Detected-jump-error-correcting quantum codes, quantum error designs, and quantum computation

G. Alber,[1] Th. Beth,[2] Ch. Charnes,[3] A. Delgado,[4] M. Grassl,[2,5] and M. Mussinger[1]
[1]*Institut für Angewandte Physik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany*
[2]*Institut für Algorithmen und Kognitive Systeme, Universität Karlsruhe, D-76128, Germany*
[3]*Department of Computer Science and Software Engineering, University of Melbourne, Parkville, Victoria 3052, Australia*
[4]*Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131, USA*
[5]*The Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, California 94720-5070, USA*
(Received 19 August 2002; published 21 July 2003)

The recently introduced detected-jump-correcting quantum codes are capable of stabilizing qubit systems against spontaneous decay processes arising from couplings to statistically independent reservoirs. These embedded quantum codes exploit classical information about which qubit has emitted spontaneously and correspond to an active error-correcting code embedded in a passive error-correcting code. The construction of a family of one-detected-jump-error-correcting quantum codes is shown and the optimal redundancy, encoding, and recovery as well as general properties of detected-jump-error-correcting quantum codes are discussed. By the use of design theory, multiple-jump-error-correcting quantum codes can be constructed. The performance of one-jump-error-correcting quantum codes under nonideal conditions is studied numerically by simulating a quantum memory and Grover's algorithm.

DOI: 10.1103/PhysRevA.68.012316 PACS number(s): 03.67.Lx, 42.50.Lc, 89.70.+c

I. INTRODUCTION

The discovery of powerful, fast quantum algorithms [1–3] launched new efforts to implement such quantum algorithms in real physical systems. Quantum algorithms simultaneously exploit two characteristic features of quantum theory; namely, the fundamental phenomenon of quantum interference and the fact that for distinguishable quantum systems the dimension of the Hilbert space increases exponentially with the number of systems. Therefore, to implement a quantum algorithm in a real quantum system we must be able to create and manipulate arbitrary superpositions of quantum states and to preserve quantum coherence during computation. Unfortunately, quantum coherence is very fragile. Typically, any coupling to an environment leads to decoherence so that quantum-mechanical superpositions are rapidly destroyed.

The urgent need to develop efficient methods to protect quantum coherence has led to the study of very general classes of quantum error-correcting codes [4–7]. The main idea is to restrict the dynamics of a quantum algorithm to a subspace of the Hilbert space, in which errors can be identified uniquely by suitable measurements and where the error operators can be inverted by unitary operations. Typically, this is achieved by an encoding of the logical information and by a suitable choice of quantum gates.

For some special cases it is also possible to design a passive error-correcting quantum code [8–10]. Such a passive quantum code relies on a subspace of the Hilbert space which is not affected by any errors at all. In this situation the unitary recovery operation is the identity operation so that an active correction of the errors is not necessary.

In principle, any type of error can be corrected by these strategies as long as enough physical qubits are available to achieve the required redundancy and one can make a large number of control measurements and perform the rapid recovery operations. However, in view of current-day experimental possibilities [11] it is generally difficult to achieve both the requirements. Therefore it is desirable to develop alternative error-correcting strategies that possibly correct a restricted class of errors only, but which tend to minimize both redundancy and the number of recovery operations. Recently, the first steps in this direction have been taken by defining a new class of *one-detected-jump-error-correcting quantum codes* which are capable of stabilizing distinguishable qubits against spontaneous decay processes into statistically independent reservoirs [12]. These codes are constructed by embedding an active error-correcting code in a passive code space and by exploiting information available on error positions. This embedding procedure leads to a significant reduction of redundancy and the number of control measurements and recovery operations.

In this paper the physical principles underlying detected-jump-error-correcting quantum codes are explored and generalized, motivated by the practical need for quantum-error-correcting codes which minimize both redundancy and the number of recovery operations. Based on these physical principles an upper bound is established on the number of logical states of a general embedded detected jump-error-correcting quantum code.

From this bound it is apparent that the recently discovered one-detected-jump-error correcting quantum codes have minimal redundancy. Based on this family of optimal one-*detected-jump-error-correcting quantum codes*, we establish links with the general notions of combinatorial design theory [13]. For this purpose the concept of a *spontaneous emission error design* is introduced. This is a powerful tool for constructing multiple-detected-jump-error-correcting quantum codes capable of stabilizing distinguishable qubits against spontaneous decay processes. As an example, we present an

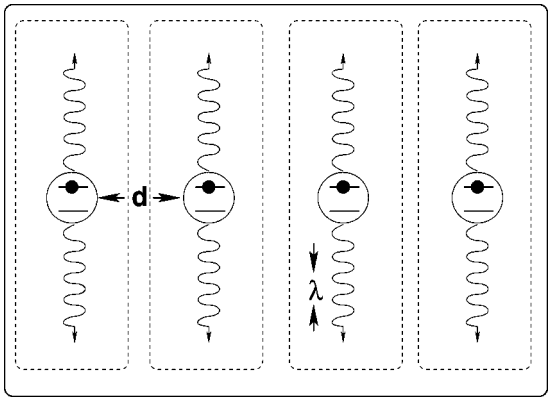


FIG. 1. Model of a quantum computer consisting of an ensemble of distinguishable two-level systems (qubits), which are well localized at positions $\mathbf{x}_\alpha$ and which are coupled to the vacuum modes of the radiation field. The mean distance **d** between adjacent qubits is assumed to be large in comparison with the wavelength of the spontaneously emitted radiation, i.e., $\lambda \ll \mathbf{d}$. Thus, the spontaneous decay processes do not affect the distinguishability of the qubits. The qubits are monitored continuously by photodetectors capable of determining the time at which a spontaneous decay process occurred and the position of the affected qubit.

embedded three-detected-jump-error-correcting quantum code.

This paper is organized as follows. In Sec. II basic physical aspects concerning the spontaneous emission of photons by qubit systems are summarized. In Sec. III the physical principles are explored, which lead to the construction of one-detected-jump-error-correcting quantum codes. The conditions for general-detected-jump-error-correcting quantum codes are given in Sec. IV. The links with combinatorial design theory are established in Sec. V. Finally, in Sec. VI numerical examples are presented, which exhibit basic stability properties of the optimal one-detected-jump-error-correcting quantum codes.

II. SPONTANEOUS DECAY OF DISTINGUISHABLE QUBITS AND PHOTON DETECTION

In this section we summarize basic facts about the dynamical description of a quantum system interacting with initially unoccupied modes of the electromagnetic field. These considerations are the starting point for the development of optimal strategies of error correction, which we pursue in the subsequent sections.

