# Network implementation of covariant two-qubit quantum operations

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(Dated: November 8, 2018)

A six-qubit quantum network consisting of conditional unitary gates is presented which is capable of implementing a large class of covariant two-qubit quantum operations. Optimal covariant NOT operations for one and two-qubit systems are special cases contained in this class. The design of this quantum network exploits basic algebraic properties which also shed new light onto these covariant quantum processes.

PACS numbers: 03.67.Mn,03.65.Ud

## I.  INTRODUCTION

It is well known that certain tasks of information processing cannot be performed perfectly on the quantum level despite the fact that they can be performed perfectly on a classical level [1, 2]. Typically, impossibilities of this kind on the quantum level hint on the existence of corresponding no-go theorems. They raise interesting questions concerning the optimality of these quantum processes with respect to particular quality measures. A prominent example in this respect is the copying of arbitrary quantum states which cannot be achieved perfectly [3]. The associated problem of determining quantum operations which can achieve this tasks in the best possible way has stimulated numerous theoretical and experimental investigations starting with the early work of Bužek and Hillery [4].

Another process of this kind is the quantum NOT transformation which is to change an arbitrary quantum state into an orthogonal one and which cannot be performed perfectly for arbitrary input states [5, 6]. Recently, the problem of optimizing quantum NOT processes has been addressed not only for arbitrary pure one-qubit input states [5] but also for pure two-qubit input states of a given degree of entanglement [7]. In this latter context the possible input states are restricted to the set of pure two-qubit states of a given degree of entanglement which does not constitute a linear vector space. Therefore, the previously mentioned impossibility arguments concerning quantum NOT operations acting on arbitrary input states do not apply. All optimal quantum operations could be determined which perform such a quantum NOT operation for all possible pure two-qubit input states of a given degree of entanglement with the same quality. It was demonstrated that these optimal two-qubit quantum NOT operations are members of a convex set of covariant (completely positive) two-qubit quantum operations. This convex set is generated by four elementary two-qubit quantum operations which form the vertices of a three-dimensional polytope. Furthermore, it could be shown that only in the case of maximally entangled pure two-qubit input states it is possible to perform such a covariant quantum NOT operation perfectly. However, so far it is still unknown how this convex set of covariant two-qubit quantum operations can be implemented in quantum networks with the help of simple elementary quantum gates.

In general, a systematic approach to the problem of designing elementary quantum gate sequences which implement a given family of covariant quantum operations is not known. In the following it is shown that for the above mentioned convex set of covariant two-qubit quantum operations this problem can be solved completely. This is due to the fact that this convex set of quantum operations has special algebraic properties which can be exploited in a convenient way. In addition, these algebraic properties shed new light on the properties of these covariant two-qubit quantum operations. With the help of additional auxiliary qubits it is possible to design a quantum network which involves a particular sequence of conditional unitary qubit gates. Depending on the preparation of the auxiliary qubits any covariant quantum operation within this convex set can be implemented by this quantum network. One of the advantages of this particular network implementation is that the sequence of conditional unitary qubit gates involved is independent of the covariant quantum operation under consideration.

This paper is organized as follows. In Sec. II basic definitions and properties of the recently introduced convex set of covariant two-qubit quantum processes [7] are summarized. The essential algebraic properties of these quantum operations which are useful for the subsequent construction of the quantum network are discussed in a subsection. Sec. III addresses the main problem how this convex set of quantum operations can be implemented unitarily by a suitable choice of auxiliary quantum systems and by an appropriate sequence of elementary quantum gates. As a main result it is shown that any covariant quantum operation of the convex set discussed in Sec. II can be implemented by a unitary master transformation which is independent of the particular quantum operation under consideration. A quantum network implementation of this main result involving controlled unitary Pauli operations is discussed in a subsection.

## II. COVARIANT TWO-QUBIT QUANTUM OPERATIONS

In this section basic aspects of all completely positive quantum process are summarized that transform pure two-qubit input states of a given degree of entanglement in a covariant way. The recently discussed optimal quantum NOT operations [7] are special cases thereof.

