The ε-expansion in the symmetry-broken phase of an interacting Bose gas at finite temperature

G Metikas, O Zobay and G Alber

Institut für Angewandte Physik, Technische Universität Darmstadt, 64289 Darmstadt, Germany

Abstract.   We discuss the application of the momentum-shell renormalization group method to the interacting homogeneous Bose gas in the symmetric and in the symmetry-broken phases. It is demonstrated that recently discussed discrepancies are artifacts of not taking proper care of infrared divergencies appearing at finite temperature. If these divergencies are taken into account and treated properly by means of the ε-expansion, the resulting renormalization group equations and the corresponding universal properties are identical in the symmetric and the symmetry-broken phases.

1. Introduction

Renormalization techniques have been employed for the study of interacting Bose gases near the critical temperature, because in this temperature regime the fluctuations dominate the mean field. The bulk of this work was written before the experimental realization of Bose-Einstein condensation (BEC) in ultracold atomic gases [1, 2, 3, 4, 5, 6, 7, 8]. It was shown that only at zero temperature the quantum nature of the three-dimensional Bose gas differentiates it from a three-dimensional two-component classical field theory. At any finite temperature the Bose gas converges to the classical theory as the fixed point of the renormalization group (RG) equations is approached.

Therefore the calculation of the universal properties of the Bose gas can be performed in the same way as for a classical theory, that is through the ε-expansion where $\varepsilon = 4 - D$ and D is the number of spatial dimensions, e.g. [9, 10, 11, 12]. In $D = 3$, although the results of the expansion up to second order in ε are in remarkable agreement with experimental values of critical exponents (measured in He^4 experiments, but due to universality applicable in the case of Bose gases as well [13]), higher-order results diverge from the experimental values. The reason is that the ε-expansion is asymptotic, as first noted in [14], and to obtain meaningful results when higher orders in ε are included, one has to make use of resummation techniques, see e.g. [15]. This way critical exponents have been calculated up to fifth order in [16, 17, 18, 19, 20, 21, 22], see also for corrections [23, 24] and improvements [25]. However, the results thus obtained

are sensitive to the way the resummation is performed and consequently somewhat ambiguous, see e.g. [26].

An alternative method is to calculate the universal properties perturbatively as series of powers of g^* (g^* being the infrared stable fixed point for the interaction g) directly in $D = 3$, as first suggested in [27]. These series are then truncated to order g^{*L} where L is the number of loops in which the calculation is performed. Though this method is fundamentally less satisfactory than the ε-expansion, see e.g. [15], it can be used in the regime of small but non-zero chemical potential. It has been employed for the calculation of critical exponents up to seventh order in g^* for $N = 0, 1, 2, 3$ [28,29,26] and for arbitrary N [30,31] where N is the number of components of the vector field. The series in g^* are again asymptotic and have to be resummed. There is in general agreement with the corresponding ε-expansion results. We will be referring to this technique as the direct method.

After the experimental realization of BEC in ultracold atomic gases [32, 33, 34], because of the renewed interest in these systems, a new generation of papers on the renormalization of Bose gases appeared. Starting with [35], a series of papers relied on the so-called momentum-shell approach [36, 37, 38, 39]. In this method, momentum shells around the cutoff are successively integrated out directly at $D = 3$ according to Wilson's method, but unlike in the direct method no expansion of the critical exponents over g^* is performed.

This apparently new method, when applied in the symmetric (normal) phase, yields universal results which, when compared to experimental values, are worse than even the first-order ε-expansion results. However, when the momentum-shell method is applied to the symmetry-broken phase, it yields results which are far better than the first-order ε-expansion and, in fact, as good as the results of the second-order ε-expansion. Based on this observation, it was assumed that the reliability of the momentum-shell method increases when it is used in the symmetry-broken phase, and a calculation of non-universal properties (for example transition temperature versus scattering length) from the symmetry-broken phase RG equations was attempted. For this reason, one may now wonder if applying the ε-expansion or the direct method in the symmetry-broken rather than in the symmetric phase as is usually done would improve the results of these methods.

