starting point inside this convex set [38].

With this hit-and-run Monte-Carlo method we randomly sampled $N = 10^8$ bipartite quantum states for different values of d_A and d_B. For each of these randomly selected states it has been tested whether it violates the sufficient condition for EPR steerability from Alice to Bob (31) for LOOs or informationally complete local (N,M)-POVMs with optimized parameters h_i. If inequality (31) is violated this quantum state is EPR steerable from Alice to Bob and is kept. Otherwise this quantum state is dismissed. This way Euclidean volume ratios $R_{S:A\to B}$ between the volumes of EPR steerable quantum states from Alice to Bob and all bipartite quantum states have been determined numerically. In view of the optimization over the parameters h_i these Euclidean volume ratios $R_{S:A\to B}$ are always larger than the corresponding ratios which are obtainable directly from a violation of inequality (23).

We summarize our numerical results in table 1. We have numerically investigated bipartite quantum states with $2 \leqslant d_A, d_B \leqslant 4$. The cases not shown in table 1 yield negligible volume ratios below our numerical accuracy. These results suggest that the detection of typical bipartite EPR steerability from Alice to Bob based on a violation of inequality (31) with LOOs or informationally complete local (N,M)-POVMs significantly underestimates the volume ratios of EPR steerable states in higher dimensional scenarios beyond two-qubit cases. It should be emphasized that in view of the peculiar scaling properties of (N,M)-POVMs discussed in section 5 the volume ratios of table 1 cannot be increased by changes to other LOOs or informationally complete local (N,M)-POVMs.

Table 1. Numerical estimates of lower bounds of the Euclidean volume ratios $R_{S:A\to B}$ between EPR steerable quantum states from Alice to Bob and all bipartite quantum states for different dimensions d_A and d_B of Alice's and Bob's quantum systems: These estimates are based on a violation of inequality (31) with Alice's local measurement being optimized by rescaling. The procedure described in [30] was used to estimate the numerical errors.

Case	$d_B = 2$	$d_B = 3$
$d_A = 2$	$5,011 \times 10^{-2}$ $\pm 1,5 \times 10^{-4}$	$1,92 \times 10^{-5}$ $\pm 4,1 \times 10^{-6}$
$d_A = 3$	$5,72 \times 10^{-5}$ $\pm 6,4 \times 10^{-6}$	0

Table 2. Numerical lower bounds on the Euclidean volume ratios $R_{S:A\to B}$ between EPR steerable quantum states from Alice to Bob and all bipartite quantum states for $d_A = 2$ and different dimensions $2 \leqslant d_B \leqslant 7$ of Bob's quantum system: These estimates are based on the approach of Das *et al* [31]. The Peres-Horodecki condition has been used as a sufficient condition for bipartite entanglement of $\tau_{A\to B}$ for $d_B > 3$. The procedure described in [30] has been used to estimate the numerical errors.

d_B	2	3	4
	$0,05167$ $\pm 1,5 \times 10^{-4}$	$0,10936$ $\pm 3,4 \times 10^{-4}$	$0,17278$ $\pm 5,6 \times 10^{-4}$
d_B	5	6	7
	$0,24009$ $\pm 8,3 \times 10^{-4}$	$0,3119$ $\pm 1,3 \times 10^{-3}$	$0,3842$ $\pm 1,5 \times 10^{-3}$

In order to quantify this possible underestimation of bipartite steerability it is of interest to compare the results of table 1 with the corresponding results of another sufficient condition for bipartite EPR steerability from Alice to Bob which has been proposed recently by Das *et al* [31]. However, this method is only applicable if Alice's quantum system is a qubit. Beyond this requirement it does not impose any restrictions on the dimensionality of Bob's quantum system. The authors have proven that given a bipartite quantum state ρ, entanglement of the mixed quantum state

$$\tau_{A\to B} := \mu\rho + \frac{(1-\mu)}{2}\mathbb{1}_2 \otimes \mathrm{Tr}_A(\rho) \tag{33}$$

