# Quantum Information Processing

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# 2 Quantum information processing and error correction with jump codes

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## 2.1 Introduction

Current developments in quantum information processing demonstrate in an impressive way the practical potential of quantum physics [1–3]. In quantum computation, for example, characteristic quantum phenomena, such as interference and entanglement, are exploited for solving computational tasks more efficiently than it is possible by any known classical means [4–7]. However, interference and entanglement are fragile phenomena which can be destroyed easily by uncontrolled interactions with an environment. In order to protect quantum information against the resulting decoherence powerful methods of error correction [8–18] have been developed over the last few years.

The main aim of quantum error correction is to reverse the perturbing influence of an uncontrollable environment. Whether such an inversion is possible or not and how it can be achieved most efficiently depends on the physical interaction between the quantum system considered and its environment. In the subsequent sections we discuss main ideas underlying a recently developed new class of error correcting quantum codes which are capable of correcting a frequently occurring class of errors arising from spontaneous decay processes [19]. In quantum optical systems such spontaneous decay processes may arise from the spontaneous emission of photons and in solid state devices, for example, they may originate from the spontaneous emission of phonons. These jump codes exploit in an optimal way information about errors which is obtained from continuous observation of the environment. It will be demonstrated that on the basis of Heisenberg- and Ising-type Hamiltonians universal quantum gates can be constructed for these jump codes. They guarantee that any error can be corrected even if it occurred during the action of one of these gates. Thus, with the help of these quantum gates it is possible to stabilize quantum information processing units against spontaneous decay processes.

This contribution is organized as follows: In Sec. 2 we summarize basic facts about the inversion of general quantum operations or generalized measurements. One of the particularly useful results arising from the systematic analysis of this general problem is an algebraic criterion for the inversion of error operators. In Sec. 3 we discuss the theoretical description of spontaneous decay processes and of continuous measurement processes by master equations.

The practical need of inverting events involving zero- and one-photon (or phonon) emission processes leads directly to one-error correcting jump codes. These quantum codes exploit in an optimal way information about error times and error positions by monitoring the environment continuously. In Sec. 4 we address the problem of stabilizing the coherent dynamics of a quantum system against spontaneous decay processes. An example of such a coherent dynamics is a quantum algorithm performed by a quantum information processing unit. In particular, we address two main problems which arise in this context. Firstly, we deal with the question how one can implement any unitary transformation entirely within the code space of a jump code without leaving it at any time. Secondly, we propose a universal entanglement gate which allows one to entangle two arbitrary basic quantum registers of a quantum information processing unit. This entanglement gate does not leave the error correcting code space of a jump code at any time. Together with the local unitary transformations which can be performed on any of the basic quantum registers it forms a universal set of quantum gates.

## 2.2  Invertible quantum operations and error correction

A typical quantum information processing unit is composed of a system of $N$ two-level quantum systems, so called qubits, which can be addressed individually. According to the linear superposition principle of quantum mechanics an arbitrary pure quantum state of such a $N$-qubit quantum register is of the form

$$|\psi\rangle = \sum_{i_1,i_2,\dots,i_N=0,1} a_{i_1 i_2 \cdots i_N} |i_N,\dots,i_2,i_1\rangle \tag{2.1}$$

with $|0_\alpha\rangle$ and $|1_\alpha\rangle$ denoting two orthogonal basis states of qubit $\alpha$. The corresponding orthonormal basis states of the $N$-qubit Hilbert space $\mathcal{H}$ are denoted $|i_N,i_{N-1},\cdots,i_1\rangle \equiv |i_N\rangle \otimes |i_{N-1}\rangle \otimes \cdots \otimes |i_1\rangle$. The complex coefficients $a_{i_1 i_2 \cdots i_N}$ fulfill the normalization condition $\sum_{i_1,\cdots,i_N=0,1} |a_{i_1 i_2 \cdots i_N}|^2 = 1$. The power of quantum computation relies on the ability to preserve the quantum coherence of such a register-state. Any coupling to an external environment which involves uncontrollable degrees of freedom may destroy linear superpositions thus causing decoherence [20]. This phenomenon which is undesirable from the point of view of quantum information processing can be overcome by quantum mechanical error correction techniques. Shor [8] demonstrated that quantum error correcting codes are possible. By now many different classes of error correcting quantum codes have been developed [9–18].

A main aim of quantum error correction is to reverse the dynamical influence of an external environment on the states of a quantum register [21–23]. The most general dynamical influence of this kind can be represented by a unitary joint evolution of a quantum register with an environment followed by a von Neumann measurement performed on the environment. If initially the quantum register and the environment are not entangled and if the various possible measurement results are discarded, this way a trace-preserving or deterministic quantum operation $\mathcal{E}$ is obtained. Its action on an arbitrary register state with density operator $\rho$ (and proper normalization $\mathrm{Tr}\rho = 1$) can be characterized by a set of Kraus-operators [24] $\{K_{lm}\}$. These Kraus- or error operators characterize all possible environmental influences which may occur and they satisfy the completeness relation $\sum_{lm} K_{lm}^\dagger K_{lm} = 1$. The quantum state resulting

from a deterministic quantum operation is given by

$$\mathcal{E}: \quad \rho \to \mathcal{E}(\rho) \quad = \quad \sum_l p_l \rho_l. \tag{2.2}$$

The labels $l$ characterize all possible measurement results which occur with probabilities $p_l = \text{Tr}(\sum_m K_{lm}^\dagger K_{lm} \rho)$. Observation of a particular measurement result, say $l$, implies that immediately afterwards the register is in the normalized state $\rho_l = \sum_m K_{lm} \rho K_{lm}^\dagger / p_l$.