### A. Basic definitions and general properties

Let us consider a general completely positive quantum operation $\Pi$ which transforms an arbitrary two-qubit input state $\rho$ in a covariant way according to

$$\Pi\left(U_1 \otimes U_2 \rho U_1^\dagger \otimes U_2^\dagger\right) = U_1 \otimes U_2 \Pi(\rho) U_1^\dagger \otimes U_2^\dagger. \quad (1)$$

Thereby, the requirement of complete positivity ensures that this transformation can be implemented in a unitary way possibly with the help of additional auxiliary quantum systems which are uncorrelated with the two-qubit system initially. If the covariance condition (1) is satisfied for arbitrary unitary one-qubit transformations $U_1, U_2 \in SU(2)$ [8], it is guaranteed that the quality of performance of a quantum NOT operation is the same for all possible pure entangled two-qubit input states of a given degree of entanglement [7, 9, 10].

Recently, it was shown [7] that all possible completely positive covariant two-qubit quantum operations $\Pi(v, x, y)$ fulfilling Eq.(1) form a three-parametric set, i.e.

$$\rho_{out} = \Pi(v, x, y)(\rho) = \sum_{i,j=0}^{3} K_{ij}(v, x, y) \rho K_{ij}^\dagger(v, x, y), \quad (2)$$

with the Kraus operators

$$K_{00}(v, x, y) = \frac{1}{4}(1 + 3x + 3v + 9y)^{\frac{1}{2}} I \otimes I,$$
$$K_{i0}(v, x, y) = \frac{1}{4}(1 + 3x - v - 3y)^{\frac{1}{2}} \sigma_i \otimes I,$$
$$K_{0i}(v, x, y) = \frac{1}{4}(1 - x + 3v - 3y)^{\frac{1}{2}} I \otimes \sigma_i, \quad (3)$$
$$K_{ij}(v, x, y) = \frac{1}{4}(1 - x - v + y)^{\frac{1}{2}} \sigma_i \otimes \sigma_j, \quad i, j \in \{1, 2, 3\},$$

the unit operator $I$ and the Pauli spin operators $\sigma_1 = X$, $\sigma_2 = Y$, and $\sigma_3 = Z$. The possible values of the three parameters $x, v$ and $y$ are restricted by the requirement of non negativity of the prefactors entering (3), i.e.

$$1 + 3x + 3v + 9y \geq 0, \qquad 1 + 3x - v - 3y \geq 0,$$
$$1 - x + 3v - 3y \geq 0, \qquad 1 - x - v + y \geq 0. \quad (4)$$

In addition, trace preservation of the quantum operation $\Pi(v, x, y)$ implies

$$\sum_{i,j=0}^{3} K_{ij}^\dagger(v, x, y) K_{ij}(v, x, y) = I. \quad (5)$$

An optimal quantum NOT operation transforms an arbitrary pure two-qubit input state with a given degree of entanglement into a not necessarily pure output state of its orthogonal complement in an optimal way. Thereby, the sets $\Omega_\alpha$ of pure two-qubit states with a given degree of entanglement $\alpha \in [0, 1/\sqrt{2}]$ are defined by [11, 12]

$$\Omega_\alpha = \Big\{ \ (U_1 \otimes U_2)\big(\alpha|0\rangle \otimes |0\rangle +$$
$$\sqrt{1 - \alpha^2}|1\rangle \otimes |1\rangle\big)\Big|U_1, U_2 \in SU(2)\Big\}. \quad (6)$$

In the special case $\alpha = 0$ the two-qubit states are separable whereas in the opposite extreme case $\alpha = 1/\sqrt{2}$ they are maximally entangled.

Let us now summarize some basic properties of such optimal quantum NOT operations [7]:

- There is a characteristic threshold value of entanglement at $\alpha_0 = \sqrt{\left(1 - \sqrt{1 - 4K}\right)/2} \approx 0.1836$ with $K = \left(8 - 3\sqrt{6}\right)/20$. For $\alpha \leq \alpha_0$ the optimal quantum NOT operation, i.e. $U_{SEP}$, is independent of the degree of entanglement $\alpha$ and is characterized by the characteristic parameters $(v = -1/3, x = -1/3, y = 1/9)$ (compare with (3)). This particular quantum operation is identical to two optimal covariant one-qubit NOT operations $u^1$ [5] applied to each of the input qubits separately, i.e. $U_{SEP} = u^1 \otimes u^1$ with

$$u^1(\rho) = \frac{1}{3}(2I - \rho). \quad (7)$$

These one-qubit NOT operations $u^1$ transform an arbitrary pure one-qubit input state into an orthogonal state in an optimal way [5].