We consider a model of a quantum computer in which N two-level atoms (qubits) interact with external laser pulses which synthesize the quantum gates underlying a quantum algorithm. These N qubits are assumed to be arranged in an array with well-defined positions $\mathbf{x}_\alpha$ $(\alpha=1,\ldots,N)$ (see Fig. 1). In addition, these N qubits are assumed to be distinguishable, which requires that their mean nearest-neighbor distance is large in comparison with the optical wavelengths involved. Their distinguishability guarantees that the dimension of their associated Hilbert space is $K_{\mathcal{H}}=2^N$ and thus scales exponentially with the number of qubits. In addition, it is assumed that these N qubits couple to the unoccupied modes of the electromagnetic field. This coupling causes spontaneous decay processes of the qubits from their excited states $|1\rangle_\alpha$, to their stable lower lying states $|0\rangle_\alpha$. Within the Born, Markov, and the rotating wave approximations the resulting dynamics of the reduced density operator $\rho(t)$ of this N-qubit system is described by the master equation [14,15]

$$\frac{d\rho(t)}{dt}=-\frac{i}{\hbar}[H_{\mathrm{eff}}\rho(t)-\rho(t)H_{\mathrm{eff}}^{\dagger}]+\sum_{\alpha=1}^{N}L_\alpha\rho(t)L_\alpha^{\dagger} \quad (1)$$

with the non-Hermitian effective Hamiltonian

$$H_{\mathrm{eff}}=H-\frac{i\hbar}{2}\sum_{\alpha=1}^{N}L_\alpha^{\dagger}L_\alpha. \quad (2)$$

Thereby, the coherent dynamics of the N-qubit system in the absence of the coupling to the vacuum modes of the electromagnetic field is described by the Hamiltonian H which incorporates the influence of the external laser pulses. In addition, we assume that the mean distance between the qubits is much larger than the wavelengths of the spontaneously emitted radiation. Therefore, to a good degree of approximation each qubit α couples to a different set of modes of the radiation field so that these sets constitute statistically independent reservoirs [15]. In Eq. (1) the coupling of qubit α to its reservoir and the resulting spontaneous decay process $|1\rangle_\alpha \rightarrow |0\rangle_\alpha$ are characterized by the Lindblad operator

$$L_\alpha=\sqrt{\kappa_\alpha}\ \mathbb{1}_{\beta\neq\alpha}\otimes|0\rangle_{\alpha\alpha}\langle 1|, \quad (3)$$

where $\mathbb{1}_{\beta\neq\alpha}$ denotes the identity on every except the αth qubit, and κ_α is the associated spontaneous decay rate.

Provided that initially the N-qubit system is in a pure state, say $|\psi_0\rangle$, a formal solution of the master Eq. (1) is given by [14]

$$\rho(t)=\sum_{n=0}^{\infty}\int_0^t dt_n\int_0^{t_n}dt_{n-1}\cdots \times\int_0^{t_2}dt_1 p(t;t_n\alpha_n,\ldots,t_1\alpha_1)|t;t_n\alpha_n,\ldots,t_1\alpha_1\rangle \times\langle t;t_n\alpha_n,\ldots,t_1\alpha_1|, \quad (4)$$

with the pure quantum state

$$|t;t_n\alpha_n,\ldots,t_1\alpha_1\rangle=\frac{e^{-iH_{\mathrm{eff}}(t-t_n)/\hbar}L_{\alpha_n}\cdots L_{\alpha_1}e^{-iH_{\mathrm{eff}}t_1/\hbar}}{p(t|t_n\alpha_n,\ldots,t_1\alpha_1)^{1/2}}|\psi_0\rangle$$

and with the probabilities

$$p(t;t_n\alpha_n,\ldots,t_1\alpha_1)=\langle\psi_0|e^{iH_{\mathrm{eff}}^{\dagger}t_1/\hbar}L_{\alpha_1}^{\dagger}\cdots L_{\alpha_n}^{\dagger} \times e^{iH_{\mathrm{eff}}^{\dagger}(t-t_n)/\hbar}e^{-iH_{\mathrm{eff}}(t-t_n)/\hbar}L_{\alpha_n}\cdots \times L_{\alpha_1}e^{-iH_{\mathrm{eff}}t_1/\hbar}|\psi_0\rangle. \quad (5)$$

It can be shown that each pure state $|t;t_n\alpha_n,\ldots,t_1\alpha_1\rangle$ describes the quantum state of the N-qubit system at time t conditioned on the emission of precisely n photons at times $t_1\leqslant\cdots\leqslant t_n$ by qubits $\alpha_1,\ldots,\alpha_n$ [14–18]. Thus, each of the pure quantum states of Eq. (4) corresponds to a possible measurement record $(t_1\alpha_1,\ldots,t_n\alpha_n)$ in an experiment in which each qubit is observed continuously by photodetectors. In the subsequent discussion, it is important to note that due to the large separation between the qubits ideally this measurement record determines not only the spontaneous decay times t_i, but also the associated positions α_i $(i=1,\ldots,n)$ of the qubits that have been affected by these decay processes. The measurement record $(t_1\alpha_1,\ldots,t_n\alpha_n)$ is observed with probability $p(t;t_n\alpha_n,\ldots,t_1\alpha_1)$. According to Eq. (4) the quantum state $|t;t_n\alpha_n,\ldots,t_1\alpha_1\rangle$ resulting from a particular measurement record is determined by two types of effects. First, the time evolution between two successive photon emission events is characterized by the non-Hermitian Hamiltonian $H_{\rm eff}$ of Eq. (2). Thus, even in the absence of any spontaneous photon emission process in a given time interval $[0,t]$, in general, an arbitrary quantum state is modified by the couplings to the vacuum modes of the electromagnetic field. Second, immediately after the spontaneous emission of a photon by qubit α the quantum state is modified by the Lindblad operator L_α. This distinction between the two different types of dynamics is crucial for the development of a useful quantum error-correcting strategy that minimizes the redundancy as well as the number of control measurements and recovery operations.

III. DESIGNING OPTIMAL ONE-DETECTED-JUMP-ERROR-CORRECTING QUANTUM CODES

In this section we introduce the main ideas leading to the construction of optimal one-detected-jump-error-correcting embedded quantum codes which are capable of stabilizing systems of distinguishable qubits against spontaneous decay processes. These quantum codes exploit the physical possibilities that are offered by the continuous observation of the distinguishable qubits by photodetectors.

How can we stabilize an N-qubit system, such as that depicted in Fig. 1, against spontaneous decay processes if we can observe the qubits continuously with the help of photodetectors? To achieve this two major tasks have to be tackled; namely, we have to correct the modifications taking place during successive photon emission events. These modifications are described by the effective Hamiltonian of Eq. (2). And, we have to invert each quantum jump $|1\rangle_\alpha\rightarrow|0\rangle_\alpha$ caused by the spontaneous emission of a photon by qubit α.

In principle, the errors taking place during two successive photon emission events can be corrected by an active error-correcting quantum code with a sufficiently rapid sequence of control measurements and recovery operations [19]. However, for practical applications such an approach is not very attractive. One faces all the problems that are related to the implementation of large numbers of control measurements and recovery operations. It is therefore much more desirable to develop passive error-correcting methods for stabilizing the N-qubit system against modifications arising from the effective Hamiltonian of Eq. (2).

The main idea is to determine an appropriate subspace D of the Hilbert space of the N distinguishable qubits in which the perturbing part of the Hamiltonian acts as a multiple of the unit operator, i.e.,

$$H_{\rm eff}|_D=H|_D-\frac{i}{2}c\mathbb{1}\Big|_D, \tag{6}$$

where c denotes a c number. (In our setting this number is real valued and positive.) If the (unperturbed) dynamics characterized by the Hamiltonian H does not take an initial pure quantum state outside this decoherence-free subspace D, this condition implies that, provided no photon is emitted in the time interval $[0,t]$, the quantum state at time t is identical with the unperturbed state [compare with Eq. (4)]. Thus, if one can find such a sufficiently high-dimensional decoherence-free subspace, the dynamics taking place between successive spontaneous photon emission events is stabilized perfectly without the need for control measurements and recovery operations. In practice, it is desirable to choose the dimension of the decoherence-free subspace to be as large as possible.