Typically a set of Kraus-operators $\{K_{lm}\}$ which defines a quantum operation (or generalized measurement) is not unique. Any two sets of Kraus-operators, say $\{\overline{K}_{\lambda\mu}\}$ and $\{K_{lm}\}$, give rise to the same quantum operation if and only if they are related by a unitary matrix $\mathcal{U}_{\lambda\mu,lm}$, i.e. $\overline{K}_{\lambda\mu} = \sum_{\lambda\mu,lm} \mathcal{U}_{\lambda\mu,lm} K_{lm}$ [25]. An important special case of deterministic quantum operations are pure ones. They are characterized by the property that for each measurement result $l$ the associated quantum state $\rho_l$ involves one Kraus-operator $\{K_l\}$ only, i.e.

$$\mathcal{E}_p: \quad \rho \to \mathcal{E}_p(\rho) \quad = \quad \sum_l p_l \rho_l \tag{2.3}$$

with $p_l = \text{Tr}(K_l^\dagger K_l \rho)$, $\rho_l = K_l \rho K_l^\dagger / p_l$ and $\sum_l K_l^\dagger K_l = 1$. Pure quantum operations correspond to situations where a maximum amount of information about the register state is extracted from the quantum state of an environment [21–23]. As a result, an initially prepared pure register state, say $|\psi\rangle$, remains pure, i.e. $|\psi\rangle \to |\psi'\rangle = K_l |\psi\rangle / \sqrt{\langle \psi | K_l^\dagger K_l | \psi \rangle}$.

A quantum operation $\mathcal{E}$ is reversible, if one can construct a deterministic quantum operation $\mathcal{R}$ such that $\mathcal{R}(\mathcal{E}(\rho)) = \rho$ for any density operator $\rho$. The recovery operation $\mathcal{R}$ is required to be deterministic because we want the reversal definitely to occur not just with some probability. In general such an inverse quantum operation cannot be constructed over the whole state space of a quantum register. The main problem in quantum error correction is to find an appropriate, sufficiently high dimensional subspace $\mathcal{C} \subset \mathcal{H}$ over which such an inversion operation can be defined.

It has been shown by Knill and Laflamme [26] and by Bennett et al. [27] that a quantum operation is reversible on a subspace $\mathcal{C}$ if and only if there exists a non-negative matrix $\Lambda_{ll'}$ such that

$$P_\mathcal{C} K_l^\dagger K_{l'} P_\mathcal{C} \quad = \quad \Lambda_{ll'} P_\mathcal{C} \tag{2.4}$$

for all possible error (or Kraus-) operators $K_l$ and $K_{l'}$. Thereby $P_\mathcal{C}$ denotes the projection operator onto the desired subspace $\mathcal{C}$ which is usually called a quantum error-correcting code space or code. Its code words which may be identified with classical bit-strings are formed by an orthonormal basis of states, say $\{|c_i\rangle, i = 1, \cdots, L\}$. The difference $r$ between the dimension of the original Hilbert space $\mathcal{H}$ and the dimension of $\mathcal{C}$, i.e. $r = 2^N - L \geq 0$, is a measure of the redundancy which has to be introduced in order to guarantee successful error correction. For the actual reversal of a quantum operation one has to identify first of all the character of the error (i.e. its syndrome) by an appropriate measurement and subsequently one has to apply an appropriate unitary recovery operation which reverses this quantum operation [21–23]. The criterion of Eq. (2.4) guarantees the existence of such a measurement process and its associated unitary recovery operation. These two basic steps, namely determination of the character of an error and subsequent application of a (nontrivial) unitary recovery operation, constitute the basic elements of any kind of active quantum error correction.

A special situation arises, if one is able to identify a subspace $\mathcal{C}'$ which fulfills not only Eq. (2.4) but also the more stringent condition

$$K_l P_{\mathcal{C}'} = \lambda_l P_{\mathcal{C}'} \tag{2.5}$$

for all possible error operators $K_l$ considered. In this case the quantity $\Lambda_{ll'}$ of Eq. (2.4) factorizes according to $\Lambda_{ll'} \equiv \lambda_l^* \lambda_{l'}$. It is apparent that in this case all the required unitary recovery operations are trivial as they are equal to the identity operation over the code space $\mathcal{C}'$. Thus, no recovery operation has to be performed at all. Such a passive error correction [16–18] is not only capable of correcting single but also multiple errors of arbitrary order. However, so far only very few physical situations are known in which sufficiently high dimensional decoherence-free subspaces (DFSs) $\mathcal{C}'$ can be constructed. In many cases the relevant DFSs are one-dimensional so that they are not of any practical interest for purposes of quantum information processing.

In practical applications one is interested in constructing error correcting methods which tend to decrease not only the number of recovery operations but which also minimize redundancy. For this purpose it may be advantageous to combine passive and active methods of quantum error correction. In the subsequent sections we discuss such a family of error correcting quantum codes which is capable of correcting spontaneous decay processes of the distinguishable qubits of a quantum information processor.