- For $\alpha > \alpha_0$ the optimal NOT operations depend on the degree of entanglement $\alpha$ and are characterized by the parameters (compare with (3))

$$y = -\frac{1}{3} \frac{2 - 31\alpha^2\beta^2 + 20\alpha^4\beta^4}{-2 - 35\alpha^2\beta^2 + 100\alpha^4\beta^4},$$
$$x + v = \frac{2}{3} \frac{4 - 29\alpha^2\beta^2 - 20\alpha^4\beta^4}{-2 - 35\alpha^2\beta^2 + 100\alpha^4\beta^4},$$
$$x, v \geq -\frac{1}{3} \quad (8)$$

with $\beta = \sqrt{1 - \alpha^2}$.

- It can be shown that perfect NOT operations can be constructed for maximally entangled input states only. These perfect covariant NOT operations form a one-parameter family specified by characteristic parameters fulfilling the conditions $y = -\frac{1}{3}, x + v = \frac{2}{3}$ with $x, v \geq -\frac{1}{3}$.

- All completely positive covariant two-qubit processes (1) form a three-dimensional convex set [7].

Any of these processes $\Pi(\mathbf{a})$ can be represented in the form

$$\Pi(\mathbf{a}) = a_{00}I + a_{11}U_{SEP} + a_{01}U_{ME}^{(1)} + a_{10}U_{ME}^{(2)} \quad (9)$$

with $a_{mn} \geq 0$ and $\sum_{m,n \in \{0,1\}} a_{mn} = 1$. The quantum operations

$$
\begin{aligned}
U_{ME}^{(1)} &= \Pi(v=1, x=-1/3, y=-1/3), \\
U_{ME}^{(2)} &= \Pi(v=-1/3, x=1, y=-1/3) \quad (10)
\end{aligned}
$$

are members of the one-parameter family of perfect NOT operations for maximally entangled input states. They are characterized by the additional property that they leave the reduced density operators of the first $(U_{ME}^{(1)})$ or second $(U_{ME}^{(2)})$ qubit unchanged. The convex set of quantum processes (9) forms a three dimensional polytope whose vertices are given by the quantum operations $I$, $U_{SEP}$, $U_{ME}^{(1)}$, and $U_{ME}^{(2)}$. This polytope contains also other interesting quantum operations, such as the universal two-qubit NOT process $\mathcal{G}_{NOT}$ studied in Ref.[6]. This latter process is the optimal NOT operation for all possible pure two-qubit input states irrespective of their degree of entanglement. Its convex decomposition is given by

$$\mathcal{G}_{NOT} = 0.6\, U_{SEP} + 0.2\, U_{ME}^{(1)} + 0.2\, U_{ME}^{(2)}. \quad (11)$$

## B. Algebraic properties

Let us now explore further algebraic properties of the covariant two-qubit processes of Eqs.(1) and (9).

The vertices $U_{SEP}$, $U_{ME}^{(1)}$, $U_{ME}^{(2)}$ of the polytope (9) are orthogonal and the operators representing these processes are traceless, i.e.

$$
\begin{aligned}
Tr\left(U_{SEP}\right) &= 0, \ \ Tr\left(U_{ME}^{(1)}\right) = 0, \ \ Tr\left(U_{ME}^{(2)}\right) = 0, \\
Tr\left(U_{ME}^{(1)}U_{SEP}\right) &= 0, \ \ Tr\left(U_{ME}^{(2)}U_{SEP}\right) = 0, \\
Tr\left(U_{ME}^{(1)}U_{ME}^{(2)}\right) &= 0. \quad (12)
\end{aligned}
$$

