

**Thermodynamic properties of confined interacting Bose gases: A renormalization-group approach**

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A renormalization-group method is developed with which thermodynamic properties of a weakly interacting, finite-size Bose gas can be investigated. Thereby effects originating from a confining potential are taken into account by periodic boundary conditions and by treating the resulting discrete energy levels of the confined degrees of freedom properly. The resulting density of states modifies the flow equations of the renormalization group in momentum space. It is shown that as soon as the characteristic length of confinement becomes comparable to the thermal wavelength of a weakly interacting and trapped Bose gas its thermodynamic properties are changed significantly. This is exemplified by investigating characteristic bunching properties of the interacting Bose gas which manifest themselves in the second-order coherence factor.

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**I. INTRODUCTION**

The experimental realization of Bose-Einstein condensation of trapped, ultracold, weakly interacting atomic gases [1,2] has renewed the interest in their thermodynamic properties. Definitely, many of these properties have already been studied more than 30 years ago in connection with the theoretical efforts to describe superfluidity of strongly interacting helium atoms. However, in these recent experimental realizations the physics of the trapping process and of the internal structure of the trapped bosons still pose interesting theoretical questions which are not yet understood completely [3].

So far the majority of theoretical approaches aiming at describing thermodynamic properties of weakly interacting, trapped Bose gases has concentrated on mean-field approaches [3]. At zero temperature these mean-field approaches lead to the well-known Gross-Pitaevski equation and to their multicomponent generalizations. These theoretical descriptions have proven useful in obtaining first quantitative understandings of many recent trapping experiments [2]. However, it is known that mean-field theories do not yield an accurate description of thermodynamic properties of interacting gases close to the transition point of a phase transition [4]. Furthermore, with improving experimental accuracy it is expected that in the near future possible deviations from mean-field results might become accessible experimentally even in regimes far away from transition points.

Motivated by these prospects in this paper a theoretical description of thermodynamic properties of confined, weakly interacting Bose gases is developed which is based on renormalization-group (RG) methods [5–7]. RG methods are known to yield descriptions of thermodynamic partition functions which transcend mean field theories and which are also capable of describing the dynamical region close to a second-order phase transition accurately. Furthermore, by these methods certain universal quantities, such as critical exponents, might even be evaluated by analytical means. Typically the evaluation of thermodynamic partition functions with the help of RG methods is only slightly more complicated than by mean-field methods. Thus, these RG methods represent a convenient theoretical approach complementing elaborate numerical Monte Carlo simulations of par-

tion functions with which one may also test limitations of mean-field theories.

In the special case of an unconfined, homogeneous three-dimensional interacting Bose gas such a RG approach has already been developed recently by Bijlsma and Stoof [8]. However, their quantitative predictions of the dependence of the transition temperature on the scattering length, for example, have been more than one order of magnitude larger than corresponding numerical predictions which are based on elaborate Monte Carlo simulations of the relevant partition function [9]. Recently this discrepancy has been clarified [10–12]. These studies are in satisfactory agreement with the RG calculations of Bijlsma and Stoof [8] and with recent experiments [13] and they attribute the numerical inaccuracies of the previous Monte Carlo approach to incorrect extrapolation procedures. This example demonstrates that despite extensive previous work there are still open questions even in the context of the traditional case of a homogeneous and unconfined interacting Bose gas. In addition, the experimental possibility to control and manipulate trapped Bose condensates has raised many interesting physical questions most of which have not been tackled yet theoretically beyond the framework of mean-field approximations.

In this paper a RG approach is developed which is capable of determining thermodynamic properties of weakly interacting and confined Bose gases. This approach aims at describing physical effects of spatial confinement beyond the local-density approximation by modeling effects originating from confinement by periodic boundary conditions. However, contrary to the usual continuum-limit approximation within our approach the required summations over all possible states of the atomic center-of-mass motion are not replaced by integrations but the discrete energy levels resulting from the confined degrees of freedom are taken into account. Thus the presented approach should yield a valid description even in cases in which the thermal wavelength of an interacting Bose gas becomes comparable to the characteristic length of confinement and in which the local-density approximation breaks down. Indeed in the subsequent treatment it is demonstrated that as soon as the confinement length becomes comparable to the thermal wavelength of the interacting Bose gas thermodynamic properties of an inter-

acting Bose gas are changed significantly. This characteristic behavior is exemplified by investigating various thermodynamic properties of an interacting Bose gas, such as its bunching properties which manifest themselves in the second-order coherence factor and which are accessible to experimental observation. Though the presented RG approach differs from the one developed by Bijlsma and Stoof [8] previously it is demonstrated that resulting critical properties, such as the dependence of the transition temperature on the scattering length, are in satisfactory agreement with these previous RG results and with other recent calculations [10–12].

