Fortschr. Phys. **49** (2001) 10–11, 901–908

Quantum Error Correction and Quantum Computation with Detected-jump Correcting Quantum Codes

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Abstract

A new class of embedded error correcting quantum codes is discussed. These detected-jump correcting quantum codes are capable of stabilizing distinguishable qubits against spontaneous decay. Due to their low redundancy and due to the small number of control measurements and recovery operations required these new error-correcting quantum codes offer interesting perspectives for quantum computation. In this contribution main ideas underlying these embedded error correcting quantum codes and their links to fundamental concepts of combinatorial design theory are discussed. In addition, possibilities are explored for implementing universal quantum gates within these code spaces.

PACS numbers: 03.67.Lx, 89.70.+c

1. Introduction

In many problems of quantum optics it is crucial to maintain quantum coherence and entanglement and to protect them against environmental influences. In particular, this is important in the areas of quantum computation and quantum communication [1–3] where these quantum phenomena are exploited for fast computation or secure transmission of secret messages. This protection of quantum coherence and entanglement against external perturbations may be achieved with the help of quantum error correction.

Recently a new class of error correcting quantum codes has been developed [4]. These detected-jump correcting quantum codes are capable of stabilizing distinguishable qubits against spontaneous decay processes which are caused by statistically independent reservoirs. This new class of quantum codes relies on embedding an active error correcting code into a decoherence free subspace and simultaneously taking into account classical information about which qubit has been affected by the environment. These quantum codes have two main advantages. Firstly, by the embedding procedure the number of necessary control measurements and recovery operations is reduced significantly. Secondly, by exploiting the classical information about the error position the redundancy required can be lowered significantly in comparison with other previously proposed embedding schemes. Due to their low redundancy, their simple structure and the small number of control measurements and recovery operations required, these new error-correcting codes offer interesting perspectives for the implementation of quantum algorithms in numerous physical systems, such as arrays of trapped ions [5] or nuclear spin systems [6]. However, for this purpose one has to be able to implement arbitrary quantum algorithms in such a way that the code space is not left at any time during the computation. In this contribution main basic ideas underlying these new embedded error correcting quantum codes are discussed and possibilities for implementing arbitrary quantum algorithms within these code spaces are explored. In addition, links between this new class of embedded quantum codes and fundamental notions of combinatorial design theory are established which are expected to be particularly useful for the development of more general embedded quantum codes with low redundancy.

This contribution is organized as follows: In Sec. 2 the theoretical description of spontaneous decay processes in the presence of continuous measurements is summarized briefly.

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In Sec. 3 basic ideas of quantum error correction are recapitulated and in Sec. 4 our recently proposed new class of one-error correcting detected-jump quantum codes [4] is introduced. In Sec. 5 we explore the possibility of constructing universal sets of quantum gates for quantum computation within these code spaces. Connections between these embedded quantum codes and fundamental notions of combinatorial design theory are finally discussed in Sec. 6.

2. Spontaneous Decay of Distinguishable Qubits and Continuous Measurements

Let us start by considering a simple model of a quantum computer [5] consisting of N distinguishable qubits which are coupled to statistically independent reservoirs and whose mean distance is much larger than the wave lengths of the photons (or phonons) which are emitted spontaneously into these reservoirs. Within the Born- and Markov approximation the time evolution of the density operator $\rho(t)$ of these N qubits can be described by the master equation [7]

$$\dot{\rho}(t) = -\frac{i}{\hbar}\,[H, \rho(t)] + \frac{1}{2}\sum_{\alpha=1}^{n}\{[L_\alpha, \rho(t)\,L_\alpha^\dagger] + [L_\alpha \rho(t), L_\alpha^\dagger]\}\,. \tag{1}$$

Thereby the Lindblad operator $L_\alpha = \sqrt{\kappa_\alpha}\,|0\rangle\,\langle 1|_\alpha$ characterizes spontaneous decay of qubit α from its excited state $|1\rangle_\alpha$ to its stable state $|0\rangle_\alpha$ with rate κ_α. The coherent part of the N-qubit dynamics is described by the Hamiltonian H. In the context of quantum computation this coherent time evolution results from the application of quantum gates which constitute a quantum algorithm. Typically, the Born-Markov approximation underlying Eq. (1) is well fulfilled in quantum optical systems. However, in many cases their application in solid state devices also requires additional assumptions, such as sufficiently high temperatures of the reservoirs involved [8].