Here we show that the distinction introduced by the momentum-shell method of [35, 36, 37, 38, 39] between RG methods applied in the symmetric and the symmetry-broken phases is an artifact of not taking care of the infrared divergence which appears at finite temperature in the interacting Bose gas theory. When this divergence is taken into account and treated properly, as in a classical theory, by means of the ε-expansion, the RG equations and the resulting universal properties are identical in the symmetric and in the symmetry-broken phases.

Furthermore, even if one focuses on the regime of small but non-zero chemical potential, where the direct method applies, the use of the direct method in the symmetry-broken rather than the symmetric phase deteriorates the results instead of

improving them.

2. Derivation of RG equations

The partition function of the homogeneous s-wave interacting Bose gas is

$$Z(\mu, \beta, V, g) \equiv \mathrm{Tr}e^{-\beta(\hat{H}-\mu\hat{N})} = \int \delta[\phi, \phi^*]e^{-S[\phi,\phi^*]} \tag{1}$$

with the Euclidean action

$$S[\phi, \phi^*] = \frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \int_V d^D\mathbf{x} \left\{ \phi^*(\tau, \mathbf{x}) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \phi(\tau, \mathbf{x}) + \frac{g}{2}|\phi(\tau, \mathbf{x})|^4 \right\}. \tag{2}$$

We give an outline of the basic steps of the renormalization procedure. More details can be found in, e.g., [35, 37, 39].

In order to implement the first step of the RG procedure (Kadanoff transformation), we split the field $\phi(x)$ into a long-wavelength component $\phi_<(x)$ (slow field) and a short-wavelength component $\delta\phi_>(x)$ (fast field). The fast field involves Fourier components which are contained only in an infinitesimally thin shell in momentum space of thickness $\Lambda e^{-l} \leq |\mathbf{p}| \leq \Lambda$ near the momentum cutoff Λ, whereas the slow field has all its Fourier components in the sphere whose center is at the origin of the momentum space and whose radius is Λe^{-l}.

We now perform the one-loop calculation of the effective theory of the slow field. We integrate out the fast field and expand the resulting effective action in powers of g keeping up to order g^2. This perturbative effective action is equal to the original action (2) (with the field ϕ replaced by the slow field $\phi_<$) plus two additional terms. The first of these additional term is proportional to g and therefore quadratic in the modulus of the slow field. The second additional term is proportional to g^2 and quartic to the modulus of the slow field. The effective Lagrangian of this theory can be cast in the form of the original Lagrangian because the additional terms produced by the integration over the fast fields have such a form that they can be considered as corrections to μ and g. Thus, after one infinitesimal integration, the chemical potential and the interaction in the effective action are

$$\mu' = \mu + d\mu = \mu - gd_D \int_{\delta V_{\mathbf{p}}} dp \; p^{D-1}f_1[\beta, E(p) - \mu],$$

$$g' = g + dg = g - g^2 d_D \int_{\delta V_{\mathbf{p}}} dp \; p^{D-1}f_2[\beta, E(p) - \mu] \tag{3}$$

where $E(p) = p^2/2m$ and

$$f_1(\beta, E) = 2N(\beta E) + 1,$$

$$f_2(\beta, E) = \frac{1 + 2N(\beta E)}{2E} + 4\beta N(\beta E)[1 + N(\beta E)] \tag{4}$$

with the Bose-Einstein distribution $N(x) = 1/(e^x - 1)$. The density of states d_D can be expressed in terms of the surface Ω_D of a unit hypersphere in D dimensions according to

$$d_D = \frac{1}{(2\pi)^D}\Omega_D \quad \text{with} \quad \Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{5}$$

The infinitesimal momentum shell which is integrated out is denoted by $\delta V_{\mathbf{p}}$. We note that the integration procedure has no effect on the inverse temperature or the slow field, i.e.,

$$\beta' = \beta, \ \phi'_< = \phi_<. \tag{6}$$

Now the only remaining difference between the form of the effective action and that of the original action is that in the original action the momentum of the field is integrated from 0 to the momentum cutoff Λ whereas in the effective action the momentum of the slow field $|\mathbf{q}'|$ is integrated from 0 to Λe^{-l}. This difference is eliminated by a trivial rescaling of the momentum $|q(l)| = |\mathbf{q}'|e^l$ which induces a trivial rescaling of the rest of the parameters of the effective action,

$$\mu(l) = \mu' e^{2l}, \ g(l) = g' e^{(2-D)l}, \ \beta(l) = \beta e^{-2l}, \ \phi_<(l) = \phi_< e^{Dl/2}. \tag{7}$$