An important special case occurs when all the N qubits have identical spontaneous decay rates, i.e., $\kappa_\alpha\equiv\kappa$. In this situation it follows that

$$\sum_{\alpha=1}^{N}L_\alpha^\dagger L_\alpha\equiv\kappa\sum_{\alpha=1}^{N}|1\rangle_{\alpha\alpha}\langle 1| \tag{7}$$

and any subspace formed by basis states involving an equal number, say $w\leqslant N$, of excited qubits is a decoherence-free subspace. For a given number of N qubits the dimension K_D of such a decoherence-free subspace is given by $K_D=\binom{N}{w}$ which is maximal if $w=\lfloor N/2\rfloor$. ($\lfloor x\rfloor$ denotes the largest integer smaller or equal to x.)

In general, the first spontaneous emission of a photon will affect the quantum state of the N-qubit system in an irreversible way. According to Eq. (4) the spontaneous emission of a photon by qubit α, for example, is described by the application of the Lindblad operator L_α, which induces a quantum jump $|1\rangle_\alpha\rightarrow|0\rangle_\alpha$. This Lindblad operator is not invertible over the decoherence-free subspace D so that this quantum jump cannot be corrected. In order to correct for this quantum jump actively, we have to restrict the dynamics to a still smaller subspace $C\subset D$ in which a unitary operator, say U_α, can be found having the property

$$U_\alpha L_\alpha|_C=\mathbb{1}|_C. \tag{8}$$

Therefore, if we still want to take advantage of passive error correction between successive photon emission events we have to construct an active error-correcting quantum code within the relevant decoherence-free subspace.

We now construct a one-detected-jump-error-correcting embedded quantum code in the special case of identical spontaneous decay rates considered above. According to the criterion given in Ref. [5], the orthogonal basis states

$\{|\boldsymbol{c}_i\rangle : i=1,\ldots,K\}$ of a subspace C constitute an active error-correcting quantum code with respect to the set of error operators $\{L_\alpha : \alpha=1,\ldots,N\}$ if and only if

$$\langle \boldsymbol{c}_i | L_\alpha^\dagger L_\beta | \boldsymbol{c}_j \rangle = \delta_{ij} \Lambda_{\alpha\beta} \tag{9}$$

for all possible values of i,j and α,β. Equation (9) states the necessary and sufficient conditions for the existence of unitary recovery operations that fulfill Eq. (8) for the error operators $\{L_\alpha : \alpha=1,\ldots,N\}$. In the physical setting this criterion states that (i) different orthogonal quantum states remain orthogonal under the action of error operators and (ii) all basis states are affected by a given pair of errors L_α and L_β in a similar way. The latter condition necessarily implies that the scalar products between states $L_\alpha|\boldsymbol{c}_i\rangle$ and $L_\beta|\boldsymbol{c}_i\rangle$ are state independent. It is plausible that a larger set of error operators leads to a more restrictive set of conditions of the type of Eq. (9). Furthermore, we also expect that more restrictive conditions lead to a higher redundancy of an active quantum code.

As an example, consider the situation where continuous observation of the N-qubit system by photodetectors does not reveal which qubit α has emitted the registered photon. This implies that the error operators that could induce a spontaneous decay process are in the set $\{L_\alpha : \alpha=1,\ldots,N\}$. It has been shown by Plenio *et al.* [20] that when the error positions are unknown, eight physical qubits are needed to encode two orthogonal logical states by an embedded quantum code. This should be compared with the optimal active one-error-correcting code using five qubits [21].

Thus, the advantage offered by using an embedded quantum code, capable of passively correcting errors between successive photon emission events leads to a significant increase of redundancy in comparison to purely active methods. However, this disadvantage can be overcome if, besides knowing the error time, information about the error position α is also available. In principle, this information can be obtained from continuous observation of the N-qubit system by photodetectors as long as the mean distance between adjacent qubits is large in comparison with the wavelength of the spontaneously emitted radiation. For this purpose, it is important that each photon emitted by one of the qubits can be detected.

How can we construct a one-detected-jump-error-correcting embedded quantum code that exploits information about the error position in an optimal way so that its redundancy is minimized? Let us concentrate again on our previously introduced example of identical spontaneous decay rates. In this setting we have a decoherence-free subspace that involves w excited qubits. This stabilizes the dynamics between successive photon emission events passively. For example, in the simple case of $N=4$ and $w=2$, the orthogonal basis states $|\boldsymbol{c}_i\rangle$ of the decoherence-free subspace are given by $\{|1100\rangle,|0011\rangle,|1010\rangle,|0101\rangle,|1001\rangle,|0110\rangle\}$. Within this six-dimensional decoherence-free subspace a possible active quantum code, capable of correcting one quantum jump at a time at a known position α, is determined by the following three orthonormal code words:

$$|\boldsymbol{c}_0\rangle = \frac{1}{\sqrt{2}}[|1100\rangle + e^{i\varphi}|0011\rangle],$$

$$|\boldsymbol{c}_1\rangle = \frac{1}{\sqrt{2}}[|01\,010\rangle + e^{i\varphi}|1010\rangle],$$

$$|\boldsymbol{c}_2\rangle = \frac{1}{\sqrt{2}}[|1001\rangle + e^{i\varphi}|0110\rangle]. \tag{10}$$

These code words are formed by all possible different pairs of basis states of the decoherence-free subspace which can be constructed by interchanging states $|0\rangle$ and $|1\rangle$. These complementary pairings are also characterized by equally weighted probability amplitudes that involve an arbitrary phase φ. It can be easily checked that this embedded quantum code fulfills the criterion of Eq. (9) for any of the Lindblad operators L_α of Eq. (3) separately, but not for two different Lindblad operators with $\alpha \neq \beta$. Thus, provided that the error time and the error position are known this embedded quantum code can correct one quantum jump at a time. In addition, errors between successive quantum jumps are corrected passively. In this way the number of control measurements and recovery operations is reduced and so is the redundancy. This is apparent by comparing our quantum code with the code proposed by Plenio *et al.* [20], which requires eight physical qubits for the encoding of two orthogonal quantum states.

The above construction based on complementary pairings can be generalized to an even number N of physical qubits; giving an infinite family of one-detected-jump-error-correcting embedded quantum codes. The construction is the following. We start from the highest-dimensional decoherence-free subspace which involves all quantum states in which half of the qubits are excited to states $|1\rangle_\alpha$. Subsequently, we construct an active quantum code within this decoherence-free subspace by complementary pairings. The dimension K_J of the resulting embedded quantum code is given by

$$K_J = (1/2)\binom{N}{N/2} \equiv \binom{N-1}{N/2-1}. \tag{11}$$

It is a remarkable fact that, provided one wants to correct errors passively between successive quantum jumps, it is not possible for an even number of physical qubits N to reduce the redundancy of such an embedded jump code (see Sec. V and Ref. [22]). Therefore, our family of embedded quantum codes has the desirable feature that it minimizes redundancy and that it requires only a small number of recovery operations. For instance, in a given time interval, say $[0,t]$, the mean number n of required recovery operations is of the order of $n \approx N\kappa t/2$.

