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Complex Chaos in Conditional Qubit Dynamics and Purification Protocols

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Abstract. Selection of an ensemble of equally prepared quantum systems, based on measurements on it, is a basic step in quantum state purification. For an ensemble of single qubits, iterative application of selective dynamics has been shown to lead to complex chaos, which is a novel form of quantum chaos with true sensitivity to the initial conditions. The Julia set of initial values with no convergence shows a complicated structure on the complex plane. The shape of the Julia set varies with the parameter of the dynamics. We present here results for the two-qubit case demonstrating how a purification process can be destroyed with chaotic oscillations.

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1. Introduction

Quantum state purification is a process where entanglement is enhanced in an ensemble of quantum states, by making measurements on part of the ensemble and using the gained information to select part of the ensemble [1]. The complete evolution of the remaining subensemble is not unitary anymore, since the measurement process extracts information from the system on one hand, and this information is fed back by the conditional selection itself. In this way, nonlinear transformations can be realized [2].

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Chaos in dynamical systems has a long history in physics [3]. Exponential sensitivity to initial conditions is the usual definition for chaos in classical physics. In quantum mechanics, however, unitary evolution of a closed system prevents initially close states to separate exponentially fast, nevertheless quantum chaos is an important field dealing with quantized counterparts of classically chaotic systems [4].

Occurrence of true chaos should be possible in a model describing quantum-classical correspondence, for example via continuous measurements, as confirmed by recent numerical results [5]. Moreover, it is possible to find a regime where the continuously measured quantum system is far from the classical limit and still it exhibits true chaos with a positive Lyapunov exponent, which was calculated by numerical simulations [6]. Continuous measurements are used to model the environmental decoherence and the measurement result are not fed back to the system, resulting in stochastic dynamics.

Recently, we have proven [7] that true chaos occurs in the conditional dynamics of qubits, in a similar arrangement proposed for quantum state purification [2]. We could calculate the positive Lyapunov exponent analytically for a simple case. The more general dynamics of a single qubit is described by a complex, nonlinear map, in this way directly realizing complex chaos [7]. This situation is different compared to classical chaotic dynamics, in the sense that classical chaotic maps are acting on the phase space with real numbers. The complex nonlinear map, behind complex chaos, is acting on the complex space representing the Hilbert space of the system. We are not aware of any other physical system directly realizing complex chaotic maps.

Conditional dynamics by measurement selection as applied in purification can be considered as feeding back results of strong (von Neumann) measurements with the selection as opposed to the idea of feeding back the results of weak measurements [10]. The latter scheme has also been suggested as a candidate for a truly chaotic quantum system, although no proof was provided.

In the present paper we focus on implications of complex chaos for the purification procedure. The structure of the paper is the following. First, we shortly review how complex chaos occurs in iterated deterministic quantum maps. For a one-qubit system we illustrate the behavior of the system by showing the non-trivial Julia sets of non-converging initial states. In the next section, we consider the two-qubit system as an example of quantum state purification and finally we conclude.

2. Iterated Deterministic Quantum Maps

The quantum XOR gate is defined as the following operation

$$XOR_{12}|i\rangle_1|j\rangle_2 = |i\rangle_1|i-j\rangle_2 \quad \text{mod } (D)\,, \tag{1}$$

where D is the dimension of the Hilbert space for both subsystems. The XOR gate is the key operation for deterministic quantum state purification protocols [2] together with a measurement. For a nonlinear transformation one needs two identical copies of the system. The application of the XOR gate followed by the measurement of the second system in the $|0\rangle$ state leaves the first system in a state defined by the following nonlinear transformation

$$\rho' = \mathcal{S}\rho\,, \qquad \rho_{ij} \xrightarrow{\mathcal{S}} N\rho_{ij}^2 \tag{2}$$

with the renormalization factor $N = 1/\sum \rho_{ii}^2$. The matrix element squaring is defined with respect to a prescribed orthonormal basis $\{|i\rangle\}$. By repeating the transformation $\mathcal{S}$ we expect that smaller diagonal matrix elements will tend to zero and the largest one survives, converging to unity. In other words, pure states can be stable fixed points of the map, leading to purification of the state.