If the initial state of the N-qubit system is pure, a formal solution of Eq. (1) is given by [7]

$$\rho(t) = \sum_{N=0}^{\infty}\sum_{\alpha_1,\ldots,\alpha_N}\int_0^t dt_N \int_0^{t_N} dt_{N-1}\ldots\int_0^{t_2} dt_1\;|\psi(t\mid\{t_i,\alpha_i\})\rangle\,\langle\psi(t\mid\{t_i,\alpha_i\})| \tag{2}$$

with

$$|\psi(t\mid\{t_i,\alpha_i\})\rangle = e^{-i[\tilde{H}(t-t_N)]/\hbar}\,L_{\alpha_N}\ldots e^{-i[\tilde{H}(t_2-t_1)]/\hbar}\,L_{\alpha_1}\,e^{-i\tilde{H}t_1/\hbar}\,|\psi(t=0)\rangle\,. \tag{3}$$

Each of the unnormalized, pure states of Eq. (2) describes the quantum state of the N-qubit system conditioned on the observation of N quantum jumps of qubits $\alpha_1,\ldots,\alpha_N$ which take place at times $t_1 \le \ldots \le t_N$ and which originate from the spontaneous emission of photons (phonons) at these times [9]. The action of these quantum jumps is represented by the Lindblad operators $L_{\alpha_N},\ldots,L_{\alpha_1}$ in Eq. (3). The norm of the quantum state $|\psi(t|t_N\alpha_N,\ldots,t_1\alpha_1)\rangle$ defines the probability with which the associated quantum trajectory $(t_1\alpha_1,\ldots,t_N\alpha_N)$ contributes to $\rho(t)$. In this quantum jump representation the time evolution between two successive quantum jumps (conditioned on the emission of no photons) is determined by the non-hermitian effective Hamiltonian $\tilde{H} = H - i(\hbar/2)\sum_{\alpha=1}^{n} L_\alpha^\dagger L_\alpha$. Eqs. (1)–(3) constitute a convenient starting point for the theoretical development of error correcting schemes which are based on the continuous observation of spontaneously decaying qubits.

3. Quantum Error Correction by Embedded Quantum Codes

A necessary prerequisite for quantum computation is the encoding of the possible (classical) logical input states of a computation, say the binary strings of the form $(i_1, \ldots, i_N)$, by orthogonal (i.e. distinguishable) and possibly also normalized quantum states of physical qubits, say $|c_1\rangle, \ldots, |c_l\rangle$. The selection of such an orthonormal computational basis is, to a large extent, arbitrary. In quantum error correction one exploits this freedom of choice for an 'appropriate' encoding which enables one to stabilize the quantum states against external perturbations. There are two different strategies for quantum error correction which have been developed so far.

Active quantum error correction [10–13] may be viewed as a generalization of classical error correcting techniques to the quantum domain. It rests on sequences of appropriately chosen control measurements which determine the character (syndrome) of an error and on the subsequent application of appropriate unitary recovery operations inverting this error. An error which can be characterized by an error operators L_a is reversible over a code space by a unitary recovery operation if and only if the basis states $|c_i\rangle$ of this code space fulfill the conditions [134]

$$\langle c_i| L_a^\dagger L_b |c_j\rangle = \Lambda_{ab}\delta_{ij} \tag{4}$$

for all possible error operators L_a and L_b under consideration. These conditions guarantee that under the action of different errors orthogonal quantum states remain orthogonal and that all states are affected by given errors in a 'similar' way. One of the disadvantages of active quantum error correction is the typically large number of control measurements which is needed for the stabilization of a quantum algorithm.