## 2.3   Quantum error correction by jump codes

### 2.3.1   Spontaneous decay and quantum trajectories

Any interaction of a quantum system with an environment whose degrees of freedom are not accessible to observation leads to decoherence. An example of such an interaction is the coupling of a quantum register to the unoccupied vacuum modes of the electromagnetic field (compare with Fig. 2.1). As a result an excited qubit can decay spontaneously by emission of a photon. For the sake of quantum information processing situations are of particular interest in which no spontaneous decay process affects the distinguishability of the qubits involved. This is guaranteed whenever the wave lengths $\lambda$ of the spontaneously emitted photons are much smaller than typical distances $d$ between adjacent qubits and therefore the qubits decay into statistically independent environments. In this case the time evolution of the state of the quantum register $\rho(t)$ is given by a quantum master equation of the form [23]

$$\frac{d\rho}{dt}(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_{\alpha} \left\{ \left[ L_\alpha, \rho(t) L_\alpha^\dagger \right] + \left[ L_\alpha \rho(t), L_\alpha^\dagger \right] \right\}. \tag{2.6}$$

Thereby the Hamiltonian $H$ describes the coherent dynamics of the quantum register in the absence of any coupling to its environment. This coherent dynamics might represent a quantum algorithm, for example. The Lindblad operators [19] $L_\alpha = \sqrt{\kappa_\alpha}|0_\alpha\rangle\langle 1_\alpha|$ with $\alpha = 1, \cdots, N$ characterize the influence of the environment on the quantum register. The spontaneous decay rate of qubit $\alpha$ is denoted by $\kappa_\alpha$. It should be mentioned that the Born- and Markov approximations underlying the derivation of Eq. (2.6) are applicable whenever the interaction
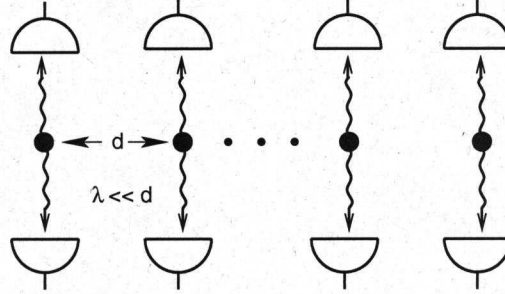
**Figure 2.1:** Schematic representation of an array of distinguishable qubits whose spontaneous emission times $t_i$ and error positions $\alpha_i$ are monitored continuously by photodetectors.

between system and environment is weak and, in addition, the environmental correlation time is small. Typically these conditions are well fulfilled for quantum optical systems. Sometimes they are also fulfilled for other quantum systems, such as solid state devices with phononic decay processes, provided the environmental temperature is sufficiently high [30].

If the initial state of a quantum register is pure, a formal solution of the quantum master equation (2.6) is given by [31, 32]

$$\rho\left(t\right) = \sum_{n=0}^{\infty} \sum_{\alpha_1,\dots,\alpha_n} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1 \left|t; t_n, \alpha_n; \dots; t_1, \alpha_1\right\rangle \tag{2.7}$$
$$\left\langle t; t_n, \alpha_n; \dots; t_1, \alpha_1\right|$$

with the unnormalized pure states

$$\left|t; t_n, \alpha_n; \dots; t_1, \alpha_1\right\rangle = e^{-\frac{i}{\hbar} H_{eff}(t-t_n)} L_{\alpha_n} \dots L_{\alpha_2} e^{-\frac{i}{\hbar} H_{eff}(t_2-t_1)} L_{\alpha_1} e^{-\frac{i}{\hbar} H_{eff} t_1} \left|t=0\right\rangle. \tag{2.8}$$

According to Eq. (2.7) the state of the register at time $t$ is unravelled into a sum of contributions which are associated with all possible numbers $n$ of spontaneously emitted photons. For a given number $n$ of emitted photons the quantum state is unravelled into a sum of all contributions which describe all possible sequences of emission events taking place at emission times $t_1 \le t_2 \le \dots \le t_n$ and affecting qubits $\alpha_1, \dots, \alpha_n$. The pure state $\left|t; t_n, \alpha_n; \dots; t_1, \alpha_1\right\rangle$ of Eq. (2.8) describes the resulting quantum state of the register [31, 32]. The quantum jumps of the qubits from their excited to their ground states due to spontaneous decay processes are characterized by the Lindblad operators $L_{\alpha_i}$. The time evolution between two successive quantum jumps with no photon emission in between is described by the effective Hamiltonian

$$H_{eff} = H - \frac{i\hbar}{2} \sum_{\alpha} L_\alpha^\dagger L_\alpha. \tag{2.9}$$

The norm of the quantum state of Eq. (2.8) yields the probability with which a particular measurement record characterized by a quantum trajectory [32–34] $(t_n, \alpha_n; \ldots; t_1, \alpha_1)$ contributes to the density operator $\rho(t)$. The formal solution of Eq. (2.7) describes the dynamics of the quantum register under the influence of the environment in cases in which the environment is monitored continuously by photodetectors [31, 32] but the measurement results are discarded. According to Sec. 2.2 the formal solution of Eq. (2.7) describes a deterministic quantum process where each quantum trajectory characterizes a particular measurement record.

## 2.3.2   Jump codes

How can we stabilize a quantum system, such as the one depicted in Fig. 2.1, against spontaneous decay processes, if we are able to monitor the distinguishable qubits continuously by photodetectors? According to Eq. (2.7) we have to tackle two major tasks. Firstly, we have to correct the modifications taking place during successive photon emission events. These modifications are described by the effective (non-hermitian) Hamiltonian of Eq. (2.9). Secondly, we have to invert each quantum jump which is caused by the spontaneous emission of a photon. These quantum jumps are described by the Lindblad operators appearing in Eq. (2.8).