The paper is organized as follows: In Sec. II the general theoretical approach is developed. Starting from the path-integral representation of the grand canonical partition function an RG method is developed for its evaluation. As effects of a confining potential are taken into account by periodic boundary conditions which still guarantee translational invariance of the problem the RG procedure is developed in momentum space. As a main approximation the random-phase approximation (RPA) [14] is used which is applied to the symmetry broken phase. In Sec. III resulting numerical results are presented. Firstly, we concentrate on characteristic properties of an unconfined homogeneous three-dimensional Bose gas which have been of interest recently. Secondly, the influence of finite-size effects is exemplified by investigating isothermal properties of the pressure and of the second order coherence factor of an interacting Bose gas.

## II. THEORETICAL TREATMENT

In this section a RG approach is developed for the evaluation of the grand canonical partition function of an interacting and confined Bose gas in  $D$  spatial dimensions. Effects of confinement are taken into account by periodic boundary conditions and by proper summations over the discrete energies associated with the confined degrees of freedom. Accordingly, the RG approach is developed in momentum space. The RG flow equations are derived for the temperature regime below the phase transition with the help of the random-phase approximation [14]. This dynamical regime is characterized by a nonzero-order parameter. Details of the derivation of these RG equations are given in Appendix A. For the sake of completeness the RG equations that apply to the dynamical region above the phase transition, i.e., in the region of a vanishing order parameter, are summarized in Appendix B. These latter equations have already been discussed in detail previously [8,14].

### A. Path-integral representation of the partition function

Thermodynamic properties of an interacting system of bosons can be described by its (grand-canonical) partition function [15]

$$Z(z, \beta) = \text{Tr}[e^{-\beta(\mathbf{H} - \mu\mathbf{N})}] \quad (1)$$

with the fugacity  $z = e^{\beta\mu}$  ( $\mu$  denotes the chemical potential) and with the inverse temperature  $\beta = 1/(kT)$  ( $k$  and  $T$  are the Boltzmann constant and temperature). The Hamiltonian

and the number operator of the interacting bosons are denoted  $\mathbf{H}$  and  $\mathbf{N}$ , respectively. For interacting bosons this partition function can be represented as a functional integral over a complex scalar field  $\Phi$ . In  $D$  spatial dimensions this path-integral representation is given by [7,15]

$$Z(z, \beta) = \int d[\Phi(\xi, \Theta), \Phi(\xi, \Theta)^*] e^{-S(\Phi, \Phi^*)} \quad (2)$$

with the (dimensionless) action

$$\begin{aligned} S(\Phi, \Phi^*) = & \int_0^{\beta/\beta_\Lambda} \int_{V\Lambda^D} d\Theta d^D \xi \\ & \times \left\{ \Phi^*(\xi, \Theta) \left[ -\frac{1}{2} \Delta_\xi - M + \frac{\partial}{\partial \Theta} \right] \Phi(\xi, \Theta) \right. \\ & + \frac{1}{2} \int_{V\Lambda^D} d^D \xi' |\Phi(\xi, \Theta)|^2 \\ & \left. \times \left[ \beta_\Lambda V \left( \frac{\xi - \xi'}{\Lambda} \right) \right] |\Phi(\xi', \Theta)|^2 \right\}. \quad (3) \end{aligned}$$

The volume within which the Bose system is confined to is denoted  $V$ . For the sake of simplicity this volume will be assumed to be of cubic shape and the resulting spatial confinement will be described by imposing periodic boundary conditions on the complex field  $\Phi(\xi, \Theta)$ . In Eq. (2) scaled quantities have been introduced which involve a yet arbitrary characteristic momentum ( $\hbar\Lambda$ ) whose associated inverse temperature is given by  $\beta_\Lambda = m/(\hbar^2\Lambda^2)$ . The mass of the interacting bosons is denoted  $m$ . The scaled chemical potential  $M$  is defined by  $M = \mu\beta_\Lambda$ . The two-body interaction potential between the bosons is denoted  $V(\mathbf{x} - \mathbf{x}')$  and  $\xi = \mathbf{x}\Lambda$  and  $\Theta = \tau/(\hbar\beta_\Lambda)$  are scaled spatial coordinates and the scaled imaginary time. The fugacity  $z$  appearing in Eq. (3) is related to the chemical potential  $M$  by [7]

$$\ln z = M - \frac{1}{2} \beta_\Lambda V(\mathbf{x} = \mathbf{0}). \quad (4)$$

The integration measure appearing in Eq. (2) is defined later (cf. Eq. (9)).