Therefore, according to Eq.(9) the coefficients $\mathbf{a} \equiv (a_{00}, a_{01}, a_{10}, a_{11})$ of an arbitrary covariant two-qubit quantum operation $\Pi(\mathbf{a})$ are given by

$$a_{00} = \frac{1}{4}Tr\left(\Pi(\mathbf{a})\right), \quad a_{11} = \frac{Tr\left(\Pi(\mathbf{a})U_{SEP}\right)}{Tr\left(U_{SEP}^2\right)},$$

$$a_{01} = \frac{Tr\left(\Pi(\mathbf{a})U_{ME}^{(1)}\right)}{Tr\left(U_{ME}^{(1)\,2}\right)}, \quad a_{10} = \frac{Tr\left(\Pi(\mathbf{a})U_{ME}^{(2)}\right)}{Tr\left(U_{ME}^{(2)\,2}\right)}. \quad (13)$$

Another interesting feature of the covariant two-qubit quantum operations (9) concerns repeated applications. If two such quantum operations are applied successively

| covariant quantum operation | $a_{00}$ | $a_{11}$ | $a_{01}$ | $a_{10}$ |
|---|---|---|---|---|
| $U_{SEP}^2$ | 1/9 | 4/9 | 2/9 | 2/9 |
| $U_{ME}^{(1)\,2}$ | 1/3 | 0 | 2/3 | 0 |
| $U_{ME}^{(2)\,2}$ | 1/3 | 0 | 0 | 2/3 |
| $U_{SEP}U_{ME}^{(1)} = U_{ME}^{(1)}U_{SEP}$ | 0 | 2/3 | 0 | 1/3 |
| $U_{SEP}U_{ME}^{(2)} = U_{ME}^{(2)}U_{SEP}$ | 0 | 2/3 | 1/3 | 0 |
| $U_{ME}^{(1)}U_{ME}^{(2)} = U_{ME}^{(2)}U_{ME}^{(1)}$ | 0 | 1 | 0 | 0 |

TABLE I: Convex decompositions of products of elementary covariant quantum operations which constitute the vertices of the polytope (9).

the resulting quantum operation is again of the form (9). Thus, these quantum operations form a half group. The coefficients of the convex decompositions of some products of the elementary quantum operations $U_{SEP}$, $U_{ME}^{(1)}$, and $U_{ME}^{(2)}$ are summarized in Table I. According to this table we have the relation

$$U_{ME}^{(1)}U_{ME}^{(2)} = U_{SEP}. \quad (14)$$

Furthermore, it is apparent that the quantum operations $U_{SEP}$, $U_{ME}^{(1)}$, and $U_{ME}^{(2)}$ commute. Finally let us point out that the considered covariant processes (9) have nontrivial limit expressions for $\Pi(a)^n$ for $n \to \infty$.

## III. QUANTUM NETWORK IMPLEMENTATION

In this section it is shown how an arbitrary covariant two-qubit quantum operation (9) can be implemented in a six-qubit quantum network by an appropriate sequence of controlled unitary gates. For this purpose it is demonstrated first that any covariant two-qubit process (9) can be implemented with the help of four auxiliary qubits by a master unitary operation. This master unitary operation is independent of the particular covariant two-qubit quantum operation under consideration. A particular covariant two-qubit process is selected by preparing the auxiliary four-qubit quantum system in a suitably chosen quantum state. In a second step a sequence of conditional (unitary) Pauli gates is constructed which implements this unitary master transformation in this six-qubit quantum network.

### A. Unitary representation with auxiliary qubits

For the purpose of implementing the covariant quantum operations (9) unitarily with the help of auxiliary qubits let us first of all introduce some useful notation. In addition to the four dimensional Hilbert space $\mathcal{H}$ of the two-qubit input states we introduce four auxiliary qubits whose Hilbert space $\mathcal{H}_{ancilla}$ is sixteen dimensional. The quantum states $|ijkl\rangle$ with $i,j,k,l \in 0,1$ are assumed to

form an orthonormal basis in this latter Hilbert space. We start from the observation that apart from normalization factors the Kraus operators of (3) are unitary. Therefore, it is convenient to introduce the corresponding sixteen renormalized unitary two-qubit operators