Let us consider now the $D = 2$ case, when each system is a qubit. The most general iterative deterministic dynamics based on matrix element squaring is achieved if we allow for an arbitrary unitary transformation in each step

$$\mathcal{R}\rho = U\rho U^\dagger\,, \tag{3}$$

with

$$U = \begin{pmatrix} \cos x & \sin x\ e^{i\phi} \\ -\sin x\ e^{-i\phi} & \cos x \end{pmatrix}, \tag{4}$$

in the prescribed basis. In this way one step of the dynamics reads

$$\rho' = \mathcal{F}\rho = \mathcal{R}\mathcal{S}\rho\,, \tag{5}$$

and repeating the transformation $\mathcal{F}$ leads to the discrete conditional time-evolution we are interested in.

The description of the dynamics simplifies considerably if we take an initial pure state of the qubit. Let us introduce the following notation

$$|\psi\rangle = N(z|0\rangle + |1\rangle)\,, \tag{6}$$

where the state is normalized by $N = (1 + |z|^2)^{-1/2}$. Thus we haved represented the qubit Hilbert space on the complex plane, where the parameter z describes the state of the qubit. The transformation $\mathcal{F}$ maps the pure state onto a pure state and transforms z as

$$z \mapsto F_p(z) = \frac{z^2 + p}{1 - p^* z^2}\,, \tag{7}$$

where $p = \tan x e^{i\phi}$ and the star denotes complex conjugation. The conditional dynamics of the qubit are thus governed by $F_p(z)$, which is a nonlinear $\mathbb{C} \to \mathbb{C}$ map with one complex parameter p. A considerable difference compared to chaotic systems in classical physics is that the underlying space is complex, here. Even the simplest nonlinear maps of the complex plane can show intricate dynamical structure, such as the famous Mandelbrot set. The study of the mathematics related to maps in one complex variable has a long history and an extensive literature (for a review, see [9]).

The traditional approach to a nonlinear map in one complex variable is to divide the complex plane of the initial values z_0 into regular and irregular points forming the Fatou and Julia sets, respectively. Regular starting points from the Fatou set will converge to a stable cycle (also elements of the Fatou set) when repeating the iteration. Initial values included in the Julia set are considered to be chaotic, leading to irregular oscillations or forming unstable cycles. Taking into account both the initial condition z_0 and the complex parameter p a four dimensional parameter space is defined. In a simplified situation, when the parameter p is set to zero, we could define and calculate the positive Lyapunov exponent [7].

The full dynamics induced by Eq. (7) take place on the Riemann sphere $\hat{\mathbb{C}}$ consisting of $\mathbb{C}$ together with the point at infinity. The physical meaning of the points 0 and ∞ for z are the two basis states of the qubit, $|1\rangle$ and $|0\rangle$, respectively. The map $F_p(z)$ is a rational function of degree two. The Julia sets corresponding to various p values can possess highly non-trivial structures. Physically speaking, the parameter p describes the rotation of the qubit state $|\psi\rangle$. Setting the parameter to $p = 1$ corresponds to a rotation of $\pi/4$ that transforms, for example, the basis states into their equal superpositions. This is a symmetric situation with respect to the basis states. The orbit of one critical point, $z_{c2} = \infty$, is part of the attractive cycle $\{-1, \infty\}$. The other critical point $z_{c1} = 0$ follows the orbit $0 \mapsto 1 \mapsto \infty$ and thus lands on the same periodic cycle. Therefore the only stable cycle for this map is the fixed point $\{-1, \infty\}$. This also proves that the map is hyperbolic [9]. In general, numerical calculation of the Julia set for quadratic rational maps is difficult. There are no generic algorithms to compute it. Here we can simply apply the criterion of convergence to the stable cycle. In Fig. 1 we show the Julia set for $p = 1$ as well as for $p = 1 + 0.1I$, $p = 1 + 0.5I$ and $p = 0.5$. The first 3 cases are similar to each other, there is one stable periodic cycle with length 2. The sets look like gradual distortion of the initial set. In the last case we have also only one stable cycle, but its length is 1. The structure of the set is qualitatively different from the previous cases. These figures illustrate well the sensitivity for initial conditions, a small change of the initial state can change the convergence properties of the dynamics.