for a value of $\mu \in [0,\ 1/\sqrt{3}\,]$ is sufficient for EPR steerability from Alice to Bob. As there are powerful methods for determining bipartite entanglement, this approach may yield better lower bounds on the volume ratios $R_{S:A\to B}$ in its regime of validity. In particular, according to Peres [39] and Horodecki [40] one may use the existence of a negative partial transpose (NPT) of $\tau_{A\to B}$ as a sufficient condition for bipartite entanglement for cases with $d_B > 3$. In cases with $d_B \leqslant 3$ this latter condition is also necessary for bipartite entanglement. However, a possible disadvantage of this approach is that it is not based on local measurements of Alice and Bob. Table 2 depicts our numerically obtained lower bounds on Euclidean volume ratios $R_{S:A\to B}$ based on the approach by Das *et al* [31] combined with NPT tests of the states $\tau_{A\to B}$ of (33) for different dimensions d_B of Bob's quantum system. A comparison with the results of table 1 demonstrates that for two qubits the detection of EPR steerability by optimized local measurements leads to a result in agreement with the one of table 2. However, in all other cases the local measurement based sufficient condition for EPR steerability from Alice to Bob significantly underestimates the volume ratios $R_{S:A\to B}$ of table 2. Furthermore, the volume ratios of table 2 also agree with the intuition suggested by the concept of EPR steerability from Alice to Bob that increasing the dimensionality of Bob's local quantum system should increase his ability to detect EPR steerability by Alice.

In view of the differences between the results of tables 1 and 2 one may ask whether the underestimated volume ratios of table 1 may still be improved by using the approach of Das *et al* [31], while detecting entanglement of the quantum state $\tau_{A\to B}$ of (33) by local measurements. Recently it has been demonstrated that local informationally complete (N,M)-POVMs are as powerful in detecting entanglement of bipartite quantum states as LOOs [34]. This is a consequence of the peculiar scaling properties characterizing local informationally complete (N,M)-POVMs and of their relation to LOOs. It has already been shown by Gittsovich and Gühne [41] that a sufficient condition for bipartite entanglement detection by LOOs is given by a violation of the inequality

$$\|C(G^A, G^B|\rho)\|_1 \leqslant \sqrt{(1 - \mathrm{Tr}_A\{(\rho^A)^2\})(1 - \mathrm{Tr}_B\{(\rho^B)^2\})}. \tag{34}$$

with G^A and G^B denoting the LOOs of Alice and Bob and ρ^A and ρ^B denoting their reduced quantum states. Using LOOs of the form (10) for Alice and Bob and the relations $\tau^A_{A\to B} = \mu\rho^A + ((1 - \mu)/2)\mathbb{1}_2$ and $\tau^B_{A\to B} = \rho^B$ for the reduced quantum states of (33), inequality (34) assumes the form

$$\frac{1}{\mu}\|C(\tilde{G}^A, \tilde{G}^B|\tau_{A\to B})\|_1 = \|C(\tilde{G}^A, \tilde{G}^B|\rho)\|_1 \leqslant \sqrt{\left(\frac{1+\mu^2}{2\mu^2} - \mathrm{Tr}_A\{(\rho^A)^2\}\right)(1 - \mathrm{Tr}_B\{(\rho^B)^2\})}. \tag{35}$$

For $\mu = 1/\sqrt{3}$ inequality (35) reduces to inequality (23) with $d_A = 2$. Thus, for cases with $d_A = 2$ a violation of inequality (23) characterizes once again the sufficient condition for EPR steerability from Alice to Bob by entanglement detection via correlation matrices of local measurements involving LOOs or informationally complete (N,M)-POVMs. This demonstrates that the approach of Das *et al* [31] combined with local measurements involving LOOs or informationally complete (N,M)-POVMs as tests for bipartite entanglement cannot improve the results of table 1.

7. Conclusions

We have applied the correlation-matrix-based sufficient condition for bipartite EPR steerability from Alice to Bob of Lai and Luo [22] to local measurements based on (N,M)-POVMs performed on arbitrary dimensional bipartite quantum systems. It has been shown that within the class of local informationally complete (N,M)-POVMs this sufficient EPR steerability condition, which is based on a violation of inequality (17), exhibits a peculiar scaling property. This implies that a violation of one and the same inequality characterizes this sufficient condition for measurements involving LOOs and for all local informationally complete (N,M)-POVMs. Thus, local informationally complete (N,M)-POVMs are as powerful as LOOs for detecting bipartite EPR steerability from Alice to Bob based on a violation of inequality (23).