For the sake of simplicity let us concentrate in this section on the case of a quantum memory without any intrinsic coherent time evolution, i.e. $H \equiv 0$ in Eq. (2.6). If we want to correct the errors taking place during successive photon emission events, we must invert the pure quantum operation which is characterized by the one-parameter family of Kraus-operators

$$K_0(t) \;=\; e^{-\sum_\alpha L_\alpha^\dagger L_\alpha t/2}. \tag{2.10}$$

Specializing the criterion of Eq. (2.4) to the case of these hermitian error operators an inversion is possible over a subspace $\mathcal{C}$ if and only if

$$P_{\mathcal{C}} K_0(2t) P_{\mathcal{C}} = \Lambda_{00}(t) P_{\mathcal{C}} \tag{2.11}$$

with $\Lambda_{00}(t) \geq 0$. Stated differently, over the code space $\mathcal{C}$ the undesired modification appearing in the effective Hamiltonian of Eq. (2.9) has to act as a (non-negative) multiple of the unit operator. Thus, the code space we are looking for is a DFS of the effective Hamiltonian with $H \equiv 0$.

In the subsequent discussion we focus on the important special case in which the spontaneous decay rates of all qubits are equal, i.e. $\kappa_\alpha \equiv \kappa$. The corresponding DFSs can be found easily because the relevant operator, i.e. $\sum_\alpha L_\alpha^\dagger L_\alpha = \kappa \sum_\alpha |1_\alpha\rangle\langle 1_\alpha|$, just enumerates the number of excited qubits. Therefore, any set of orthonormal states which all involve the same number of excited qubits constitutes a passive error correcting code space for the Kraus-operators $K_0(t)$. The dimension $D$ of a DFS involving $N$ physical qubits $k$ of which are excited, i.e. a DFS – $(N, k)$, is given by $D = \binom{N}{k} \equiv N!/[k!(N-k)!]$. For a given number of physical qubits $N$ this dimension is maximal, if half of the qubits are excited, i.e. for $k = [N/2]$. ($[x]$ denotes the largest integer smaller or equal to $x$.) Such a DFS of maximal dimension involving four physical qubits, for example, is formed by the set of code words $\{|1100\rangle, |0011\rangle, |1010\rangle, |0101\rangle, |1001\rangle, |0110\rangle\}$.

In general, arbitrary linear superpositions of code words of such a DFS cannot be stabilized against quantum jumps arising from spontaneous decay processes. If we also want to invert each individual quantum jump, we have to find an appropriate subspace $\mathcal{C}' \subseteq \mathcal{C}$ over which any of the quantum operations appearing in Eq. (2.7) is reversible. For this purpose we note, that within any DFS $- (N, k)$ the time evolution between successive quantum jumps is proportional to the unit operator, i.e. $e^{-\sum_\alpha L_\alpha^\dagger L_\alpha t/2}|_{\mathcal{C}} \equiv e^{-k\kappa t/2} P_{\mathcal{C}}$. Therefore, we have to find appropriate subspaces $\mathcal{C}' \subseteq \mathcal{C}$ over which the Lindblad operators appearing in Eq. (2.8) are reversible. The details of the construction of an active error correcting quantum code capable of correcting one quantum jump at a time, for example, depends very much on whether the error position is known or not. In the case of an unknown error position one has to fulfill the criterion of Eq. (2.4) for all possible Lindblad operators $L_\alpha$ with $\alpha \in \{1, \cdots, N\}$. Plenio et al. [22] have been able to find such a code which requires at least eight physical qubits for the encoding of one logical qubit, i.e. for two orthonormal logical states. In contrast, if the error position $\alpha$ of a quantum jump characterized by Lindblad operator $L_\alpha$ is known, the redundancy of such an active one-error correcting quantum code which is embedded into a passive code can be lowered significantly.

The simplest example of such an embedded quantum code or jump code which is capable of correcting one error at a time can be constructed with the help of four physical qubits [19]. The (unnormalized) code words of this particular jump code represent a logical qutrit and are given by

$$|c_0\rangle = |0011\rangle + e^{i\varphi}|1100\rangle, \; |c_1\rangle = |0101\rangle + e^{i\varphi}|1010\rangle, \; |c_2\rangle = |0110\rangle + e^{i\varphi}|1001\rangle \quad (2.12)$$

with an arbitrary phase $\varphi$. Obviously, the code words of this jump code consist of four-qubit states in which half of the qubits are excited. The equal number of excited qubits involved in this code guarantees that the effective time evolution between successive quantum jumps is corrected passively. A characteristic feature of this quantum code is the complementary pairing of states with equal probabilities. This latter property guarantees the validity of the necessary and sufficient conditions of Eq. (2.4) provided the error position is known. This one-error correcting jump code involves three logical states and four physical qubits two of which are excited. Therefore, let us call it jump code $1 - JC(4, 2, 3)$. This construction of a one-error correcting embedded quantum code can be generalized easily to any even number $N$ of physical qubits. Thus, any jump code $1 - JC(N, N/2, \binom{N-1}{N/2-1})$ can be constructed by an analogous complementary paring of $N$-qubit states half of which are excited. This way one obtains $\binom{N-1}{N/2-1}$ orthogonal code words which form a one-error correcting embedded quantum code for spontaneous decay processes. It can be shown that this family of one-error correcting quantum codes is optimal in the sense that their redundancy cannot be reduced any further [19]. Asymptotically, for large numbers of physical qubits the effective number of logical qubits $L_q$ which can be encoded by the jump code $1 - JC((N, N/2, \binom{N-1}{N/2-1}))$ is given by $L_q \equiv \log_2\binom{N-1}{N/2-1} = N - \log_2\sqrt{N} + O(1)$. In addition, a far reaching link between these jump codes and fundamental structures of combinatorial design theory [36] can be established. Exploiting this link jump codes can be constructed which are even capable of correcting more than one error at a time [19,37].