In passive quantum error correction [14–16] this latter disadvantage is overcome by an encoding in which all code words $|c_i\rangle$ are affected by the possible errors L_a in the same way, i.e. $L_a |c_i\rangle = \lambda_a |c_i\rangle$. Thus, one encodes the logical information in a highly degenerate common eigenspace of all the error operators, i.e. in a decoherence free subspace. However, typically a practicable passive quantum code can be found only for a few, very special error operators. Nevertheless, provided passive error correction is possible, it offers the advantage of stabilizing a quantum algorithm against perturbations without any need for control measurements and recovery operations.

Embedded quantum codes [17] combine the advantages of both methods. They are based on an active error correcting quantum code which is constructed within a decoherence free subspace. Thereby one aims at correcting passively as many of the errors as possible and at using active quantum error correction for the remaining errors. Embedded quantum codes are particularly useful for the stabilization of quantum algorithms as they reduce the number of control measurements and recovery operations significantly.

4. Detected-Jump Correcting Quantum Codes

The dissipative dynamics of N distinguishable qubits as described by Eq. (1) can be stabilized against spontaneous decay in an effective way by embedded quantum codes [14–18]. For this purpose one constructs first of all a decoherence free subspace which stabilizes the time evolution between two successive quantum jumps passively. This time evolution (which is conditioned on the emission of no photons) is described by the effective Hamiltonian $\tilde{H}$ appearing in Eq. (3). In a second step one inverts the quantum jumps, characterized by the Lindblad operators L_α ($\alpha = 1, \ldots, N$) in Eq. (3), with the help of an active quantum error correcting code which is constructed within this decoherence free subspace.

An example of such a one-error correcting embedded quantum code has been developed by Plenio et al. [19]. It applies to the important special case of equal decay rates of all the qubits and it is capable of protecting one logical qubit against spontaneous decay. This one-error correcting embedded quantum code is based on knowledge of the time t_1 at which a quantum jump has occured. As the error position is assumed to be unknown this encoding has to fulfill the conditions of Eq. (4) for all possible error positions, i.e. for $\mathbf{a}, \mathbf{b} \in \{\alpha_1, \ldots, \alpha_N\}$. It has been demonstrated that this encoding requires at least eight physical qubits for the stabilization of one logical qubit. Thus, the reduction of control measurements and recovery operations due to the decoherence free subspace involved in this encoding procedure gives rise to a significant increase of redundancy in comparison with an encoding based entirely on active methods.

In the case of distinguishable qubits whose mean distance exceeds the wave lengths of the spontaneously emitted photons (or phonons) it is in principle possible to determine not only the jump time, say t_1, at which a spontaneous decay process has taken place but also its error position, say α_1. In quantum optical systems, for example, this may be achieved by continuous observation of the spontaneously emitted photons or of the induced photon recoil. With the help of this additional information about error positions the redundancy of such embedded quantum codes can be reduced significantly [4]. If besides the jump time t_1 also information about the error position, say α_1, is available one has to correct one error operator only, namely L_{α_1}. As a consequence, the active quantum error correcting code which has to be constructed within the decoherence free subspace has to fulfill Eqs. (4) for $\alpha = \beta \equiv \alpha_1$ only, i.e.

$$\langle c_i| \, L_\alpha^\dagger L_\alpha \, |c_j\rangle = \Lambda_\alpha \delta_{ij}. \tag{5}$$

This violation of the conditions of Eq. (4) for $\alpha \neq \beta$ offers the possibility to construct embedded quantum codes with a significantly smaller degree of redundancy.