With the help of a hit-and-run Monte-Carlo algorithm we have determined lower bounds on the Euclidean volume ratios of EPR steerable bipartite quantum states from Alice to Bob and all bipartite quantum states for low dimensions of Alice's and Bob's quantum systems. These numerical results explore the statistical typicality of locally detectable bipartite EPR steerability from Alice to Bob based on a violation of inequality (31). They demonstrate that, except for the case of two qubits, the sufficient condition for bipartite EPR steerability from Alice to Bob resulting from a violation of inequality (31) tends to underestimate the Euclidean volume ratios between EPR steerable bipartite quantum states from Alice to Bob and all bipartite quantum states significantly. Our numerical investigations also demonstrate that the recently introduced approach of Das *et al* [31], which relates bipartite EPR steerability from Alice's qubit to Bob's arbitrary dimensional qudit to bipartite entanglement, can be more efficient provided methods for detecting bipartite entanglement are used which transcend local measurements. However, besides not being based on local measurements a further disadvantage of this latter approach is that its validity is restricted to cases in which Alice's quantum system is a qubit. Therefore, further research is required for the development of efficient measurement-based methods for the detection of EPR steerability and for the exploration of its intricate relation to entanglement.

Acknowledgments

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

This appendix outlines the derivation of the general scaling relation (24) of section 5 between the 1-norms of the correlation matrices of arbitrary LOOs and local informationally complete (N,M)-POVMs. The general relations between informationally complete (N,M)-POVMs and orthonormal hermitian operators bases presented in this appendix have been obtained recently in an investigation on the detection of typical bipartite entanglement with the help of local measurements [34].

We consider a d-dimensional Hilbert space $\mathcal{H}_d = (\mathrm{Span}(\mathcal{B}), \langle\cdot|\cdot\rangle)$ with orthonormal basis $\mathcal{B} = \{|1\rangle, \cdots, |d\rangle\}$ and an associated arbitrary basis of hermitian linear operators $G = (G_1, \cdots, G_{d^2})^T$ with $G_\mu = G_\mu^\dagger$, $\mu \in \{1, \cdots, d^2\}$ acting on this Hilbert space. Let us also assume that this operator basis is orthonormal with respect to the Hilbert-Schmidt (HS) scalar product, i.e. $\langle G_\mu | G_\nu\rangle_{HS} := \mathrm{Tr}_{AB}\{G_\mu G_\nu\} = \delta_{\mu,\nu}$ so that it spans the Hilbert space $\mathcal{H}_{d^2} = (\mathrm{Span}(G), \langle\cdot|\cdot\rangle_{HS})$.

An arbitrary (N,M)-POVM, say $\Pi = \{\Pi_1, \cdots, \Pi_{NM}\}$ with $\Pi(i) \geqslant 0$, $i(\alpha, a) := (\alpha - 1)M + a$, $\alpha \in \{1, \cdots, N\}$, $a \in \{1, \cdots, M\}$, $i \in \{1, \cdots, NM\}$, can be expanded in this orthonormal hermitian operator basis, i.e. $\Pi = G^T S$ with the $d^2 \times (NM)$ matrix S of real-valued coefficients. Using the orthonormality of the basis G, for $N \geqslant 2$ conditions (7) and (8) characterizing (N,M)-POVMs can be rewritten in the form

$$(S^T S)_{i(\alpha,a),j(\alpha',a')} = \Gamma\delta_{i(\alpha,a),j(\alpha',a')} - \frac{\Gamma}{M}\left(\bigoplus_{\alpha=1}^{N} J_\alpha\right)_{i(\alpha,a),j(\alpha',a')} + \frac{d}{M^2} J_{i(\alpha,a),j(\alpha',a')} \tag{A1}$$

with $\Gamma = (xM^2 - d)/(M(M-1))$ (cf (18)), the $(NM) \times (NM)$ matrix J of all ones, i.e. $J_{i(\alpha,a),j(\alpha',a')} = 1$, and with the $M \times M$ block matrices J_α of all ones, i.e. $(J_\alpha)_{i(\alpha,a),j(\alpha',a')} = \delta_{\alpha,\alpha'}$. The spectrum of the positive semidefinite symmetric $(NM) \times (NM)$ matrix (A1) is given by

$$\mathrm{Sp}(S^T S) = \left\{\Gamma^{(N(M-1))}, \frac{dN}{M}^{(1)}, 0^{(N-1)}\right\} \tag{A2}$$

with the exponents indicating the multiplicities of the eigenvalues. For $N = 1$ the zero eigenvalue no longer appears in the spectrum of S^{TS}. Therefore, according to (9) for informationally complete (N,M)-POVMs the dimension D of the eigenspace of the nonzero eigenvalues is given by