Provided the decay rates of all qubits are equal, error position and error time can be determined perfectly and recovery operations are applied immediately after the observation of

a quantum jump, spontaneous decay processes can be corrected perfectly with these jump codes. But in reality, typically none of these conditions is fulfilled precisely. However, numerical simulations demonstrate that quantum states can be stabilized against various types of imperfections still to a high degree even if some of these conditions are not fulfilled perfectly [38].

## 2.4   Universal quantum gates in code spaces

The previously discussed error correcting jump codes allow one to stabilize a quantum memory against spontaneous decay processes. However, in order to be useful also for purposes of quantum information processing and quantum computation two major additional requirements have to be fulfilled. Firstly, one should be able to manipulate pure quantum states in such a way that a chosen error correcting code space is not left at any time during the performance of a quantum algorithm. This can be achieved by using a universal set of quantum gates which operates entirely within an error correcting code space and which is implemented by a set of Hamiltonians leaving this code space invariant. Such a Hamiltonian implementation of universal quantum gates guarantees that any quantum algorithm which is implemented with the help of these quantum gates does not leave this code space at any time even during the application of one of these quantum gates. Secondly, analogous to classical computer architecture, it is desirable to develop quantum information processors which are based on small quantum registers and, in addition, to design quantum gates in such a way that in each step at most two basic quantum registers are entangled. This ensures that the same set of quantum gates can be used for an arbitrarily large quantum information processing unit. In the subsequent sections we present an example how a quantum information processing unit meeting these two major requirements can be constructed on the basis of elementary four-qubit registers each of which constitutes a local qutrit of the jump code $1 - JC(4, 2, 3)$.

### 2.4.1   Universal sets of quantum gates for qudit-systems

Universal sets of quantum gates for qubit-systems were considered by D. DiVincenzo [39], A. Barenco et al. [40] and S. Lloyd [41]. These authors have shown that with a few Hamiltonians acting on single qubits and with one particular two-qubit Hamiltonian it is possible to generate any unitary transformation for a quantum register consisting of qubits. All possible one-qubit operations are members of the continuous group $SU(2)$ (suppressing a trivial $U(1)$ operation) and the two qubit operation entangles any two separable qubits. The lowest dimensional member of our previously discussed jump codes, namely the $1 - JC(4, 2, 3)$-code, provides a logical qutrit and therefore the most general unitary qutrit-operations needed for quantum information processing within this code space are members of the continuous group $SU(3)$ (again suppressing a trivial $U(1)$ operation). Thus the natural question arises which set of quantum gates is universal and thus capable of generating an arbitrary unitary transformation within the state space of a qutrit.

Jean-Luc and Ranee Brylinski [42] derived a generalization of the results of D. DiVincenzo, A. Barenco et al. and S. Lloyd. In particular, they demonstrated that for $d$-dimensional elementary data carriers, so called qudits, every N-qudit gate can be obtained by combinations

of all one-qudit gates and a certain two-qudit entanglement gate. In particular, these authors call a collection $\mathcal{G}$ of one-qudit and two-qudit gates universal (exactly universal), if every $N-$qudit gate with $N \geq 2$ can be approximated with arbitrary accuracy (represented exactly) by a circuit made up of $N$-qudit gates of this collection $\mathcal{G}$. A (unitary) two-qudit gate $V$ is called primitive, if it maps separable pure states again to separable pure states. Thus, if $|x\rangle$ and $|y\rangle$ are qudit-states, we can find qudit-states $|u\rangle$ and $|v\rangle$ such that $V|x\rangle|y\rangle = |u\rangle|v\rangle$. If $V$ is not primitive, it is called imprimitive. Suppose we are given a two-qudit gate $V$. Then the collection of all one-qudit gates together with $V$ is universal if and only if $V$ is imprimitive. In particular, J.-L. and R. Brylinski [42] have proved the useful criterion that, if a (unitary) two-qudit gate $V$ is diagonal in a computational basis, i.e. $V|j\rangle|k\rangle = \exp(i\theta_{jk})|j\rangle|k\rangle$, $V$ is primitive if and only if we have

$$\theta_{jk} + \theta_{pq} \equiv \theta_{jq} + \theta_{pk} \pmod{2\pi} \tag{2.13}$$

for all possible values of $j, k, p, q$.

In general, the difficulty of finding an appropriate set of Hamiltonians by which one can generate a universal set of quantum gates operating entirely within an error correcting code space depends on the physical interactions available. Typical physical two-body interaction Hamiltonians which are expected to be realizable in laboratory are Heisenberg and Ising Hamiltonians $H_{He}$ and $H_{Is}$, i.e.

$$H_{He} = \sum_{\alpha\beta} C_{\alpha\beta}(t)(\sigma_\alpha^{(x)}\sigma_\beta^{(x)} + \sigma_\alpha^{(y)}\sigma_\beta^{(y)} + \sigma_\alpha^{(z)}\sigma_\beta^{(z)}), \quad H_{Is} = \sum_{\alpha\beta} D_{\alpha\beta}(t)\sigma_\alpha^{(z)}\sigma_\beta^{(z)}. \tag{2.14}$$