3. Purification Protocol

Purification of quantum states usually aims at increasing entanglement within a system. A system consisting of two qubits was considered with conditional dynamics for purification [2]. One would expect that sensitivity found in the one-qubit case will also occur here and affect purification. Let us consider the following iteration acting on two-qubit states

$$\rho' = \mathcal{F}\rho = \mathcal{R}_1\mathcal{R}_2\mathcal{S}\rho\,. \tag{8}$$

Here $\mathcal{S}$ is the element squaring defined in Eq. (2) with the index i running from 1 to 4 through the elements of the product basis of the qubits $\{|j\rangle|k\rangle\}$, $(j, k = 0, 1)$ and the rotation $\mathcal{R}_m$ acts on the mth qubit, with parameters $x_m\,, \phi_m$ as defined in

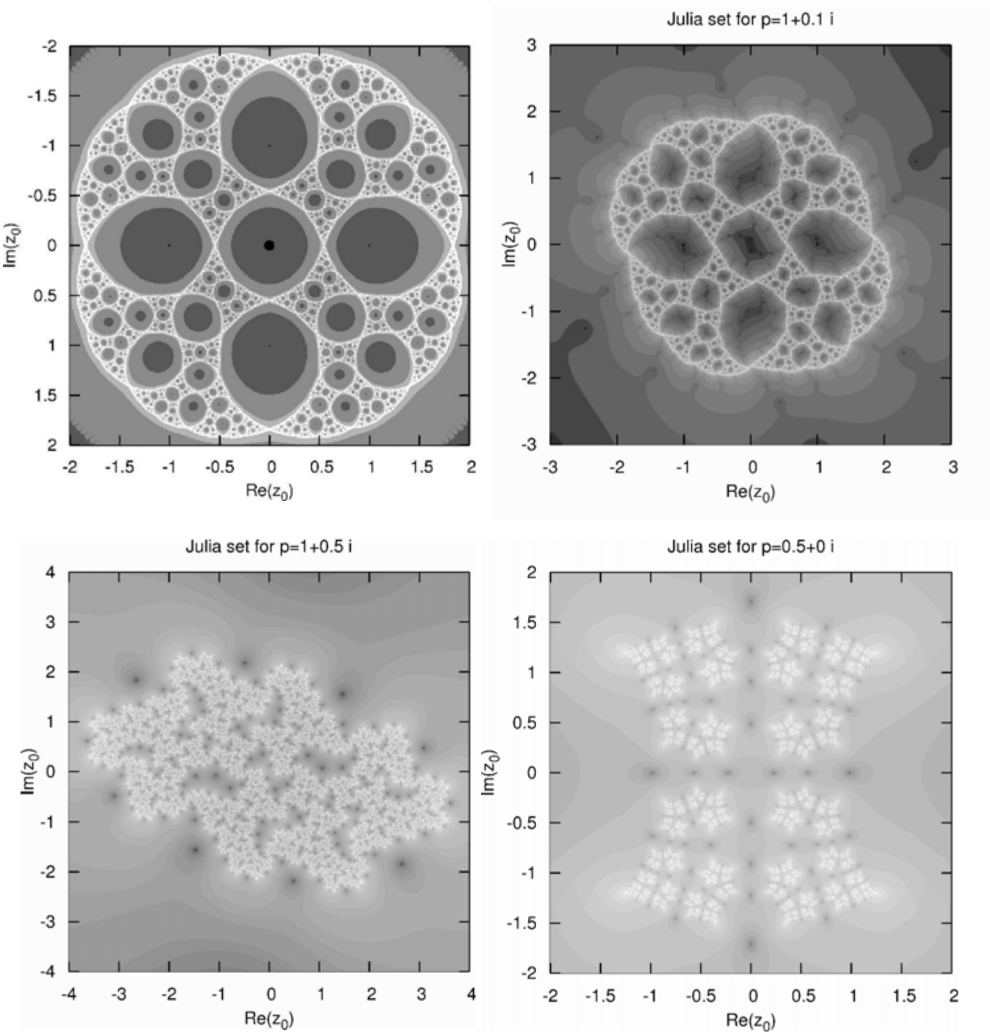

Fig. 1. Julia sets for the nonlinear map (7). The parameter is set to $p = 1$ on the upper left, $p = 1 + 0.1I$ on the upper right, $p = 1 + 0.5I$ on the lower left and $p = 0.5$ for the lower right picture. Grayscale indicates how fast the map converges to the stable cycle $\{-1, \infty\}$ (dark – fast, grey – slow convergence, white – no convergence)

Eqs. (3), (4). Now, the parameters of the two (local) complex rotations span $\mathbb{C}^2$, and the initial state can be any valid two-qubit density operator. Obviously, this is an even much larger parameter space to explore, which includes the one-qubit pure states as a special case.

In order to demonstrate how a purification process may be affected by the chaotic nature of this nonlinear map, we choose a target state $|\psi_{\text{target}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This state is part of a second order fixed point of the map when all four rotation angles equal to $\pi/4$. To characterize purification we apply the fidelity defined by the overlap with the target state $F = Tr(\rho\rho_{\text{target}})$. Let us start from a mixed initial state slightly different from $|\psi_{\text{target}}\rangle$, described by the density matrix

$$\rho_0 = \begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.45 & 0.445 & 0 \\ 0 & 0.445 & 0.45 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