Due to the trace operation involved in the evaluation of the partition function of Eq. (1) the functional integration in Eq. (2) has to be performed over all complex fields  $\Phi(\xi, \Theta)$  which are periodic in the imaginary scaled time  $\Theta$ , i.e.,  $\Phi(\xi, \Theta) = \Phi(\xi, \Theta + \beta/\beta_\Lambda)$  [7]. If one also imposes spatial periodic boundary conditions which still preserve the translational invariance of the problem it is convenient to transform the functional integral of Eq. (2) into the momentum-frequency representation. This is achieved by Fourier transforming the complex fields according to

$$\Phi(\xi, \Theta) = \sum_{\mathbf{m}, n} \left[ \frac{e^{i\mathbf{k}_\mathbf{m} \cdot \xi}}{\sqrt{V\Lambda^D}} \right] \left[ \frac{e^{-i\omega_n \Theta}}{\sqrt{\beta/\beta_\Lambda}} \right] \varphi_{\mathbf{m}n} \quad (5)$$

with the Matsubara frequencies  $\omega_n = 2\pi n\beta_\Lambda/\beta$  and with the scaled wave vectors  $\mathbf{k}_\mathbf{m}^{(i)} = 2\pi\mathbf{m}^{(i)}/(L_i\Lambda)$  ( $\mathbf{m}^{(i)}$  denotes a

vector whose components are  $0, \pm 1, \pm 2, \dots$ . Thereby the quantities  $L_i$  ( $i = 1, \dots, D$ ) denote the lengths of the confining cubic volume. Defining  $(\hbar\Lambda)$  as the ultraviolet momentum cutoff of the path integral, which is much larger than all physically significant momenta, implies that the scaled wave vectors appearing in Eq. (5) are restricted to the region  $0 \leq |\mathbf{k}_m| \leq 1$ . Inserting this Fourier decomposition into Eq. (3) one obtains

$$S(\Phi, \Phi^*) = \sum_{\mathbf{m}n} [-i\omega_n + \epsilon_{\mathbf{m}} - M] |\varphi_{\mathbf{m}n}|^2 + (V\Lambda^D)^{-1} (\beta/\beta_\Lambda)^{-1} \times \frac{1}{2} \sum_{\mathbf{m}_1 n_1; \dots; \mathbf{m}_4 n_4} G(\mathbf{k}_1 + \mathbf{k}_3) (\varphi_{-\mathbf{m}_1 - n_1})^* \quad (6)$$

$$\times (\varphi_{-\mathbf{m}_2 - n_2})^* \varphi_{\mathbf{m}_3 n_3} \varphi_{\mathbf{m}_4 n_4} \times \delta(\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \mathbf{m}_4) \times \delta(n_1 + n_2 + n_3 + n_4) \quad (7)$$

with  $\epsilon_{\mathbf{m}} = \mathbf{k}_m^2/2$ . ( $\delta$  denotes the Kronecker-delta function.) The two-particle interaction is characterized by its Fourier components

$$G(\mathbf{k}) = \int_{V\Lambda^D} d^D \xi e^{i\mathbf{k}\xi} \beta_\Lambda V(\xi/\Lambda). \quad (8)$$

In the subsequent discussion we assume that the two-particle interaction is of short range so that  $G(\mathbf{k})$  is independent of momentum, i.e.,  $G(\mathbf{k}) = G$ . Thus, introducing the momentum cutoff  $(\hbar\Lambda)$  the path integral of Eq. (2) becomes [7,16]

$$Z(z, \beta) = \left[ \prod_{\mathbf{m}n} \int \frac{d^2 \varphi_{\mathbf{m}n}}{N_{\mathbf{m}n}} \right] e^{-S(\Phi, \Phi^*)}. \quad (9)$$

The normalization factors

$$N_{\mathbf{m}n} = \frac{\pi}{\zeta_{\mathbf{m}} \beta_\Lambda / \beta - i\omega_n} \quad (10)$$

with

$$2 \sinh(\zeta_{\mathbf{m}}/2) = e^{(\beta/\beta_\Lambda)(\epsilon_{\mathbf{m}} - M)/2}$$

guarantee that in the limit of vanishing interactions, i.e.,  $G \equiv 0$ , the partition function reduces to the well-known expression for the ideal Bose gas [7], i.e.,

$$Z(z, \beta) = \prod_{\mathbf{m}} \{ [1 - e^{-(\beta/\beta_\Lambda)(\epsilon_{\mathbf{m}} - M)}]^{-1} \}. \quad (11)$$

As expected on physical grounds,  $Z(z, \beta)$  should become independent of the momentum cutoff  $(\hbar\Lambda)$  for  $\beta/\beta_\Lambda, 1/M \gg 1$ . Obviously this requirement is fulfilled for the ideal gas.