Thereby, $\sigma_\alpha^{(x)}, \sigma_\alpha^{(y)}, \sigma_\alpha^{(z)}$ denote the three Cartesian components of the Pauli spin operators of qubit $\alpha$ and the quantities $C_{\alpha\beta}(t)$ and $D_{\alpha\beta}(t)$ denote coupling coefficients of qubits $\alpha$ and $\beta$. These latter coefficients are assumed to be tunable arbitrarily. If it is not possible to realize particular linear combinations or commutators of these Hamiltonians by appropriate tunings of these coupling coefficients, one may use appropriate products, such as

$$e^{i(t_1 H_1 + t_2 H_2)} = \left(e^{i\frac{t_1}{n}H_1} e^{i\frac{t_2}{n}H_2}\right)^n + O\left(\frac{1}{n}\right),$$

$$e^{i(i[t_1 H_1, t_2 H_2])} = \left(e^{i\frac{t_1}{\sqrt{n}}H_1} e^{i\frac{t_2}{\sqrt{n}}H_2} e^{-i\frac{t_1}{\sqrt{n}}H_1} e^{-i\frac{t_2}{\sqrt{n}}H_2}\right)^n + O\left(\frac{1}{\sqrt{n}}\right). \tag{2.15}$$

According to Eqs.(2.15) one needs infinite products for representing unitary transformations corresponding to sums or commutators of Hamiltonians exactly. However, it can be shown that in many cases exact representations can also be obtained which involve finite products only [43, 44].

## 2.4.2 Universal one-qutrit gates

In this section we address the question how arbitrary unitary transformations can be implemented in the error correcting code spaces of jump codes with the help of Heisenberg-type and Ising-type Hamiltonians. Thereby the Hamiltonians considered are expected to leave these code spaces invariant so that during the application of an arbitrary sequence of unitary

transformations the error correcting code space is not left at any time. This requirement guarantees that any error due to a spontaneous decay process occurring during the processing of a quantum state can be corrected. As an example we consider the implementation of arbitrary unitary transformations in the lowest dimensional one-error correcting jump code, i.e. the $1 - JC(4, 2, 3)$-code [45].

Two classes of two-particle Hamiltonians of the Heisenberg- and Ising-type acting on physical qubits will be needed for this construction, namely

$$E_{\alpha\beta} = \frac{1}{2} \left( P_{\alpha,\beta} + \sigma_\alpha^{(x)} \sigma_\beta^{(x)} + \sigma_\alpha^{(y)} \sigma_\beta^{(y)} + \sigma_\alpha^{(z)} \sigma_\beta^{(z)} \right) \text{ and } F_{\alpha\beta} = \frac{1}{2} \left( P_{\alpha,\beta} + \sigma_\alpha^{(z)} \sigma_\beta^{(z)} \right)$$
(2.16)

with $\alpha, \beta = 1, \cdots, N$. Any member of this family of two-particle Hamiltonians acts on the physical qubits $\alpha$ and $\beta$ only leaving all other qubits unaffected. The terms of Eqs.(2.16) involving the projection operator $P_{\alpha,\beta}$ represent an energy shift of the two qubits. The residual interaction terms are of Heisenberg- and Ising-type. ¿From these Hamiltonians we can select the six members $E_{12}, E_{23}, E_{13}, F_{12}, F_{13}, F_{13}$, for example. Their action on the code words of the jump code $1 - JC(4, 2, 3)$ with $\varphi = 0$ (compare with Eq. (2.12)) can be represented by the matrices

$$E_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.17)$$

$$F_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Accordingly, the unitary transformations resulting from the Hamiltonians $E_{12}, E_{13}$ and $E_{23}$ swap two codewords and change the phase of the third code word. The unitary transformations resulting from the Hamiltonians $F_{12}, F_{13}$ and $F_{23}$ change the phase of exactly one of the three code words. It is straight forward to demonstrate that the six operators

$$C_{12}^+ = E_{23} - F_{23}, \quad C_{13}^+ = E_{13} - F_{13}, \quad C_{23}^+ = E_{12} - F_{12}, \quad (2.18)$$
$$C_{12}^- = i[C_{13}^+, C_{23}^+], \quad C_{13}^- = i[C_{12}^+, C_{23}^+], \quad C_{23}^- = i[C_{12}^+, C_{13}^+]$$

and the two operators $F_{12}$ and $F_{13}$ form a basis for the Lie Algebra of the continuous group $SU(3)$. Thus by an appropriate linear combination of these eight generators any unitary transformation belonging to the continuous group $SU(3)$ can be represented on the code space of the jump code $1 - JC(4, 2, 3)$.

### 2.4.3  A universal entanglement gate

In computer science it is common practice to use basic registers of a fixed size and to scale an information processing unit by using several of these basic registers. Consequently, on the one hand an algorithm consists of the manipulation of single basic registers, and on the other hand of the interaction between any two of these registers at a time. Such an architecture ensures that the same set of gates can be used for an arbitrarily scaled device. If one applies
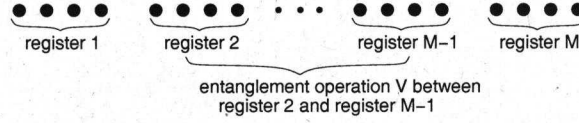
**Figure 2.2:** Schematic representation of an array of qubits consisting of $M$ basic registers. Each basic register carries a logical qutrit of the jump code $1 - JC(4,2,3)$. Any two basic registers, such as registers 2 and $(M-1)$, can be entangled by the entanglement gate $V$.