$$F_{2i+j\ 2k+l} = \sigma_{2i+j} \otimes \sigma_{2k+l} \quad (15)$$

with $\sigma_0 = I$ and $i, j, k, l \in \{0, 1\}$. From these latter unitary two-qubit operators we can construct the unitary master transformation

$$\mathcal{U} = \sum_{i,j,k,l \in \{0,1\}} F_{2i+j\ 2k+l} \otimes |ijkl\rangle\langle ijkl| \quad (16)$$

which operates on all six-qubits of the Hilbert space $\mathcal{H} \otimes \mathcal{H}_{ancilla}$. Let us assume that initially the four auxiliary qubits are prepared in the mixed quantum state

$$\Sigma(\mathbf{a}) = \sum_{i,j,k,l \in \{0,1\}} \frac{a_{\text{sgn}(i+j)\ \text{sgn}(k+l)}}{3^{\text{sgn}(i+j)+\text{sgn}(k+l)}} |ijkl\rangle\langle ijkl| (17)$$

with the normalization $a_{00} + a_{01} + a_{10} + a_{11} = 1$ and with $\text{sgn}(x) = x/|x|$ denoting the signum-function ($\text{sgn}(0) = 0$). Depending on the values of the coefficients $\mathbf{a} \equiv (a_{00}, a_{01}, a_{10}, a_{11})$ any covariant quantum process $\Pi(\mathbf{a})$ can be implemented unitarily with the help of the unitary master transformation (16) by preparing the auxiliary four qubits in the quantum state (17) initially and by disregarding these four auxiliary qubits after the unitary transformation, i.e.

$$\left(a_{00}I + a_{01}U_{ME}^{(1)} + a_{10}1U_{ME}^{(2)} + a_{11}U_{SEP}\right)(\rho) \equiv$$
$$\Pi(\mathbf{a})(\rho) = \text{Tr}_{ancilla}\left\{\mathcal{U}\rho \otimes \Sigma(\mathbf{a})\mathcal{U}^\dagger\right\}. \quad (18)$$

This unitary implementation of the covariant quantum operations (9) is a main result of our paper. It can be proved in a straightforward way by inserting Eqs.(16) and (17) into Eq.(18).

Before addressing the general problem of implementing an arbitrary quantum operation of the form of Eq.(18) by elementary quantum gates in this six-qubit quantum network let us consider the unitary implementation of the covariant quantum operation $\Pi(a_{00} = 0 = a_{10} = a_{11}, a_{01} = 1) = U_{ME}^{(1)}$ as an example. For this purpose the auxiliary four-qubit quantum system has to be prepared in the mixed quantum state $\Sigma(a_{00} = 0 = a_{10} = a_{11}, a_{01} = 1) = (1/3)\{|0001\rangle\langle0001| + |0010\rangle\langle0010| + |0011\rangle\langle0011|\}$. Thus, Eq. (18) yields

$$\Pi(a_{00} = 0 = a_{10} = a_{11}, a_{01} = 1) = \frac{F_{01}}{\sqrt{3}}\rho\frac{F_{01}^\dagger}{\sqrt{3}} +$$
$$\frac{F_{02}}{\sqrt{3}}\rho\frac{F_{02}^\dagger}{\sqrt{3}} + \frac{F_{03}}{\sqrt{3}}\rho\frac{F_{03}^\dagger}{\sqrt{3}} = U_{ME}^{(1)}. \quad (19)$$

## B. Network implementation with conditional Pauli gates

Let us now implement the unitary master transformation (16) by a quantum circuit in the six-qubit quantum network which involves four auxiliary qubits. According to Eq.(16) the quantum circuits have to be designed in such a way that, whenever the four auxiliary qubits are prepared in a particular quantum state of the computational basis $|ijkl\rangle$ $(i, j, k, l \in \{0, 1\})$, the unitary transformation $F_{2i+j\ 2k+l}$ is acting onto the two target qubits of the main system with Hilbert space $\mathcal{H}$. In order to achieve this goal let us introduce elementary conditional unitary five-qubit quantum gates $C(U)$ which involve four control qubits and one target qubit and whose action on an arbitrary quantum state $|\psi\rangle$ of the target qubit and a quantum state of the computational basis of the four control qubits $|ijkl\rangle$ is given by

$$C(U)|\psi\rangle_{target}\otimes|ijkl\rangle_{control} = U^{i\cdot j\cdot k\cdot l}|\psi\rangle_{target}\otimes|ijkl\rangle_{control} \quad (20)$$