As an example, let us consider again the special case of equal spontaneous decay rates of all the qubits, i.e. $\kappa_\alpha = \kappa_\beta \equiv \kappa$. A passive error correcting code stabilizing the dynamics between successive quantum jumps can be constructed from all states with the same number of excited qubits. Thus, for an even number N of qubits the corresponding code space with the maximal possible dimension of magnitude $d = \binom{N}{N/2} \equiv N!/[(N/2)!]^2$ involves all quantum states in which $(N/2)$ qubits are excited. For $N = 4$, for example, this decoherence free subspace contains the states $\{|1100\rangle, |0011\rangle, |1010\rangle, |0101\rangle, |1001\rangle, |0110\rangle\}$. Within this six-dimensional decoherence free subspace we can construct three (unnormalized) logical basis states by complementary pairing, namely

$$|c_0\rangle = |1100\rangle + e^{i\varphi}\,|0011\rangle\,, \qquad |c_1\rangle = |1010\rangle + e^{i\varphi}\,|0101\rangle\,,$$
$$|c_2\rangle = |1001\rangle + e^{i\varphi}\,|0110\rangle\,, \tag{6}$$

which involve an arbitrary (relative) phase φ. It turns out that this particular encoding within this six-dimensional decoherence free subspace enables one to correct one spontaneous decay event at a time actively provided one knows the error position. Thus, these three orthogonal basis states define the simplest example of a one-error correcting quantum code which involves four physical qubits two of which are excited. We call it a one-error correcting detected-jump quantum code, i.e. a $1 - JC(4,2,3)$ code. In Sec. 5 we will take advantage of the yet unspecified phase appearing in Eq. (6) for constructing a set of universal quantum gates which are suitable for quantum computation in this code space.

This $1 - JC(4,2,3)$-code can be generalized easily to any arbitrary even number N of physical qubits. Starting from a decoherence free subspace which involves $(N/2)$ excited qubits one constructs the code words of the corresponding embedded quantum code again by complementary paring analogous to Eq. (6). This way one obtains the class of all possible one-error correcting $1 - JC\left(N, N/2, \binom{N-1}{N/2-1}\right)$ codes. It can be shown that these one-error correcting embedded quantum codes are optimal in the sense that their redundancy cannot be reduced any further. In order to estimate their redundancy we can compare the number of physical qubits N with the number of effective logical qubits L_q which can be encoded. As a measure for this effective number of logical qubits we can use the logarithm of the number of logical states. Thus for a large number of physical qubits we obtain the relation

$$L_q \equiv \log_2 \binom{N-1}{N/2-1} \approx N - \log_2 \sqrt{N} + O(1) \tag{7}$$

which demonstrates that asymptotically the number of logical and physical qubits differ only by a term of order $O(\log_2 \sqrt{N})$.

5. Quantum Computation within the Code Space of a 1 – *JC*(4,2,3)-code

Any of the previously discussed $1 - JC\left(N, N/2, \binom{N}{N/2-1}\right)$-codes offers interesting perspectives for stabilizing quantum algorithms against spontaneous decay of physical qubits. However, for this purpose it is necessary to find universal sets of quantum gates which guarantee that an arbitrary quantum algorithm can be implemented entirely within any of these code spaces [20, 21]. As a first step towards a general treatment of this problem it is demonstrated by the subsequent considerations that any unitary operation acting on the three logical states of our previously discussed $1 - JC(4, 2, 3)$-code can be implemented by a universal set of quantum gates which operate entirely within this code space.

For this purpose let us start from the rather general assumption that the coherent dynamics of the four physical qubits constituting our $1 - JC(4, 2, 3)$-code can be controlled by a Heisenberg Hamiltonian of the form [6, 22] $H(\{J\}) = \sum_{ij} J_{i,j}\mathbf{S}_i \cdot \mathbf{S}_j$ and that the coupling constants J_{ij} between physical qubits i and j can be tuned arbitrarily. ($\mathbf{S}_i = (\sigma_x^i, \sigma_y^i, \sigma_z^i)$ denotes the Pauli vector acting on qubit i.) With a particular tuning of the coupling constants it is possible to implement the state swapping operator $E_{i,j} = (\mathbf{1} + \mathbf{S}_i \cdot \mathbf{S}_j)/2$ which exchanges the states of the physical qubits i and j. (Note that the unit operator refers to the state spaces of qubits i and j only.) By choosing $\varphi = \pi$ in our $1 - JC(4, 2, 3)$-code (compare with Eq. (6)), it is apparent from Table 1 that the set of all possible state swapping operators acting on the four physical qubits leaves the code space of our $1 - JC(4, 2, 3)$-code invariant.