## B. Renormalization-group approach for the evaluation of the partition function

One of the simplest methods of evaluating the partition function of Eq. (9) is the mean-field approximation [16]. Thereby one expands the field  $\Phi(\xi, \Theta)$  appearing in Eq. (2) quadratically around the most probable, uniform field configuration

$$\bar{\varphi} = \sqrt{(V\Lambda^D)(\beta/\beta_\Lambda)M/G}, \quad (12)$$

which is determined by the requirement  $\delta S(\Phi, \Phi^*)|_{\Phi = \Phi^* = \bar{\varphi}/\sqrt{(V\Lambda^D)(\beta/\beta_\Lambda)}} = 0$ . The resulting Gaussian integrals of this quadratic expansion can be performed in a straightforward way. For  $G > 0$  this most probable field configuration is nonzero only if  $M > 0$ .

A more sophisticated and more accurate method of evaluating the partition function makes use of renormalization-group methods. The basic idea of the renormalization-group procedure is to perform the integrations in Eq. (9) successively [5]. In each step only field components  $\delta\varphi_{\mathbf{m}n}$  are integrated out whose momenta are located in an infinitesimally small momentum shell around the maximum momentum  $(\hbar\Lambda)$ , i.e., for which  $e^{-l} < |\mathbf{k}_m| < 1$  with  $0 < l \ll 1$ . All other small-momentum field components  $\varphi_{\mathbf{m}n}$  outside this momentum shell, which constitute the field  $\phi_{<}$ , are left unchanged. If only an infinitesimal momentum shell is integrated out, a quadratic expansion of  $S(\Phi, \Phi^*)$  with respect to the fields  $\delta\varphi_{\mathbf{m}n}$  is sufficient for the further evaluation of the partition function [17].

The desired aim of this integration over the large-momentum field components is to obtain a new scaled partition function which is similar to the original one except for possible renormalizations of the characteristic parameters  $M$  and  $G$  [4,5]. In particular this new scaled partition function has to have the same momentum cutoff  $(\hbar\Lambda)$ . By repeated application of this transformation one eventually integrates out all momentum components and obtains the value of the partition function. However, in general this desired aim can be achieved only approximately. Two commonly used approximation schemes involved are perturbation theory [5] and the random-phase approximation (RPA) [14]. In our subsequent treatment the RPA is employed. It is particularly useful in the case of a nonvanishing most probable configuration  $\bar{\varphi}$  [8].

In the subsequent development we separate the zero-temperature contribution from the grand thermodynamic potential  $\Omega(M, \beta) = -\ln Z(z, \beta)/(\beta/\beta_\Lambda)$  according to

$$\Omega(M, \beta) = \Omega(M, \beta \rightarrow \infty) + \omega(M, \beta). \quad (13)$$

For the zero-temperature contribution  $\Omega(M, \beta \rightarrow \infty)$  accurate approximations are available. In the simplest form of the mean-field approximation this zero-temperature contribution is given by [16]

$$\Omega(M, \beta \rightarrow \infty) = -(V\Lambda^D) \frac{M^2}{2G}. \quad (14)$$

Note that the depletion term [18] is not included in Eq. (14). This term is of order  $O(\sqrt{Na^3/V})$  and is thus negligibly small for a weakly interacting Bose gas.

It is shown in Appendix A that on integrating out momentum components of the fields  $\varphi_{\mathbf{m}l}$  which are located in the momentum shell  $e^{-l} \leq |\mathbf{k}_{\mathbf{m}}| \leq 1$  ( $0 < l \ll 1$ ) the RPA yields the differential scaling relation

$$\frac{d\omega(l)}{dl} = \frac{(V\Lambda^D)}{(\beta/\beta_\Lambda)} d(l) e^{-Dl} \ln(1 - e^{-\lambda(l)}) \quad (15)$$

with  $\omega(M, \beta) \equiv \omega(l \rightarrow \infty)$ . Thereby the quantity  $\lambda(l) = [\beta(l)/\beta_\Lambda] \sqrt{\epsilon_{>} [\epsilon_{>} + 2M(l)]}$  reflects the Bogoliubov dispersion relation at each step of the renormalization with the scaled inverse temperature  $\beta(l) = \beta e^{-2l}$ . The differential scaling of the chemical potential  $M(l)$  and of the coupling strength  $G(l)$  is governed by the RG equations