In terms of the rescaled parameters, equations (3) assume the form

$$\mu(l) = e^{2l}\mu - e^{2l}g d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \ p^{D-1} f_1[\beta, E(p) - \mu],$$

$$g(l) = e^{(2-D)l}g - e^{(2-D)l}g^2 d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \ p^{D-1} f_2[\beta, E(p) - \mu]. \tag{8}$$

The system of (8) becomes autonomous if one solves it together with

$$\beta(l) = e^{-2l}\beta. \tag{9}$$

We perform the integrations in (8) over the infinitesimal momentum shell $\delta V_{\mathbf{p}}$ and keep terms only up to first order in l, i.e.,

$$\mu(l) = \mu + 2\mu l - g d_D \Lambda^D f_1[\beta, E(\Lambda) - \mu]l,$$
$$g(l) = g + (2-D)gl - g^2 d_D \Lambda^D f_2[\beta, E(\Lambda) - \mu]l. \tag{10}$$

Repeating the above procedure of integrating out shells of high-momentum and rescaling, we find the RG equations for the chemical potential and the interaction,

$$\frac{d\mu(l)}{dl} = 2\mu(l) - g(l)d_D \Lambda^D f_1[\beta(l), E(\Lambda) - \mu(l)],$$

$$\frac{dg(l)}{dl} = (2-D)g(l) - g(l)^2 d_D \Lambda^D f_2[\beta(l), E(\Lambda) - \mu(l)] \tag{11}$$

where the number of renormalization steps l is a continuous parameter running from 0 (no shells integrated out) to $l^* = \infty$ (all shells integrated out). The trivially rescaled quantities defined in (7) also become functions of l.

Setting $g = 0$ and consequently $g(l) = 0$ in (11) one can study the ideal Bose gas both at zero ($\beta = \infty$) and at finite temperature ($\beta \neq \infty$), e.g. [38]. One can also study the zero-temperature, interacting Bose gas, e.g. [40,41,8,42]. Finally, one can study the problem in its full generality, the finite-temperature interacting Bose gas.

In reference [8], the symmetric phase of the finite-temperature interacting Bose gas in an arbitrary number of dimensions was considered. It was shown that, for any finite temperature, the interacting Bose gas is driven towards a two-component classical system, as the fixed point of the RG equations is approached, see also [1,2,3,4,5,6,7].

Our formalism is different from that of [8] or [1, 2, 3, 4, 5, 6, 7], so it may be of some interest to show that we can come to the same conclusion. In the course of the following calculation we will also see how the finite temperature theory we are examining here develops an infrared divergence.

The RG method which we have been using is perturbative over the interaction g. Therefore, as in the zero-temperature interacting gas case, e.g. [43], we examine the behaviour of (11) near the fixed point $(\mu^*, \beta^*) = (0, 0)$ of the unperturbed system, that is the finite-temperature ideal gas. Near this fixed point the Bose-Einstein distribution can be expanded as follows

$$N[\beta(E(p) - \mu)] = \frac{1}{\beta[E(p) - \mu]}, \tag{12}$$

and therefore equations (8) near the fixed point become

$$\mu(l) = e^{2l}\mu - e^{2l}g d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \, p^{D-1} \, 2\frac{1}{\beta[E(p) - \mu]},$$

$$g(l) = e^{(2-D)l}g - e^{(2-D)l}g^2 d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \, p^{D-1} \left\{ \frac{9}{2}\frac{1}{E(p) - \mu} + 5\frac{1}{\beta(E(p) - \mu)^2} \right\}. \tag{13}$$