To stabilize any quantum algorithm against spontaneous decay processes using an embedded one-detected-jump-error-correcting quantum code, three requirements have to be met. First, one has to be able to register the time t and position α of each spontaneous decay event that takes place dur-

ing the performance of the quantum algorithm. As indicated schematically in Fig. 1 this can be achieved by continuous observation of the N-qubit system with photodetectors. In principle, an identification of the perturbed qubit α is possible provided the mean nearest-neighbor spacing of the qubits is large in comparison with the wavelengths of the radiation emitted spontaneously. However, in practice, the error position α might not be determined so easily due to imperfect detection efficiencies of the photodetectors. Therefore, in actual applications shelving techniques [23,24] might be useful, which amplify each spontaneously emitted photon to such an extent that it can be detected with an efficiency arbitrarily close to unity.

Second, we have to ensure that each spontaneous decay event is corrected immediately by application of an appropriated unitary transformation which inverts the effect of the Lindblad operator L_α. In practice, this inversion has to be performed on a time scale that is small in comparison with the natural time scale of the quantum algorithm and with the mean spontaneous decay time. Third, one has to ensure that the sequence of quantum gates that constitute the quantum algorithm does not leave the code space at any time. This can be done by encoding the logical information within the code space, and by developing a universal set of quantum gates which leaves this code space invariant.

Ideally these quantum gates are implemented by suitable Hamiltonians. This ensures that the code space is left invariant even during the application of one of these universal quantum gates. Such universal sets of Hamiltonian-induced quantum gates have already been developed for decoherence-free subspaces D of the kind discussed above [8–10]. But, in general, unitary gates based on swapping Hamiltonians need not be universal on the embedded quantum code, or the swapping Hamiltonians do not leave the embedded quantum code invariant. The solution of this intricate and yet unsolved problem is beyond the scope of the present work. However, some preliminary results have already been obtained recently [25–27].

So far we have shown that any Lindblad operator L_α of the form of Eq. (3) can be inverted by our one-detected-jump-error correcting quantum codes. We provide an example of a unitary transformation that achieves this inversion in the case of the one-detected-jump-error-correcting quantum code involving four physical qubits. A possible sequence of quantum gates capable of inverting a spontaneous decay process affecting qubit $\alpha=2$, for example, is depicted in Fig. 2. This example demonstrates the basic fact that it is indeed possible to perform a unitary inversion of the Lindblad operator L_2 provided Eq. (9) is fulfilled for $\alpha=\beta=2$.

IV. GENERAL DETECTED-JUMP-ERROR CORRECTING QUANTUM CODES

In this section we discuss general notions of higher-order detected-jump-error-correcting quantum codes. Provided error times and error positions of $d>1$ spontaneous emission events are known, they can be corrected by a single sufficiently fast recovery operation with the help of such a d-detected-jump-error-correcting quantum code. It is apparent that at a constant error rate the cumulative correction of $d>1$ spontaneous emission events requires fewer recovery operations than the correction of each individual error separately. This reduction of the number of required recovery operations offers significant advantages for the stabilization of quantum memories or quantum algorithms, in particular, in cases in which the application of sufficiently fast recovery operations is difficult and costly. In this section we concentrate on basic conditions these multiple-error-correcting jump codes have to fulfill. Examples constructed on the basis of general ideas from combinatorial design theory are presented in Sec. V.

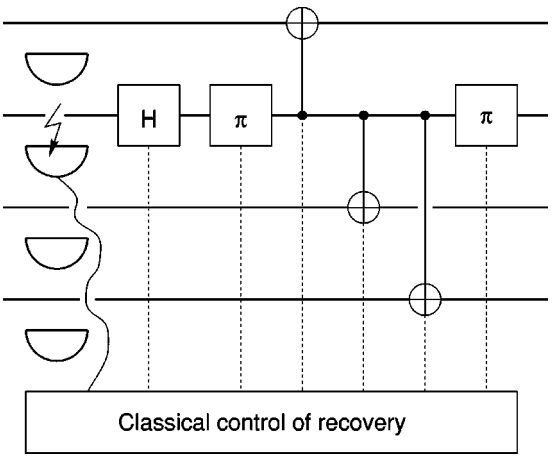


FIG. 2. Gates to invert a quantum jump (indicated by a bolt) of a physical qubit. The four physical qubits involved form a $(4,3,1)_2$ code and are numbered in increasing order from top to bottom.

To define general detected-jump-error-correcting quantum codes, we introduce some notation. For a set of positions $E=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, we denote by $\mathcal{J}_E$ the operator

$$\mathcal{J}_E := \mathcal{J}_{\{\alpha_1,\alpha_2,\ldots,\alpha_n\}} := L_{\alpha_1} L_{\alpha_2} \cdots L_{\alpha_n}.$$

The associated error times $t_1,\ldots,t_n$ are no longer mentioned explicitly, but it is understood that they are known. Note that the operators $L_{\alpha_1},\ldots,L_{\alpha_n}$ commute, because the α_i are pairwise different. Since, by Eq. (5), the errors that involve two equal indices, say $\alpha_1=\alpha_2$, cannot occur.

As discussed in Sec. III, for $H\equiv 0$ all the states that are superpositions of states with a constant number of excited qubits are common eigenstates of the nonunitary effective time evolution (6) between quantum jumps. A subspace C of such a decoherence-free subspace with orthonormal basis $\{|\boldsymbol{c}_1\rangle,\ldots,|\boldsymbol{c}_K\rangle\}$ is called a d-detected-jump-error-correcting quantum code, and is denoted by $C=(N,K,d)_w$ if the following condition holds for sets E of jump positions with at most d elements and for all basis states $|\boldsymbol{c}_i\rangle$ and $|\boldsymbol{c}_j\rangle$:

$$\langle \boldsymbol{c}_i | \mathcal{J}_E^\dagger \mathcal{J}_E | \boldsymbol{c}_j \rangle = \delta_{ij} \lambda(E). \tag{12}$$

The notation $C=(N,K,d)_w$ is motivated by classical coding theory. Similarly, the notation d-$JC(N,w,K)$ is motivated by notations from the design theory (see Ref. [12]).

The validity of this statement follows from the general conditions on quantum error-correcting codes (cf. Ref. [5]).

Since we know on which positions the jump operator $\mathcal{J}_E$ acts, only products of the form $\mathcal{J}_E^\dagger \mathcal{J}_E$ have to be considered.

There is a natural connection with combinatorics. For a basis state $|x_1 \cdots x_N\rangle$ of N qubits, the positions that are in state $|1\rangle$ define a subset of $\{1,\ldots,N\}$. Furthermore, a collection of such subsets corresponds to an equally weighted superposition of basis states.