During the evolution, the fidelity grows in every second step, tending to 1, while the other member of the limit cycle is an orthogonal state. This can be considered as a purification process. If we now slightly change the angles of both qubit rotations $x_1 = x_2$, the oscillations become metastable, and after a finite number of iterations irregular behavior emerges as shown in Fig. 2. Note that initially the fidelity grows and, as expected, numerical simulations show that the metastable region increases as the perturbation of the angles is decreased. Numerical evidence suggests chaotic dynamics also in the two-qubit case, but a rigorous proof is still lacking.

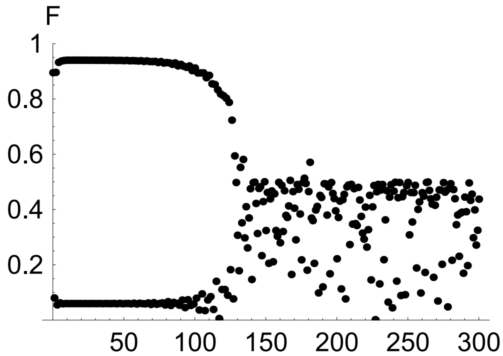

Fig. 2. Fidelity of purification for a pair of qubits by the nonlinear map (8), with rotation angles $x_1 = x_2 = 0.293\pi$, $\phi_1 = \phi_2 = \pi/4$ and initial state defined in Eq. (9) with fidelity ($F_0 = 0.895$). Irregular dynamics set in after a rather long period-two transient

4. Conclusions

Complex chaos realized by measurement conditioned iterative deterministic quantum maps takes place in an ensemble, where the ensemble size decreases exponentially with time if we started with a finite number of copies. This is due to the fact that the measured part of the system cannot take part in the dynamics anymore. In purification protocols only the first few steps are used. A possible application of high sensitivity to small differences in the initial states is to apply it as a Schrödinger microscope and discriminate between to close state.

There are several open questions about the conditions leading to complex chaos. When we include mixed states, the parameter space further increases. The proof of chaos is missing here. One could deviate from von Neumann measurements by measuring a larger part of the ensemble jointly and use the gained information either for some form of feedback, as proposed in [10] or as a selection criterion, like in the presented scheme.

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