We observe that, at $(\mu = 0, \beta = 0)$, the last term in each of the above equations is infrared divergent for all p and all D. This divergence can be treated by redefining variables; we recast the above equations in terms of the new variable $\tilde{g} = g/\beta$,

$$\mu(l) = e^{2l}\mu - e^{2l}\tilde{g}\beta d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \, p^{D-1} \, f_1[\beta, E(p) - \mu],$$

$$\tilde{g}(l) = e^{(4-D)l}\tilde{g} - e^{(4-D)l}\tilde{g}^2\beta d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \, p^{D-1} \, f_2[\beta, E(p) - \mu] \tag{14}$$

which can be also written in a differential form

$$\frac{d\mu(l)}{dl} = 2\mu(l) - \tilde{g}(l)\beta(l)d_D\Lambda^D f_1[\beta(l), E(\Lambda) - \mu(l)],$$

$$\frac{d\tilde{g}(l)}{dl} = (4 - D)\tilde{g}(l) - \tilde{g}(l)^2\beta(l)d_D\Lambda^D f_2[\beta(l), E(\Lambda) - \mu(l)]. \tag{15}$$

These are exactly the equations (4.6) of [8] for the chemical potential and the interaction supplemented by equations (4.12) of [8] in the classical regime.

Near the unperturbed fixed point $(\mu^*, \beta^*) = (0, 0)$, Eqs. (14) reduce to

$$\mu(l) = e^{2l}\mu - e^{2l}\tilde{g} d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \, p^{D-1} \, 2\frac{1}{E(p) - \mu},$$

$$\tilde{g}(l) = e^{(4-D)l}\tilde{g} - e^{(4-D)l}\tilde{g}^2 d_D \int_{\Lambda e^{-l}}^{\Lambda} dp \, p^{D-1} \, 5\frac{1}{[E(p) - \mu]^2}. \tag{16}$$

We note that apart from some numerical coefficients, equations (16) are identical with the equations ensuing from the classical Landau-Ginzburg-Wilson (LGW) reduced Lagrangian, see, e.g., equations (6.40)-(6.42) of [43].

The different coefficients are due to the fact that the Bose gas theory involves a complex field, whereas the classical LGW theory uses a two-dimensional vector field. As we will see later, this difference is insignificant in the sense that the two theories have the same universal behaviour.

Finally, it is convenient to cast (15) in terms of dimensionless variables $M = \beta_\Lambda \mu$, $\bar{G} = m^2 \Lambda^{D-4} \tilde{g} d_D / \hbar^4$, $E_> = \beta_\Lambda E(\Lambda) = 1/2$ (so that we keep track of the energy terms), and $b = \beta/\beta_\Lambda$, with $\beta_\Lambda = m/(\hbar^2 \Lambda^2)$, which yields

$$\frac{dM(l)}{dl} = 2M(l) - \bar{G}(l)b(l)f_1[b(l), E_> - M(l)],$$

$$\frac{d\bar{G}(l)}{dl} = (4 - D)\bar{G}(l) - \bar{G}(l)^2 b(l) f_2[b(l), E_> - M(l)]. \tag{17}$$

It may not be immediately obvious that the variable $\bar{G}$ is dimensionless. However recalling that $g = \Omega_D \hbar^2 a^{D-2}/m$, see [44], we can rewrite it as $\bar{G} = \Omega_D (a\Lambda)^{D-2}/b$ which is clearly dimensionless.

2.1. ε-Expansion

Since equations (16) have the same structure as the classical RG equations, the same difficulties in the infrared regime appear. In particular around $\mu = 0$ and for $D < 4$ the integral in the equation for the interaction is infrared divergent. It is interesting to note that this divergence originates in the terms

$$\frac{N[\beta(E(p) - \mu)]}{E(p) - \mu} \qquad \text{and} \qquad N[\beta(E(p) - \mu)]^2$$

of $f_2[\beta, E(p) - \mu]$, see equation (4). As in the classical case, this divergence is cured by performing the ε-expansion. We identify $4 - D$ in the equation for the interaction with ε. Furthermore we assume that μ and $\tilde{g}$ (and therefore M and $\bar{G}$) are of the same order and expand the RG equations (17) up to second order in these variables which yields

$$\frac{dM(l)}{dl} = 2M(l) - \bar{G}(l)b(l) \left\{ 2N[b(l)E_>] + 2b(l)N[b(l)E_>](1 + N[b(l)E_>])M(l) \right\},$$

$$\frac{d\bar{G}(l)}{dl} = \varepsilon\bar{G}(l) - \bar{G}(l)^2 b(l) \left\{ \frac{1 + 2N[b(l)E_>]}{2E_>} + 4b(l)N[b(l)E_>](1 + N[b(l)E_>]) \right\}. \tag{18}$$