Let $\mathcal{B}^{(1)},\ldots,\mathcal{B}^{(K)}$ be K disjoint sets of subsets $X \subseteq \{1,\ldots,N\}$, where each subset X contains w elements. Identifying the set X and the binary word $\boldsymbol{x}$ where $\boldsymbol{x}_i = 1$ if $i \in X$ and $\boldsymbol{x}_i = 0$ otherwise, we define the states

$$|\boldsymbol{c}_i\rangle = \frac{1}{\sqrt{|\mathcal{B}^{(i)}|}} \sum_{\boldsymbol{x} \in \mathcal{B}^{(i)}} |\boldsymbol{x}\rangle. \tag{13}$$

Thereby $|\mathcal{B}^{(i)}|$ denotes the number of elements (cardinality) of the set $\mathcal{B}^{(i)}$. These orthonormal states span a d-detected-jump-error-correcting quantum code $C=(N,K,d)_w$, provided that for all sets of jump positions E with no more than d elements and all sets $\mathcal{B}^{(i)}$ the following condition holds:

$$\frac{|X \in \mathcal{B}^{(i)}: E \subseteq X\}|}{|\mathcal{B}^{(i)}|} = \lambda(E). \tag{14}$$

We note that the disjointness of the sets $\mathcal{B}^{(i)}$ implies condition (12) for $i \neq j$. Furthermore, rewriting the operator $\mathcal{J}_E^\dagger \mathcal{J}_E$ as $\mathcal{J}_E^\dagger \mathcal{J}_E = \Sigma_{X \supseteq E} |\boldsymbol{x}\rangle\langle \boldsymbol{x}|$ shows that for $i=j$ and the states (13) the expectation value $\langle \boldsymbol{c}_i | \mathcal{J}_E^\dagger \mathcal{J}_E | \boldsymbol{c}_i \rangle$ equals the expression in Eq. (14).

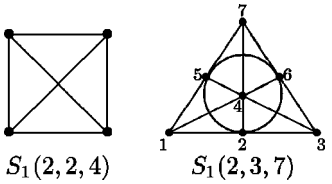
V. BLOCK DESIGNS AND JUMP CODES

In this section we will show that d-detected-jump-error-correcting quantum codes are naturally connected with d designs. These are combinatorial structures that have been extensively studied for many decades, cf. Ref. [13]. To denote this class of combinatorial structures we introduce some notations using the language of finite incidence structures. Let V be a set of N elements, called *points*, say $V=\{1,\ldots,N\}$, where $N>0$ is an integer and $|V|=N$. For $0 \leqslant w \leqslant N$ the class of w subsets of V containing w elements will be denoted by $\binom{V}{w}$. In a suggestive way its cardinality, i.e.,

$$\left|\binom{V}{w}\right| = \binom{|V|}{w} = \binom{N}{w},$$

is just the binomial coefficient $\binom{N}{w}$. An incidence structure in V is specified by a distinguished class $\mathcal{B}$ of subsets of V. The elements B of $\mathcal{B}$ are called *blocks* (or sometimes *lines*) of the incidence structure. If $\mathcal{B} \subseteq \binom{V}{w}$, we say that $\mathcal{B}$ has constant block size w. As an example, any undirected graph is an incidence structure of block size 2, if we choose V as the set of points of the graph and the points that are directly connected by an edge as blocks.

For any point $\alpha \in V$, the class $\mathcal{B}_\alpha := \{B \in \mathcal{B}: \alpha \in B\}$ denotes the class of blocks containing the point α (or "the lines through α"). If $|\mathcal{B}_\alpha| = r$ is constant for all $\alpha \in V$ the incidence structure $\mathcal{B}$ is called regular. For a graph, $|\mathcal{B}_\alpha|$ is the degree $d(\alpha)$ of the vertex α, i.e., the number of edges on which α lies. The incidence structure $\mathcal{B}$—as well as the graph itself—is called regular, if $d(\alpha)$ is constant. If there exists a constant λ_d such that for all $E=\{\alpha_1,\ldots,\alpha_d\} \in \binom{V}{d}$ the class $\mathcal{B}_E := (((\mathcal{B}_{\alpha_1})_{\alpha_2}) \ldots)_{\alpha_d}$ of all blocks containing the set of d points has size $|\mathcal{B}_E| = \lambda_d$, the incidence structure is called *d regular*. A *d design* is a d-regular incidence structure with constant block size w. It is denoted by d-(N,w,λ_d) design or as $S_{\lambda_d}(d,w,N)$ [13]. Graphs that correspond to 2-designs are depicted in Fig. 3.



$S_1(2,2,4)$ $S_1(2,3,7)$

FIG. 3. Examples of 2-designs.

The 2-design $S_1(2,3,7)$ is known as Fano plane. The lines of this plane can be taken to be

$$\mathcal{B} = \{\{1,2,3\},\{1,4,6\},\{1,5,7\},\{2,4,7\},$$
$$\{2,5,6\},\{3,4,5\},\{3,6,7\}\}.$$

From the graphical picture it is clear that any 2-subset of $\{1,2,\ldots,7\}$ is contained in exactly one block of $\mathcal{B}$, hence we have $\lambda_2 = 1$.

The preceding discussion leads to the notion of d-spontaneous emission error designs which we denote by d-SEED(N,w,K) [22]. The essential property of these combinatorial objects is the *local multiplicity* $\lambda(E)$ of a subset E of $\{1,\ldots,N\}$ containing at most d elements, which is defined by

$$\frac{|\mathcal{B}_E^{(i)}|}{|\mathcal{B}^{(i)}|} = \lambda(E), \tag{15}$$

where $\mathcal{B}^{(1)},\ldots,\mathcal{B}^{(K)}$ are disjoint subsets of $\binom{V}{w}$. Any d-SEED(N,w,K) produces a d-detected jump-error-correcting quantum code $C=(N,K,d)_w$ using the encoding defined in Eq. (13).

A class of 1-SEEDs can be constructed from the finite analog of affine geometries. For any prime power q, there is a finite field $GF(q)$ of size q. For $m,b \in GF(q)$, one defines the lines $\ell_{m,b} := \{(x,mx+b): x \in GF(q)\}$ and $\ell_{\infty,b} = \{(b,y): y \in GF(q)\}$. Each line contains q points $(x,y) \in GF(q)^2$. For $m \in GF(q) \cup \{\infty\}$, we define the classes of *parallel* lines $\mathcal{B}^{(m)} := \{\ell_{m,b}: b \in GF(q)\}$. For a given point $(x,y) \in GF(q)^2$, each block $\mathcal{B}^{(m)}$ contains exactly one line, which contains this point. Hence the blocks $\mathcal{B}^{(m)}$ form a 1-SEED$(q^2,q,q+1)$.

Another construction of SEEDs is given by the concept of *resolvable designs*. The blockset of a resolvable design can be partitioned in such a way that each part of the partition is again a design. A well-known example is given by the solu-

tion to Kirkman's schoolgirl problem (see Ref. [13] Example 5.8 [13]:

$$\mathcal{B}^{(1)}=\{\{1,2,3\},\{4,8,12\},\{5,10,14\},\{6,9,15\},\{7,11,13\}\},$$

$$\mathcal{B}^{(2)}=\{\{1,4,5\},\{2,9,11\},\{3,13,15\},\{6,8,14\},\{7,10,12\}\},$$

$$\mathcal{B}^{(3)}=\{\{1,6,7\},\{2,8,10\},\{3,12,14\},\{4,9,13\},\{5,11,15\}\},$$

$$\mathcal{B}^{(4)}=\{\{1,8,9\},\{2,5,7\},\{3,13,14\},\{4,10,15\},\{6,11,12\}\},$$

$$\mathcal{B}^{(5)}=\{\{1,10,11\},\{2,12,15\},\{3,4,6\},\{5,8,13\},\{7,9,14\}\},$$

$$\mathcal{B}^{(6)}=\{\{1,12,13\},\{2,5,6\},\{3,9,10\},\{4,11,14\},\{7,8,15\}\},$$

$$\mathcal{B}^{(7)}=\{\{1,14,15\},\{2,4,7\},\{3,8,11\},\{5,9,12\},\{6,10,13\}\},$$

partitioning the 1-design $S_7(1,3,15)$ into seven 1-designs.