Therefore, any sequence of unitary transformations of the form $\exp\left(it \sum_{i,j} a_{i,j} E_{i,j}\right)$ with real-valued coefficients a_{ij} will also leave this code space invariant. Starting from this obser-

Table 1 Action of the state swapping operators $E_{i,j}$ on the basis states of the $1 - JC(4, 2, 3)$ code

$E_{i,j}$	$\|c_0\rangle$	$\|c_1\rangle$	$\|c_2\rangle$
$E_{1,2}$	$\|c_0\rangle$	$-\|c_2\rangle$	$-\|c_1\rangle$
$E_{1,3}$	$-\|c_2\rangle$	$\|c_1\rangle$	$-\|c_0\rangle$
$E_{1,4}$	$-\|c_1\rangle$	$-\|c_0\rangle$	$\|c_2\rangle$
$E_{2,3}$	$\|c_1\rangle$	$\|c_0\rangle$	$\|c_2\rangle$
$E_{2,4}$	$\|c_2\rangle$	$\|c_1\rangle$	$\|c_0\rangle$
$E_{3,4}$	$\|c_0\rangle$	$\|c_2\rangle$	$\|c_1\rangle$

vation it is straightforward to demonstrate that any unitary transformation of the group $SU(3)$ can be implemented within our three-dimensional code space by an appropriate sequence of these state swapping operations. This possibility relies on the fact that the eight hermitian operators

$$\begin{aligned}
A_{11} &= \frac{1}{3}\,(E_{12}+E_{34}) - \frac{1}{6}\,(E_{13}+E_{24}) - \frac{1}{6}\,(E_{14}+E_{23})\,,\\
A_{22} &= -\frac{1}{6}\,(E_{12}+E_{34}) + \frac{1}{3}\,(E_{13}+E_{24}) - \frac{1}{6}\,(E_{14}+E_{23})\,,\\
B^{+}_{12} &= \frac{1}{2}\,(E_{23}-E_{14})\,, \qquad B^{+}_{13} = \frac{1}{2}\,(E_{24}-E_{13})\,, \qquad B^{+}_{23} = \frac{1}{2}\,(E_{34}-E_{12})\,,\\
B^{-}_{12} &= i[B^{+}_{13}, B^{+}_{23}]\,, \qquad B^{-}_{13} = i[B^{+}_{12}, B^{+}_{23}]\,, \qquad B^{-}_{23} = i[B^{+}_{12}, B^{+}_{13}]\,,
\end{aligned} \tag{8}$$

for example, are linearly independent generators of the group $SU(3)$. From the additional relations

$$e^{i(\alpha A+\beta B)} = \lim_{n\to\infty} \{e^{i\frac{\alpha}{n}A}\, e^{i\frac{\beta}{n}B}\}^n\,, \qquad e^{i(i[\alpha A,\beta B])} = \lim_{n\to\infty} \{e^{i\frac{\alpha}{\sqrt{n}}A}\, e^{i\frac{\beta}{\sqrt{n}}B}\, e^{-i\frac{\alpha}{\sqrt{n}}A}\, e^{-i\frac{\beta}{\sqrt{n}}B}\}^n\,,$$

which are valid for arbitrary hermitian operators A and B, it becomes apparent that any unitary transformation of the group $SU(3)$ can be approximated with the help of the state swapping operators E_{ij} within the $1-JC(4,2,3)$-code space to any given degree of accuracy.

6. Combinatorial Design Theory and Many-Error Correcting Embedded Quantum Codes

Interesting links can be established between the one-error correcting detected-jump quantum codes of Sec. 4 and fundamental notions of combinatorial design theory. These links are expected to be particularly useful for the further development of many-error correcting embedded quantum codes with low redundancy.

In order to demonstrate some basic aspects of these links let us consider the previously discussed optimal $1-JC(4,2,3)$-code as an example. This embedded quantum code has been constructed within the six-dimensional decoherence free subspace which involves all quantum states of four qubits two of which are excited. These six quantum states can be represented graphically by the system of four points and six lines depicted in Fig. 1. Each point in this diagram represents a qubit. Each basis state of this decoherence free subspace is represented by a line connecting the two qubits which are excited.