This system has a trivial fixed point at $(M^*, \bar{G}^*) = (0, 0)$ with eigenvalues

$$\lambda_1 = 2,$$

$$\lambda_2 = \varepsilon = 4 - D. \tag{19}$$

Therefore, the eigenspace of λ_2 corresponds to the unstable direction for $D < 4$, to the marginal one for $D = 4$, and to the stable direction for $D > 4$. There is also a non-trivial fixed point

$$(M^*, \bar{G}^*) = \left[\frac{\varepsilon}{10 - 2\varepsilon}, \frac{5\varepsilon}{(10 + 2\varepsilon)^2} \right] = \left[\frac{\varepsilon}{10} + O(\varepsilon^2), \frac{\varepsilon}{20} + O(\varepsilon^2) \right].$$

Up to first order in ε, its eigenvalues are

$$\lambda_1 = 2 - \frac{2}{5}\varepsilon,$$

$$\lambda_2 = -\varepsilon = D - 4, \tag{20}$$

and consequently the eigenspace of λ_2 corresponds to the stable direction for $D < 4$, to the marginal one for $D = 4$, and to the unstable direction for $D > 4$.

For the case of physical interest, $D = 3$, we set $\varepsilon = 1$ in the expressions which we have already expanded up to first order in ε. A simple example of a universal property that we can now calculate is the critical exponent for the correlation length $\nu = 1/\lambda_1 = 0.600 + O(\varepsilon^2)$. This is exactly the same as the result found in the ε-expansion study of a classical two-component LGW theory in $D = 3$, e.g. [9, 10, 11], see also [7]. A finite temperature interacting Bose gas in three spatial dimensions belongs to the same universality class as a two-component classical field theory in three spatial dimensions [7].

2.2. Direct Method

We set $D = 3$ directly in (11) or equivalently in (17). The fixed point is

$$(\mu^*, \tilde{g}^*) = (E(\Lambda)/6, 5\pi^2 E(\Lambda)^2/(18\Lambda^3))$$

or equivalently in dimensionless units $(M^*, \bar{G}^*) = (1/12, 5/144)$. We then linearize around the fixed point and calculate the eigenvalues

$$\lambda_1 = \frac{1}{50}\left(-1728\bar{G}^* + \sqrt{2985984\bar{G}^{*2} + 57600\bar{G}^* + 625} + 75\right),$$

$$\lambda_2 = \frac{1}{50}\left(-1728\bar{G}^* - \sqrt{2985984\bar{G}^{*2} + 57600\bar{G}^* + 625} + 75\right). \tag{21}$$

Finally, we expand the critical exponent $\nu = 1/\lambda_1$ up to first order in $\bar{G}^*$ because we are performing a one-loop calculation,

$$\nu = \frac{1}{2} + \frac{72\bar{G}^*}{25} + O(\bar{G}^{*2}) = 0.600 + O(\bar{G}^{*2}).$$

This is the same result as for a two-component classical theory, see e.g. [30, 31].

2.3. Momentum-Shell Method

This approach is used in [35], see also [37, 36], and corresponds to setting $D = 3$ directly in (11) or equivalently in (17) as in the direct method. In this case the RG equations (11) coincide with equations (11a) and (11b) of [35]. The fixed point is the same as in the direct method. However now we set $\bar{G}^* = 5/144$ in the eigenvalues

$$\lambda_1 = \frac{3 + \sqrt{249}}{10},$$

$$\lambda_2 = \frac{3 - \sqrt{249}}{10} \tag{22}$$

and without expanding we find $\nu = 1/\lambda_1 \approx 0.532$.