this idea to a quantum processing unit, the basic registers are formed by an appropriate number of qubits. In addition, if one wants to correct errors originating from spontaneous decay processes, the simplest basic register has to consist of four physical qubits which form a jump code $1 - JC(4,2,3)$. Thus, an appropriate quantum information processing unit capable of stabilizing quantum algorithms against spontaneous decay processes would consist of an array of such four-qubit clusters (compare with Fig. 2.2). We have already demonstrated in the previous section that any unitary transformation within such a four-qubit basic quantum register can be implemented with the help of Heisenberg- and Ising-type Hamiltonians. Here we present a universal entanglement gate which is capable of entangling two arbitrary four-qubit basic registers and which is based on Ising-type Hamiltonians. Together with the unitary transformations discussed in the previous section this entanglement gate forms a universal set of quantum gates for a quantum information processing unit which is based on four-qubit registers. In addition, the presented entanglement gate ensures that all errors due to spontaneous decay processes can be corrected even if they take place during the application of a quantum gate.

Let us consider first of all the nine tensor product states which are associated with two basic four-qubit registers. These states are constituted by the product states of two jump codes $1 - JC(4,2,3)$, namely

$$
\begin{aligned}
|00\rangle_L &= |00110011\rangle + |11001100\rangle + |00111100\rangle + |11000011\rangle, \\
|01\rangle_L &= |00110101\rangle + |11001010\rangle + |00111010\rangle + |11000101\rangle, \\
|02\rangle_L &= |00110110\rangle + |11001001\rangle + |00111001\rangle + |11000110\rangle, \\
|10\rangle_L &= |01010011\rangle + |10101100\rangle + |01011100\rangle + |10100011\rangle, \\
|11\rangle_L &= |01010101\rangle + |10101010\rangle + |01011010\rangle + |10100101\rangle, \\
|12\rangle_L &= |01010110\rangle + |10101001\rangle + |01011001\rangle + |10100110\rangle, \\
|20\rangle_L &= |01100011\rangle + |10011100\rangle + |01101100\rangle + |10010011\rangle, \\
|21\rangle_L &= |01100101\rangle + |10011010\rangle + |01101010\rangle + |10010101\rangle, \\
|22\rangle_L &= |01100110\rangle + |10011001\rangle + |01101001\rangle + |10010110\rangle.
\end{aligned}
$$

The linear subspace spanned by these states is denoted by $\mathcal{C}_9$. It is apparent that these states are linear superpositions of code words of the one-error correcting jump code $1 - JC(8,4,35)$. Let us assume that it is possible to implement the Ising-type Hamiltonian

$$
H_{ent} = 1/2(F_{26} + F_{36} + F_{27} + F_{37}) \tag{2.19}
$$

by an appropriate tuning of the coupling coefficients of Eq. (2.14). This Hamiltonian leaves the code space of the jump code $1 - JC(8,4,35)$ invariant so that any spontaneous decay process can be corrected. Let us denote the linear subspace spanned by the eight orthonormal ~~subspace~~ states $\{|00\rangle_L, |01\rangle_L, |02\rangle_L, |10\rangle_L, |11\rangle_L, |12\rangle_L, |20\rangle_L, |21\rangle_L\}$ by $A$ and the state spanned by the two orthonormal states $\{|22+\rangle_L = |01100110\rangle + |10011001\rangle$ and $|22-\rangle_L = |01101001\rangle + |10010110\rangle\}$ by **B**. With this notation the action of the Hamiltonian can be represented by $H_{ent} = P_A \oplus 2|22+\rangle_{LL}\langle 22 + |$ with $P_A$ denoting the projection operator onto subspace $A$. Therefore, the Hamiltonian $H_{ent}$ acts in the subspaces $A$ and $B$ differently. Applying this Hamiltonian for the (dimensionless) time $\tau$ yields the unitary transformation

$$U(t) = e^{-iH\tau} = e^{-i\tau}P_A \oplus (e^{-i2\tau}|22+\rangle_{LL}\langle 22 + | + |22-\rangle_{LL}\langle 22 - |). \qquad (2.20)$$

Though states $|22+\rangle_L$ and $|22-\rangle_L$ are affected differently by this Hamiltonian the unitary transformation of Eq. (2.20) does not leave the one-error correcting code space $1 - JC(8,4,35)$ at any time. Therefore, any spontaneous emission event can be corrected. In order to implement an entanglement operation within the tensor product space of two basic four-qubit registers we choose the (dimensionless) interaction time so that $\tau = \pi$. This implies that all code words in subspace $A$ are multiplied by a factor $(-1)$ and states $|22+\rangle_L$ and $|22-\rangle_L$ are both multiplied by a factor $(+1)$. Applying an additional global factor of $(-1)$ results in the conditional phase gate $V$

$$V = P_A - |22\rangle_{LL}\langle 22|. \qquad (2.21)$$

This conditional phase gate is a universal entanglement gate because, consistent with the notation of Eq. (2.13), $\theta_{ij} = 0$ for all $(i,j) \neq (2,2)$ and $\theta_{22} = \pi$. Therefore, $\theta_{12} + \theta_{21} = 0 \not\equiv \pi = \theta_{11} + \theta_{22} \pmod{2\pi}$ and according to the criterion of Eq. (2.13) $V$ is a universal entanglement gate.