We conclude this section by constructing a three-jump-correcting code. The permutation group $G=\langle(1\,2)(3\,4),(5\,6)(7\,8),(1\,2\,3)(5\,6\,7)\rangle$ of order 48 acts on the 4-element subsets of $\{1,2,\ldots,8\}$. The orbits under G of the sets $\{1,2,5,6\}$, $\{1,3,5,6\}$, and $\{1,4,5,6\}$ are mutually disjoint. Direct calculation shows that they satisfy the local multiplicity condition (15). Hence the sets $\mathcal{B}^{(1)}:=\{1,2,5,6\}^G$, $\mathcal{B}^{(2)}:=\{1,3,5,6\}^G$, and $\mathcal{B}^{(3)}:=\{1,4,5,6\}^G$ define an $(8,3,3)_4$ jump code. The corresponding (not normalized) basis states are given by

$$\begin{aligned}|c_1\rangle=&|00\,110\,011\rangle+|00\,111\,100\rangle+|01\,010\,101\rangle\\&+|01\,011\,010\rangle+|01\,100\,110\rangle+|01\,101\,001\rangle\\&+|10\,010\,110\rangle+|100\,110\,01\rangle+|10\,100\,101\rangle\\&+|10\,101\,010\rangle+|11\,000\,011\rangle+|11\,001\,100\rangle,\end{aligned}$$

$$\begin{aligned}|c_2\rangle=&|00\,110\,110\rangle+|00\,111\,001\rangle+|01\,010\,011\rangle\\&+|01\,011\,100\rangle+|01\,100\,101\rangle+|01\,101\,010\rangle\\&+|10\,010\,101\rangle+|10\,011\,010\rangle+|10\,100\,011\rangle\\&+|10\,101\,100\rangle+|11\,000\,110\rangle+|11\,001\,001\rangle,\end{aligned}$$

$$\begin{aligned}|c_3\rangle=&|00\,110\,101\rangle+|00\,111\,010\rangle+|01\,010\,110\rangle\\&+|01\,011\,001\rangle+|01\,100\,011\rangle+|01\,101\,100\rangle\\&+|10\,010\,011\rangle+|10\,011\,100\rangle+|10\,100\,110\rangle\\&+|10\,101\,001\rangle+|11\,000\,101\rangle+|11\,001\,010\rangle.\end{aligned}$$

Further examples of d-SEEDs are discussed in Ref. [22]. In that paper, there are also general bounds on the parameters of jump codes derived. In particular, the dimension K of a d-detected-jump-error-correcting $(N,K,d)_w$ code is bounded above by

$$K\leqslant\min\left\{\binom{N-d}{w-d},\binom{N-d}{w}\right\}. \tag{16}$$

For completeness, we repeat the main ideas of the proof.

The dimension of the space spanned by the basis states of N qubits, where w of them are in the excited state, is $\binom{N}{w}$. This implies the bound $K\leqslant\binom{N}{w}$. A jump on j positions reduces the number of excitations to $w-j$. After the jump, the j positions where the jump occurred are zero. There are

$$\binom{N-j}{w-j}$$

such basis states. A jump must not reduce the dimension of the code, hence

$$K\leqslant\binom{N-j}{w-j}.$$

For $0\leqslant j\leqslant d$ possible quantum jumps the lowest upper bound is achieved for $j=d$ as

$$\binom{N}{w}=\frac{N(N-1)\cdots(N-d+1)}{w(w-1)\cdots(w-d+1)}\binom{N-d}{w-d}.$$

To obtain the second upper bound in Eq. (16), we note that starting with a jump code $C=(N,K,d)_w$, applying σ_x to all qubits yields a jump code $\overline{C}=(N,K,d)_{N-w}$. Note that, σ_x interchanges ground and excited states, hence the code $\overline{C}$ lies in the decoherence-free subspace with $N-w$ excitations. Also, the linear space spanned by the operators $\mathcal{J}_E^\dagger\mathcal{J}_E$ for all subsets E with no more than d elements is invariant under conjugation by σ_x on all qubits. This holds as $\sigma_x L_\alpha^\dagger L_\alpha\sigma_x=|0\rangle_{\alpha\alpha}\langle 0|=\mathbb{1}-|1\rangle_{\alpha\alpha}\langle 1|$. Thus, for the code $\overline{C}$ we obtain the bound

$$K\leqslant\binom{N-d}{N-w-d}=\binom{N-d}{w}.$$

If there is no restriction on the number w of excited states, choosing $w=\lfloor N/2\rfloor$ maximizes the upper bound of Eq. (16), i.e.,

$$K\leqslant\binom{N-d}{\lfloor N/2\rfloor-d}. \tag{17}$$

As mentioned above, the upper bound of Eq. (17) is achieved for $d=1$ and for an even number N of qubits. A table of lower bounds (obtained by constructions from d-SEEDs) and upper bounds for small values of N and d is provided in Ref. [22].

VI. STABILITY PROPERTIES OF THE ONE-DETECTED-JUMP-ERROR-CORRECTING QUANTUM CODES

The detected-jump-error-correcting quantum codes constructed in the preceding sections can stabilize quantum algorithms provided three conditions are satisfied. First, the decay rates of the qubits are equal. Second, the time and position of each quantum jump are detected with 100% efficiency. Third, the appropriate unitary recovery operations are applied perfectly and instantaneously immediately following the detection of an error. Experimentally these requirements can only be approximated. Therefore, the natural question

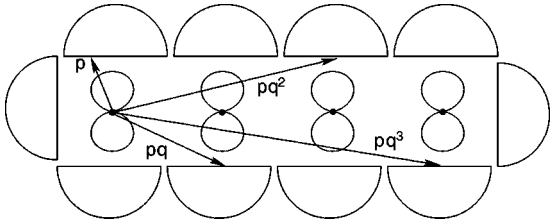


FIG. 4. Model for a 4-qubit register where jumps are detected with a given error parameter q.

arises how nonideal conditions influence the robustness of the embedded error-correcting quantum codes.

In this section several types of imperfections affecting the ideal performance of the one-detected-jump-error correcting embedded quantum codes of Sec. III are studied numerically. For this purpose we investigate the stabilization of a quantum memory and of a simple Hamiltonian dynamics against spontaneous decay processes. The effective two-level Rabi Hamiltonian considered, i.e.,

$$H = i\hbar\Omega(|x_0\rangle\langle v| - |v\rangle\langle x_0|), \tag{18}$$

can be viewed as modeling the quantum dynamics of the ideal Grover search algorithm [2] in the limit of a large number of qubits. Thereby, the Rabi frequency Ω can be related to the characteristic time τ_{op} required for performing an oracle operation and to the number of qubits N according to $\Omega = 2/(\tau_{op}\sqrt{N})$ [28]. Here, $|v\rangle$ denotes an equally weighted superposition of all (orthonormal) code words that may be used as an initial state in Grover's quantum search algorithm. The final state we are searching for is denoted $|x_0\rangle$. For this choice of $|v\rangle$ the states $|v\rangle$ and $|x_0\rangle$ are not orthogonal, because $\langle v|x_0\rangle = 1/\sqrt{N}$. However, if the number of qubits N becomes large their overlap tends to zero. According to the Hamiltonian Eq. (18) and consistent with Grover's quantum algorithm [2], after the interaction time $\tau = \pi/(2\Omega)$, the initial state $|v\rangle$ is transformed to the final state $|x_0\rangle$.