3. Symmetry-broken phase

If we spontaneously break the global U(1) symmetry of (2) by introducing the most probable configuration $\bar{\phi}$, we find

$$S[\phi, \phi^*] = -\beta V\left[\mu n_0 - \frac{n_0^2 g}{2}\right]$$

$$+ \frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \int_V d^D\mathbf{x} \; \phi^*(\tau,\mathbf{x}) \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 - \mu + 2gn_0 \right] \phi(\tau,\mathbf{x})$$

$$+ \frac{gn_0}{2\hbar} \int_0^{\hbar\beta} d\tau \int_V d^D\mathbf{x} \; [\phi^*(\tau,\mathbf{x})\phi^*(\tau,\mathbf{x}) + \phi(\tau,\mathbf{x})\phi(\tau,\mathbf{x})]$$

$$+ \frac{g\overline{\phi}}{\hbar} \int_0^{\hbar\beta} d\tau \int_V d^D\mathbf{x} \; [\phi^*(\tau,\mathbf{x})\phi^*(\tau,\mathbf{x})\phi(\tau,\mathbf{x}) + \phi^*(\tau,\mathbf{x})\phi(\tau,\mathbf{x})\phi(\tau,\mathbf{x})]$$

$$+ \frac{g}{2\hbar} \int_0^{\hbar\beta} d\tau \int_V d^D\mathbf{x} \; \phi^*(\tau,\mathbf{x})\phi^*(\tau,\mathbf{x})\phi(\tau,\mathbf{x})\phi(\tau,\mathbf{x}) \tag{23}$$

where $n_0 = |\overline{\phi}|^2 = \mu/g$ is the condensate density, and $\phi(\tau,\mathbf{x})$ now denotes the fluctuation around the most probable configuration $\overline{\phi}$.

It is possible to write down the RG equations for the symmetry-broken phase. The calculation is significantly more complicated than in the symmetric phase, for details on the derivation of the RG equations see [35] for $D = 3$ and [37, 38] for arbitrary D. We focus here on the RG equations for the chemical potential and the interaction

$$\frac{dM(l)}{dl} = 2M(l) - \bar{G}(l)b(l) \left\{ \frac{2E_>^3 + 6M(l)E_>^2 + M(l)^3}{2\Delta(l)^3}[2N(b(l)\Delta(l)) + 1] - 1 \right.$$

$$\left. + \frac{M(l)(2E_> + M(l))^2}{\Delta(l)^2}b(l)N(b(l)\Delta(l))[N(b(l)\Delta(l)) + 1] \right\},$$

$$\frac{d\bar{G}(l)}{dl} = (4 - D)\bar{G}(l) - \bar{G}(l)^2b(l) \left\{ \frac{(E_> - M(l))^2}{2\Delta(l)^3}[2N(b(l)\Delta(l)) + 1] \right.$$

$$\left. + \frac{(2E_> + M(l))^2}{\Delta(l)^2}b(l)N(b(l)\Delta(l))[N(b(l)\Delta(l)) + 1] \right\} \tag{24}$$

where $\Delta(l) = \sqrt{E_>^2 + 2M(l)E_>}$. We note that the above RG equations coincide with (17) of the symmetric phase for $M = 0$.

3.1. ε-Expansion

At the fixed point $(\mu^*, \beta^*) = (0,0)$ of the unperturbed system an infrared divergent term appears in the course of the derivation of the RG equation for the interaction. This term is exactly the same as in the symmetric case, i.e.,

$$\int_{\Lambda e^{-l}}^{\Lambda} dp \; p^{D-1} \, 5\frac{1}{(E(p) - \mu)^2},$$

but now originates in the terms

$$\frac{(E(p) - \mu)^2}{\delta(p)^3}N(\beta\delta(p)) \qquad \text{and} \qquad \frac{(2E(p) + \mu)^2}{\delta(p)^2}N(\beta\delta(p))^2$$

with $\delta(p) = \sqrt{E(p)^2 + 2\mu E(p)}$.

We can easily apply the ε-expansion technique to equations (24) in order to cure the infrared divergence. As we have seen in the symmetric phase, all we have to do is identify $4 - D$ with ε in the equation for the interaction and then expand the chemical potential equation up to first order in M and the equation for the interaction up to

zeroth order in M. We thus find the ε-expansion RG equations

$$\frac{dM(l)}{dl} = 2M(l) - \bar{G}(l)b(l)\left\{2N[b(l)E_>] + 2b(l)N[b(l)E_>](1 + N[b(l)E_>])M(l)\right\},$$

$$\frac{d\bar{G}(l)}{dl} = \varepsilon\bar{G}(l) - \bar{G}(l)^2 b(l)\left\{\frac{1 + 2N[b(l)E_>]}{2E_>} + 4b(l)N[b(l)E_>](1 + N[b(l)E_>])\right\}.$$

(25)

Comparing (25) to (18) we see that they are exactly the same! In other words, when we perform the momentum-shell integrations together with the ε-expansion, the symmetric and symmetry-broken phases yield exactly the same RG equations, and consequently identical universal properties.