This system of points and lines has a few interesting properties, namely

(1) any two points define a unique line;
(2) there are at least two points on each line;
(3) there are three points which are not on a line;
(4) to each line g and each point P not contained in g there exists a uniquely determined line h which has no point in common with g (axiom of parallels).

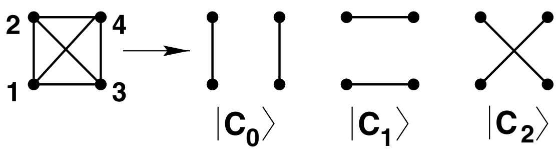

Fig. 1. Affine plane of four points over the binary field and its associated parallelisms

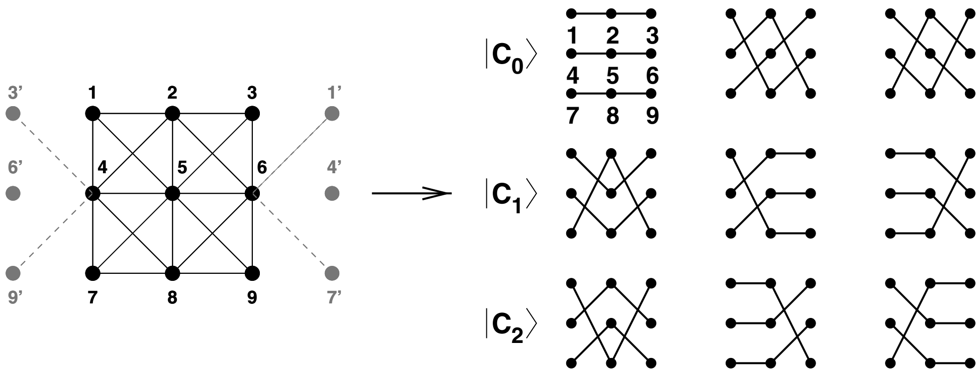

Fig. 2. Affine plane of nine points and a corresponding two-error correcting detected-jump quantum code

In combinatorial design theory a structure fulfilling these axioms is called an affine plane. The three code words of our previously discussed $1 - JC(4, 2, 3)$-code (compare with Eq. (6)) correspond to the three possible parallel pairs or parallelisms of the affine plane of Fig. 1. Thus, the affine plane of Fig. 1 may be viewed as a generating design for the parallelisms which are associated with the basis states of the $1 - JC(4, 2, 3)$-code.

These analogies suggest to pursue these connections with basic notions of combinatorial design theory further by using its powerful methods for the construction of more general many-error correcting embedded quantum codes. In particular, these analogies suggest that for this purpose we have to investigate parallelisms over finite incidence structures [23]. Thus it appears natural to start from a finite affine plane of nine points for the construction of a two-error correcting detected-jump quantum code (compare with Fig. 2).

Indeed, as apparent from Fig. 2 some of the parallelisms of this affine plane together with their permutations constitute a detected-jump correcting quantum code capable of correcting up to two quantum jumps simultaneously.

7. Conclusions

In summary, detected-jump correcting quantum codes are well suited for stabilizing distinguishable qubits against spontaneous decay. Their small redundancy offers interesting perspectives for quantum computation. However, for this purpose it is necessary to find universal sets of quantum gates which operate entirely within these code spaces. It has been demonstrated that state swapping operations constitute such a set of universal quantum operations for our three-dimensional $1 - JC(4, 2, 3)$-code. Furthermore, the discussed connections with combinatorial design theory provide powerful methods for developing more general many-error correcting embedded quantum codes with low redundancy.

Acknowledgements

This work is supported by the DFG within the SPP 'Quanteninformationsverarbeitung'. One of us (A.D.) acknowledges financial support by the DAAD. We are grateful to Thomas Beth, Chris Charnes and Markus Grassl for numerous stimulating discussions.

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