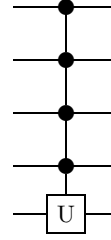(compare with Fig. (1)). In other words, the unitary



FIG. 1: Quantum circuit representation of the elementary controlled unitary operation $C(U)$ which involves four control and one target qubit. Thereby, $U$ denotes a unitary operation acting on the single target qubit which is performed if and only if the control qubits are in state $|1111\rangle_{control}$.

operation $U$ acts on the target state $|\psi\rangle_{target}$ if and only if the four control qubits are prepared in the state $|1111\rangle_{control}$. Universal quantum gates which are capable of implementing such controlled unitary operations were studied extensively in Ref. [13], for example.

With the help of the controlled unitary operations $C(U)$ also other controlled operations can be realized in a straightforward way. Suppose one wants to implement a five-qubit quantum gate in which the target qubit is transformed by a unitary transformation $U$ if and only if the first, second, and third (control) qubits are in state $|0\rangle$ and the fourth control qubit is in state $|1\rangle$ of the computational basis. As apparent from Fig. 2 this quantum gate may be realized by acting with a Pauli spin operator $X$ onto the control qubits one, two, and three before and after the application of the controlled unitary quantum gate $C(U)$.

Also multi-target conditional unitary quantum gates can be realized with the help of the elementary quantum gate $C(U)$. Such multi-target gates are natural gen-
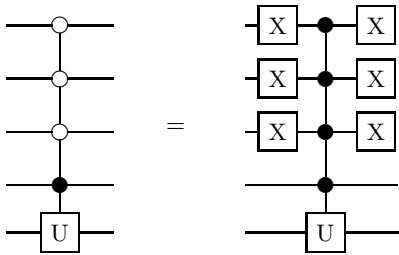
FIG. 2: Controlled unitary operation with a unitary operation $U$ acting on the target qubit if and only if the first, second, and third control qubits are in state $|0\rangle$ and the fourth control qubit is in state $|1\rangle$ of the computational basis.

eralizations of the one-qubit controlled quantum gates just introduced. In a general $d$-target conditional unitary quantum gate a set of unitary operations, say $\{U_i\}_{i=1}^d$, are performed on $d$ target qubits simultaneously if and only if the control qubits are prepared in prescribed quantum states. In Fig. 3 a two-target conditional quantum gate is depicted in which the unitary operations $U$ and $V$ are performed on the first and the second target qubit if and only if the first and the second control qubits are prepared in state $|0\rangle$ and the third and fourth control qubits are prepared in state $|1\rangle$ of the computational basis.
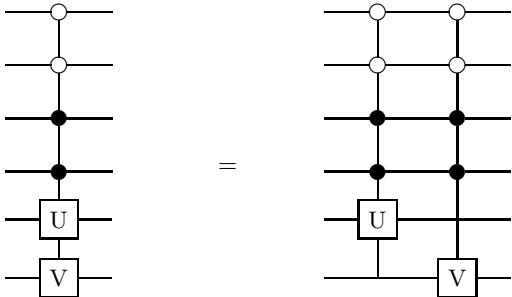


FIG. 3: Circuit implementation of a two-target quantum gate which performs an operation $U$ on the fifth qubit and an operation $V$ on the sixth qubit conditional on the first two qubits being in state $|0\rangle$ and qubits three and four being in state $|1\rangle$ of the computational basis.

With the help of such two-target conditional quantum gates a simple sequence of conditional two-target Pauli gates can be designed in our six-qubit quantum system which performs the master unitary transformation (16). The circuit scheme of this network is depicted in Fig. 4. The first four qubits constitute the control qubits of the auxiliary quantum system. According to Eq.(18) these auxiliary qubits have to be prepared in the quantum state (17) initially. The two input qubits of the main quantum system are prepared in an arbitrary quantum state $\rho$. The dynamics of the composite six-qubit quantum system are governed by the master unitary transformation (16) which is implemented by the network displayed in figure 4. The action of this dynamics on the two qubits of the main quantum system after having discarded the four auxiliary qubits is given by the quantum operation (18).