3.2. Direct Method

As in the symmetric phase, it consists of setting directly $D = 3$ in (24). The non-trivial fixed point is $(M^*, \bar{G}^*) = (1/2, 1/4)$. The eigenvalues of (24), when it is linearized around the fixed point, are

$$\lambda_1 = \frac{9\pi^2 - 12\bar{G}^* + \sqrt{3}(-16\bar{G}^{*2} + 24\pi^2\bar{G}^* + 3\pi^4)^{1/2}}{6\pi^2},$$

$$\lambda_2 = \frac{9\pi^2 - 12\bar{G}^* - \sqrt{3}(-16\bar{G}^{*2} + 24\pi^2\bar{G}^* + 3\pi^4)^{1/2}}{6\pi^2},$$

(26)

and therefore

$$\nu = 0.500 + O(\bar{G}^{*2}).$$

3.3. Momentum-Shell Method

We set directly $D = 3$ in (24) as in the direct method. The resulting RG equations, when recast in the dimensionful variables μ and g, coincide with (29a), (29b) of [35], see also equations (16), (17) of [37]. We now substitute $\bar{G}^*$ in the eigenvalues of the direct method

$$\lambda_1 = \frac{3 + \sqrt{33}}{6},$$

$$\lambda_2 = \frac{3 - \sqrt{33}}{6},$$

(27)

and without expanding we find $\nu = 1/\lambda_1 \approx 0.686$.

4. Comparison and Conclusion

The first-order ε-expansion and the momentum-shell method used in [35] and in subsequent papers [36,37,38] do not yield identical universal properties. As we have seen, the first-order ε-expansion gives the value $\nu = 0.600 + O(\varepsilon^2)$ for the critical exponent of the correlation length both in the symmetric and in the symmetry-broken phases. However the momentum-shell method gives $\nu = 0.532$ in the symmetric and $\nu = 0.686$ in

the symmetry-broken phase. These results are to be compared to the experimental value $\nu = 0.670$, see e.g. [13]. It seems as if the momentum-shell method in the symmetry-broken phase yields the best result (in fact, almost as good as the second order in the ε-expansion result $\nu = 0.655 + O(\varepsilon^3)$, see e.g. [15]). However, the results do not always justify the method used to obtain them. For example, in subsection 2.1, had we not first expanded $\nu = 1/\lambda_1$ up to first order in ε and then set $\varepsilon = 1$ but used the unexpanded formula (20) for λ_1, we would have found $\nu = 0.625$. Although this is closer than $\nu = 0.532$ to the experimental result $\nu = 0.670$, it is clearly incorrect.

The main point of this paper is that the momentum-shell method results depend on whether the RG calculation is performed in the symmetric or the symmetry-broken phase, whereas the first order ε-expansion results do not. The dependence of the momentum-shell results on the phase is an artifact of not avoiding the infrared divergence of the Bose gas theory. Because for any finite temperature the Bose gas theory has the same infrared behaviour as a classical two-component theory, the ε-expansion as we know it from classical papers, e.g. [11], can cure the infrared divergence and yield reliable results.

Furthermore we note that, even if we do not worry about the infrared divergence, and use the direct method, applying it in the symmetry-broken phase makes the result for the critical exponent of the correlation length worse rather than improving it as one would have expected from the momentum-shell method.

The above discussion and comparison of values does not take into account the effect of several approximations (derivative expansion, polynomial truncation of the effective action, assumptions about the sharpness or smoothness of the infrared cutoff separating the fast from the slow modes) which were employed in the course of the calculation. For discussions on these approximations see [45,36,39]. However since these approximations were used in the part of the calculation which is common for all three methods, that is the derivation of the RG equations (17) (or equivalently (11)) and (24), a comparison between these methods is still valid.

Acknowledgments

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