## 2.5 Summary and outlook

We discussed main ideas underlying a recently introduced class of error correcting quantum codes, the so called jump codes, which are capable of correcting spontaneous decay processes originating from the coupling of distinguishable qubits to statistically independent environments. These quantum codes exploit information about error times and error positions in an optimal way by monitoring the environment continuously. We also addressed the practical question how these error correcting quantum codes can be used for stabilizing a quantum algorithm against these types of errors. For this purpose we presented a set of universal quantum gates which guarantees that any error due to a spontaneous decay process can be corrected even if it occurred during the application of one of these quantum gates. This is possible because these quantum gates are based on Heisenberg- and Ising-type Hamiltonians which leave the code space of a jump code invariant.

Though our discussion concentrated on one-error correcting quantum jump codes, the already mentioned connection with basic concepts of combinatorial design theory may offer interesting perspectives also for the construction of multiple-error correcting jump codes with minimal redundancy. Such optimal multiple-error correcting quantum codes are expected to

be particularly useful for stabilizing the dynamics of quantum information processing units against environmental influences.

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## Bibliography

[1] D. Bouwmeester, A. Ekert and A. Zeilinger (Eds.), *The physics of quantum information*, Springer Berlin (2000).

[2] M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000).

[3] G. Alber, Th. Beth, M. Horodecki, P. Horodecki, R. Horodecki, M. Rötteler, H. Weinfurter, R. Werner, and A. Zeilinger, *Quantum Information*, Springer, Berlin (2001).

[4] D. Deutsch, Proc. Roy. Soc. Lond. A **400**, 97 (1985).

[5] W. Shor, In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, edited by S.Goldwasser (IEEE Computer Society, Los Alamitos, CA, 1994), p 124.

[6] D. Simon, In *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science, 1994, Los Alamitos, California* (IEEE Computer Society Press, New York 1994) p. 116.

[7] L. K. Grover, In *Proceedings of the 28th Annual Symposium on the Theory of Computing, 1996, Philadelphia, Pennsylvania* (ACM Press, New York, 1996) p. 212; Phys. Rev. Lett. **79**, 325 (1997).

[8] P. W. Shor, Phys. Rev. A **52**, R2493 (1995).

[9] H. Mabuchi and P. Zoller, Phys. Rev. Lett. **76**, 3108 (1996).

[10] R. Laflamme, C. Miquel, J.P. Paz, W. Zurek, Phys. Rev. Lett. **77** 198 (1996).

[11] A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996).

[12] A. Ekert and C. Macchiavello, Phys. Rev. Lett. **77**, 2585 (1996).

[13] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).

[14] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).

[15] T. Pellizzari, Th. Beth, M. Grassl, and J. Müller-Quade, Phys. Rev. A. **54**, 2698 (1996).

[16] L. M. Duan and G. C. Guo, Phys. Rev. Lett. **79**, 1953 (1997).

[17] P. Zanardi, Phys. Rev. Lett. **79**, 3306 (1997).

[18] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).

[19] G. Alber, Th. Beth, Ch. Charnes, A. Delgado, M. Grassl, M. Mussinger, Phys. Rev. Lett. **86**, 4402 (2001).

[20] D. Giulini, K. Kiefer, J. Kupsch, I. O. Stamatescu, and H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer Verlag, Berlin, 1996).

[21] M. A. Nielsen, C. M. Caves, Phys. Rev. A **55**, 2547 (1997).

[22] M. A. Nielsen, C. M. Caves, B. Schumacher, H. Barnum, Proc. Roy. Soc. Lond. A **454**, 277 (1998), e-print: quant-ph/9706064.

[23] C. M. Caves, J. of Superconductivity **12**, 707 (1999) e-print: quant-ph/9811082.

[24] K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory*, Springer Verlag Berlin (1983).

[25] M.-D. Choi, Linear Algebra and Its Applications **10**, 285 (1975).

[26] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).

[27] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).

[28] H. J. Carmichael, *Statistical Methods in Quantum Optics 1,* Springer Verlag, Berlin (1999).

[29] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[30] U. Weiss, *Quantum Dissipative Systems,* Series in Modern Condensed Matter Physics, Vol. 2, World Scientific, Singapore (1993).

[31] B. R. Mollow, Phys. Rev. A **12**, 1919 (1975).

[32] H. J. Carmichael, in *An Open Systems Approach to Quantum Optics*, Springer Verlag, Berlin (1993).

[33] J. Dalibard, Y. Castin, K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).

[34] R. Dum, A.S. Farkins, P. Zoller and C.W. Gardiner, Phys. Rev. A **46** , 4382 (1992).

[35] M. B. Plenio, V. Vedral and P. L. Knight, Phys. Rev. A **55**, 67 (1997).

[36] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Vol. 1 and 2, Cambridge University Press (1999).

[37] T. Beth, Ch. Charnes, M. Grassl, G. Alber, A. Delgado, M. Mussinger, to be published.

[38] G. Alber, Th. Beth, Ch. Charnes, A. Delgado, M. Grassl, M. Mussinger, in preparation

[39] D. P. DiVincenzo, Phys. Rev. A **51**, 1015 (1995).

[40] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A **52**, 3457 (1995).

[41] S. Lloyd, Phys. Rev. Lett. **75** 346 (1995).

[42] J. L. Brylinski and R. Brylinski, e-print quant-ph/0108062, to appear in *Mathematics of Quantum Computation*, CRC Press (2002).

[43] D. D'Alessandro, e-print: quant-ph/0111133, to appear in System and Control Letters, Elsevier Science.

[44] Robert M. Zeier, Thomas Beth, private communication.

[45] G. Alber, M. Mussinger and A. Delgado, Fortschr. Phys. **49**, 901 (2001).

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