A. Imperfect detection of error positions

Let us first of all consider situations where the jump position can be detected with a given nonzero error rate only. Such an imperfection might occur if, for example, a photon emitted by a particular trapped ion is detected by the photodetector associated with a different ion (compare with Fig. 4). The probability to detect the emitted photon at the correct position is denoted by p. The probability that an emitted photon is detected falsely by the next nearest neighbor is given by $q \times p$. Analogously, the probability of detecting the photon by the nth nearest neighbor is $q^n \times p$ with the normalization condition $p \times (1 + q + q^2 + \cdots) \equiv 1$.

The influence of this type of imperfection on a quantum memory, i.e., $H \equiv 0$ in Eq. (1), is depicted in Fig. 5. A state $|\psi(0)\rangle$ of the jump code $C = (4,3,1)_2$ is propagated according to Eq. (4) with a quantum Monte Carlo simulation of the quantum trajectories [17]. Each realization of this simulation represents an individual quantum trajectory in which the initial pure state $|\psi(0)\rangle$ is propagated accroding to Eq. (4). If a jump is detected, the appropriate recovery operation is applied. In the case of a correct detection, the quantum state of the memory is recovered perfectly. In the case of a false detection, the quantum state of the memory leaves the code space. Therefore, in this simulation the full Hilbert space of four physical qubits has to be taken into account. As a measure of fidelity the squared absolute value of the overlap between the state $|\psi(\tau)\rangle$ after a time $\tau = \pi/(2\kappa)$ and the initial state $|\psi(0)\rangle$ of the memory is plotted as a function of the error parameter q. For the evaluation of this fidelity, one has to average over a statistical ensemble of quantum trajectories.

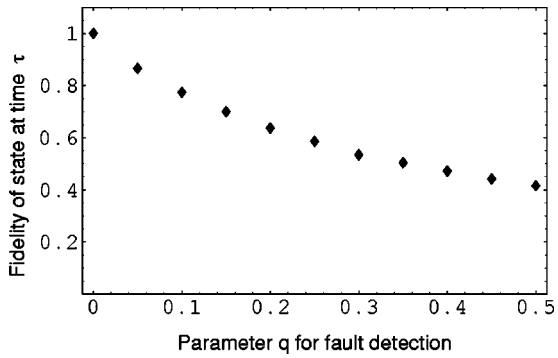


FIG. 5. Fidelity $|\langle\psi(\tau)|\psi(0)\rangle|^2$ of a quantum memory after time $\tau = \pi/(2\kappa)$ for an error parameter q.

B. Unequal spontaneous decay rates

For $\kappa_\alpha \equiv \kappa$ the orthonormal basis states $|c_i\rangle$ of the code $C \subset D$ are degenerate eigenvectors of the operator $\Sigma_\alpha L_\alpha^\dagger L_\alpha$ appearing in the effective Hamiltonian of Eq. (2). This property ensures that these states form a passive code for the effective time evolution between successive quantum jumps. The existence of such degenerate eigenstates of the operator $\Sigma_\alpha L_\alpha^\dagger L_\alpha$ relies on the assumption that the decay rates of all qubits are equal. Although this physical situation can be realized in a laboratory, it is of interest to investigate what happens if this condition of equal decay rates is violated. In this latter case, our code does not correct errors between successive quantum jumps passively.

For this purpose let us consider the Rabi Hamiltonian of Eq. (18) which describes the ideal quantum dynamics. In addition, we assume that the decay rates of the physical qubits are selected randomly according to a Gaussian distribution whose mean value $\overline{\kappa}$ is equal to the characteristic Rabi frequency Ω.

To study the resulting time evolution we choose the $(6,10,1)_3$ code based on complementary pairings (see Sec. III). In Fig. 6 the fidelity of the quantum state $\rho(\tau)$ is depicted as a function of the variance $\Delta\kappa$ of the Gaussian distribution. The fidelity is defined as the overlap $\langle x_0|\rho(\tau)|x_0\rangle$ between the (mixed) system state $\rho(\tau)$ at time $\tau = \pi/(2\Omega)$ and the desired state $|x_0\rangle\langle x_0|$ which would result from the ideal dynamics at this particular interaction time. In this numerical simulation the master equation (1) was integrated up to time $\tau = \pi/(2\Omega)$, whereas each jump operator L_α was replaced by a sequence consisting of L_α and an immediately applied unitary recovery operation U_α [see Eq. (8)]. In this

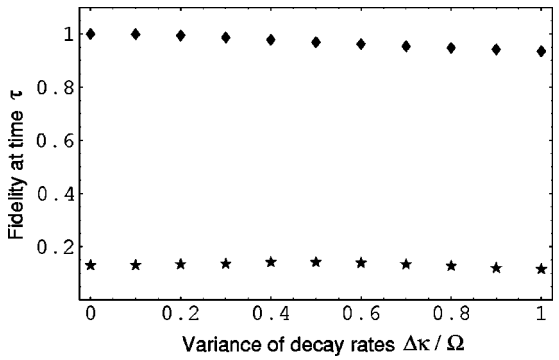


FIG. 6. Dependence of the fidelity of an effective Rabi oscillation at time $\tau=\pi/(2\Omega)$ on the variance $\Delta\kappa$ of the decay rates: The mean value of the decay rates is $\bar{\kappa}=\Omega$, the variances are chosen from Gaussian distributions with $\Delta\kappa=0.1\ldots1.0\times\Omega$. The diamonds show the simulation using the one-detected-jump-error correcting quantum code $(6,10,1)_3$, the stars show the corresponding results for six qubits without encoding.

numerical quantum Monte Carlo simulation it was assumed that the recovery operations are performed perfectly. It is apparent that the code stabilizes the quantum dynamics successfully, despite the fact that the $(6,10,1)_3$ code is not a perfect one-detected-jump-error correcting quantum code for this situation.

C. Time delay between error detection and recovery operation

Immediately after the detection of a spontaneous emission event, the qubits are described by a quantum state belonging to a subspace involving one excited qubit less than the original code space. This subspace also constitutes a passive error-correcting code. Therefore, a time delay between the detection and the application of a recovery operation does not lead to an additional error caused by the effective time evolution H_{eff}, provided the ideal quantum dynamics characterized by H is not affected. Nevertheless, this time delay must be short in comparison with the mean time between two successive spontaneous emission events. Otherwise, a second spontaneous emission may map the state of the system onto another subspace, from which a recovery is no longer possible. The numerical quantum Monte Carlo results

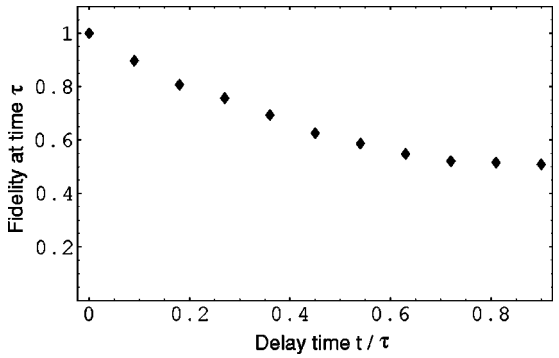


FIG. 7. Fidelity $|\langle\psi(\tau)|x_0\rangle|^2$ for an effective Rabi oscillation at time $\tau=\pi/(2\Omega)$ with a delay time t between detection and recovery. The decay rate is $\kappa=0.5\Omega$.