## IV. CONCLUSION

A six-qubit quantum network implementation of all possible two-qubit quantum operations was presented which transform all pure two-qubit input states of a given degree of entanglement in a covariant way. An advantage of this particular implementation is that it is based on a sequence of conditional Pauli gates which does not depend on the quantum operation under consideration. A particular covariant quantum operation is selected by preparing the four auxiliary qubits in an appropriate quantum state. The implementation presented rests on special algebraic properties of these covariant two-qubit

quantum operations. Analogous approaches exploiting similar algebraic properties may also turn out to be useful for network implementations of other covariant quantum processes.
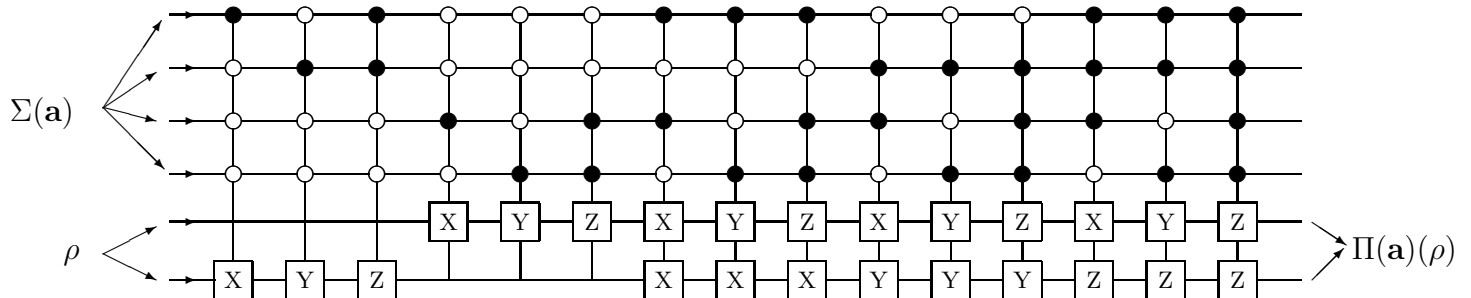
FIG. 4: Network consisting of 15 conditional unitary one- and two-target Pauli gates which performs the unitary master transformation (16) on a six-qubit quantum system. Initially the four auxiliary qubits are prepared in state $\Sigma(\mathbf{a})$ of (17) and the two qubits of the main quantum system are prepared in an arbitrary quantum state $\rho$. After the application of these quantum gates the two qubits of the main quantum system are in the quantum state $\Pi(\mathbf{a})(\rho)$ of (18).

[1] M. A. Nielsen, I. l. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] A. K. Pati, Phys. Rev. A **66**, 062319 (2002).

[3] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).

[4] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).

[5] V. Bužek, M. Hillery, and R. F. Werner, Phys. Rev. A **60**, R2626 (1999).

[6] P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).

[7] J. Novotný, G. Alber, I. Jex, Phys. Rev A 73, 062311 (2006).

[8] L. C. Biedenharn and J. D. Louck, Angular momentum in Quantum Physics (Addison-Wesley, Reading, Massachusetts, 1981)

[9] R. F. Werner, Phys. Rev. A **58**, 1827 (1998); M. Keyl and R. F. Werner, J. Math. Phys. **40**, 3283-3299.

[10] L.-P. Lamoureux, P. Navez, J. Fiurášek, and N. J. Cerf, Phys. Rev. A **69**, 040301-1 (2004).

[11] J. Novotný, G. Alber, and I. Jex, Phys. Rev. A **71**, 042332 (2005).

[12] R. Demkowicz-Dobrzanski, M. Lewenstein, Aditi Sen, Ujiwal Sen, and D. Bruß, Phys. Rev. A **73**, 032313 (2006).

[13] A. Barenco, Ch. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin, H. Weinfurter, Phys. Rev. A **52**, 3457-3467 (1995).