$$D = N(M-1) + 1 = d^2. \tag{A3}$$

The spectral representation of this symmetric matrix is given by

$$(S^T S)_{i,j} = \sum_{\mu=1}^{N(M-1)+1} X_{i,\mu}\Lambda_\mu X_{\mu,j}^T \tag{A4}$$

with

$$\Lambda_1 = \frac{d}{M^2}NM, \quad X_{i,1} = \frac{1}{\sqrt{NM}},$$
$$\Lambda_\nu = \Gamma, \quad \sum_{a=1}^{M} X_{i(\alpha,a),\nu} = 0 \tag{A5}$$

for $i \in \{1, \cdots, NM\}$, $\nu \in \{2, \cdots, N(M-1)+1\}$. The $(NM) \times (N(M-1)+1)$ matrix $X_{i,\mu}$ fulfills the orthogonality condition

$$\sum_{i=1}^{NM} (X^T)_{\mu,i} X_{i,\nu} = \delta_{\mu,\nu}. \tag{A6}$$

As a consequence of (9), for an informationally complete (N,M)-POVM the most general form of the $d^2 \times (NM)$ matrix S which is consistent with (7) and (8) is given by

$$S_{\mu,i} = \sum_{\mu'=1}^{d^2} O_{\mu,\mu'}^T \sqrt{\Lambda_{\mu'}} X_{\mu',i}^T \tag{A7}$$

with the arbitrary real-valued orthogonal $d^2 \times d^2$ matrix O, i.e. $OO^T = O^T O = P_{d^2}$. Thereby, P_d^2 denotes the projection operator onto the $(N(M-1)+1)$-dimensional eigenspace of nonzero eigenvalues of the linear operator S^{TS} acting in the Hilbert space $\mathcal{H}_{NM}$. Note that in the case of an informationally complete (N,M)-POVM this subspace is isomorphic to the Hilbert space $\mathcal{H}_{d^2}$. Let us finally also add the constraint (2) which yields

$$\mathbb{1}_d = \sum_{a=1}^{M} \Pi(i(\alpha, a)) = \sqrt{d}\sum_{\mu=1}^{d^2} G_\mu O_{\mu,1}^T \tag{A8}$$

where we have taken into account the constraints (A5) on the eigenvectors of S^{TS}. This relation together with the constraints (A5) also implies condition (6). Therefore, all requirements defining an informationally complete

(N,M)-POVM $\Pi = G^T S$, namely (2), (6), (7), (8) and (9), are fulfilled. It should be mentioned that so far the positive semidefiniteness of the POVM elements $\Pi(i(\alpha, a))$ has not been taken into account. Therefore, it can be concluded that, provided an informationally complete (N,M)-POVM exists, it is related to an orthonormal hermitian operator basis G by $\Pi = G^T S$ with the matrix elements of S being given by (A7).

With the help of (A7) it is straightforward to relate the correlation matrix of two local informationally complete (N,M)-POVMs for Alice and Bob, say Π^A and Π^B, to the correlation matrix of two LOOs, say G^A and G^B. Using relation (A7) for these local bases we find

$$\begin{aligned} C(\Pi^A, \Pi^B|\rho) &= (S^A)^T C(G^A, G^B|\rho) S^B \\ &= X^A \sqrt{\Lambda^A} O^A C(G^A, G^B|\rho)(O^B)^T \sqrt{\Lambda^B} (X^B)^T. \end{aligned} \tag{A9}$$

For the corresponding 1-norm we obtain the result

$$\|C(\Pi^A, \Pi^B|\rho)\|_1 = \|\sqrt{\Lambda^A} O^A C(G^A, G^B|\rho)(O^B)^T \sqrt{\Lambda^B}\|_1 \tag{A10}$$

with Λ^A and Λ^B denoting the diagonal matrix of nonzero eigenvalues of Alice and Bob. Using (A8), the degeneracy of all eigenvalues for $\mu \neq 1$, i.e. $\Lambda_{\mu\neq 1} = \Gamma$, and the invariance of the 1-norm under orthogonal transformations this expression simplifies to

$$\|C(\Pi^A, \Pi^B|\rho)\|_1 = \sqrt{\Gamma_A}\sqrt{\Gamma_B}\|C(G^A, G^B|\rho)\|_1. \tag{A11}$$

The above arguments demonstrate that this scaling relation is valid for arbitrary informationally complete local (N,M)-POVMs Π^A and Π^B and LOOs G^A and G^B.

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