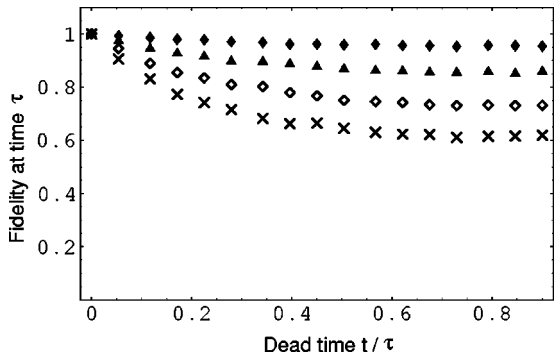


FIG. 8. Dependence of the fidelity $|\langle\psi(\tau)|x_0\rangle|^2$ at time $\tau=\pi/(2\Omega)$ on the dead time t of a photodetector. The ideal dynamics are governed by an effective Rabi oscillation with Rabi frequency Ω. For the encoding the one-detected-jump-error-correcting quantum code $(6,10,1)_3$ is used. The fidelity is depicted for four different values of $\bar{\kappa}$: $\bar{\kappa}=0.25\Omega$ (black diamonds), $\bar{\kappa}=0.5\Omega$ (triangles), $\bar{\kappa}=0.75\Omega$ (white diamonds), $\bar{\kappa}=1.0\Omega$ (crosses).

of Fig. 7 demonstrate that, as long as the delay between detection and correction is not too large compared with the mean decay time, error correction is still possible.

D. Dead times of the photodetectors

Another important condition for correct implementation of a detected-jump-error-correcting quantum code is the ability to observe the environment of each qubit *continuously*. However, immediately after the detection of a spontaneous emission event, typically the detector is not able to respond to another photon. During the latent response time of a photodetector, a second spontaneous emission event can take place, which may destroy quantum coherence. In Fig. 8 the dependence of the fidelity $|\langle\psi(\tau)|x_0\rangle|^2$ on the response time of a photodetector is depicted for various decay rates κ.

It is apparent from the numerical quantum Monte Carlo results of Fig. 8 that the detected jump-error-correcting quantum code can stabilize an algorithm as long as the response time of the photodetectors is small in comparison with the average time between successive spontaneous emission events.

VII. CONCLUSIONS

We have studied quantum error-correcting codes that exploit additional information about the locations of the errors. This information is obtained by continuously monitoring the system [29]. Errors caused by the resulting nonunitary dynamics are corrected passively by embedding the error-correcting code in a decoherence-free subspace. To construct such codes, we have established connections to the design theory. The numerical simulations presented demonstrate that the jump codes discussed can stabilize quantum systems even in cases of imperfect detections and recovery operation.

ACKNOWLEDGMENTS

This work is supported by the DFG (SPP ''Quanteninformationsverarbeitung'') and by Grant No. IST-2001-38869 of the European Commission.

[1] P.W. Shor, in *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science, 1994, Los Alamitos, California* (IEEE Computer Society Press, New York, 1994), p. 124.
[2] L.K. Grover, in *Proceedings of the 28th Annual ACM Symposium on the Theory of Computing* (ACM, New York, 1996), p. 212; Phys. Rev. Lett. **79**, 325 (1997); **80**, 4329 (1998).
[3] D. Simon, in *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science, 1994, Los Alamitos, California* (Ref. [1]), p. 116.
[4] P.W. Shor, Phys. Rev. A **52**, R2493 (1995).
[5] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
[6] A. Ekert and C. Macchiavello, Phys. Rev. Lett. **77**, 2585 (1996).
[7] E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. **84**, 2525 (2000).
[8] P. Zanardi and M. Rassetti, Mod. Phys. Lett. B **11**, 1085 (1997).
[9] L.M. Duan and G.C. Guo, Phys. Rev. Lett. **79**, 1953 (1997).
[10] D.A. Lidar, I.L. Chuang, and K.B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
[11] C.A. Sackett *et al.*, Nature (London) **404**, 256 (2000).
[12] G. Alber, Th. Beth, Ch. Charnes, A. Delgado, M. Grassl, and M. Mussinger, Phys. Rev. Lett. **86**, 4402 (2001).
[13] Th. Beth, D. Jungnickel, and H. Lenz, *Design Theory*, 2nd ed. (Cambridge University Press, Cambridge, 1999).
[14] B.R. Mollow, Phys. Rev. A **12**, 1919 (1975).
[15] H.J. Carmichael, *Statistical Methods in Quantum Optics I* (Springer, Berlin, 1999).
[16] H.J. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer, Berlin, 1993).
[17] R. Dum, A.S. Parkins, P. Zoller, and C.W. Gardiner, Phys. Rev. A **46**, 4382 (1992); C.W. Gardiner, A.S. Parkins, and P. Zoller, *ibid.* **46**, 4363 (1992).
[18] J.K. Breslin, G.J. Milburn, and H.M. Wiseman, Phys. Rev. Lett. **74**, 4827 (1995).
[19] T. Pellizzari, Th. Beth, M. Grassl, and J. Müller-Quade, Phys. Rev. A **54**, 2698 (1996).
[20] M.B. Plenio, V. Vedral, and P.L. Knight, Phys. Rev. A **55**, 67 (1997).
[21] R. Laflamme, C. Miquel, J.P. Paz, and W.H. Zurek, Phys. Rev. Lett. **77**, 198 (1996).
[22] Th. Beth, Ch. Charnes, M. Grassl, G. Alber, A. Delgado, and M. Mussinger, Designs, Codes Cryptogr. **29**, 51 (2003).
[23] W. Nagourney, J. Sandberg, and H. Dehmelt, Phys. Rev. Lett. **56**, 2797 (1986).
[24] Th. Sauter, W. Neuhauser, R. Blatt, and P.E. Toschek, Phys. Rev. Lett. **57**, 1696 (1986).
[25] G. Alber, M. Mussinger, and A. Delgado, Fortschr. Phys. **49**, 901 (2001).
[26] G. Alber, M. Mussinger, and A. Delgado, in *Quantum information Technology*, edited by G. Leuchs and Th. Beth (Wiley-VCH, New York, 2003), p. 31.
[27] K. Khodjasteh and D.A. Lidar, Phys. Rev. Lett. **89**, 197904 (2002).
[28] M. Mussinger, A. Delgado, and G. Alber, New J. Phys. **2**, 19 (2000).
[29] For another recent proposal which exploits quantum feedback control see Ch. Ahn, H.M. Wiseman, and G.J. Milburn, Phys. Rev. A **67**, 052310 (2003).