$$\begin{aligned} \frac{dM(l)}{dl} &= 2M(l) + d(l) [\beta(l)/\beta_\Lambda] \\ &\times \frac{\{\coth[\lambda(l)/2] - 1\}}{2\lambda(l)} [2M(l)G(l) - G(l)2\epsilon_{>}] \\ &- d(l) [\beta(l)/\beta_\Lambda]^3 \frac{M(l)G(l)}{2\lambda(l)^2} \left\{ \frac{1}{2 \sinh^2[\lambda(l)/2]} \right. \\ &\left. + \frac{\{\coth[\lambda(l)/2] - 1\}}{\lambda(l)} \right\} [2\epsilon_{>} + M(l)]^2, \quad (16) \end{aligned}$$

$$\begin{aligned} \frac{dG(l)}{dl} &= -(D-2)G(l) + d(l) [\beta(l)/\beta_\Lambda] \\ &\times \frac{\{\coth[\lambda(l)/2] - 1\}}{2\lambda(l)} 3G(l)^2 - d(l) [\beta(l)/\beta_\Lambda]^3 \\ &\times \frac{G(l)^2}{2\lambda(l)^2} \left\{ \frac{1}{2 \sinh^2[\lambda(l)/2]} + \frac{\{\coth[\lambda(l)/2] - 1\}}{\lambda(l)} \right\} \\ &\times [2\epsilon_{>} + M(l)]^2. \quad (17) \end{aligned}$$

The (scaled) energy of the eliminated (infinitesimal) momentum shell is denoted  $\epsilon_{>} = 1/2$ . The number of states per unit volume inside such an infinitesimal momentum shell  $e^{-l} \leq |\mathbf{k}_{\mathbf{m}}| \leq 1$  ( $0 < l \ll 1$ ) is given by  $[d(l)(\Lambda^D l)]$ . In the continuum limit, in  $D$  spatial dimensions one obtains the  $l$ -independent density

$$d(l) = \frac{\Omega_D}{(2\pi)^D} \quad (18)$$

with  $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$  denoting the surface of a  $D$ -dimensional sphere. ( $\Gamma$  denotes the Euler-Gamma function.)

Equations (15)–(17) constitute the set of differential RG equations from which the temperature-dependent part of the grand thermodynamic potential  $\omega(M, \beta) \equiv \omega(l \rightarrow \infty)$  can be evaluated. They have to be solved subject to the initial conditions

$$\begin{aligned} M(l=0) &= M, \\ G(l=0) &= G, \\ \omega(l=0) &= 0. \end{aligned} \quad (19)$$

The RG equations (15)–(17) are valid for an arbitrary number of spatial dimensions  $D$  of the interacting Bose gas as long as effects of confinement can be described by periodic boundary conditions. Effects of confinement are characterized by the quantity  $d(l)$  which reduces to the result of Eq. (18) in the continuum limit, i.e., in the absence of any confining influence. If one or more degrees of freedom are confined,  $d(l)$  is modified as one has to take into account the resulting discrete energy levels in the confined degree of freedom. In a case where one degree of freedom of the interacting Bose gas is confined, for example, one obtains

$$\begin{aligned} d(l) &= \frac{\Omega_D}{(2\pi)^D} \frac{1}{2} \sum_{M=-\infty}^{\infty} \int_{-1}^1 dz e^{izM\Lambda e^{-l}L_z} \\ &= \frac{\Omega_D}{(2\pi)^D} \frac{\pi + 2\pi[L_z e^{-l}\Lambda/(2\pi)]}{L_z e^{-l}\Lambda}. \quad (20) \end{aligned}$$

Thereby  $L_z$  is the length of the confining (square-well) potential and  $[x]$  denotes the largest integer which is smaller or equal to  $x$ .

For dimensions less than or equal to 2 one has to reconsider the validity of the  $\mathbf{k}$ -independent approximation  $G(\mathbf{k}) = G$  of the interparticle potential. If one assumes a spherically symmetric short-ranged interparticle interaction then at low energies one may expand  $G(\mathbf{k}) = G_0 + G_2 \mathbf{k}^2 + G_4 \mathbf{k}^4 + \dots$ . It is straightforward to demonstrate that for  $l \rightarrow \infty$  [which implies  $\beta(l) = \beta e^{-2l} \rightarrow 0$ ] the trivial scaling of  $G_k$  ( $k=0, 2, 4, \dots$ ) is given by  $G_k(l) = G_k e^{[2-(D-2+k)l]}$ . This implies that already at  $D=2$  not only  $G_0$  but also  $G_2$  is no longer irrelevant. Thus for a reliable description of the temperature dependence of two-dimensional Bose gases also the  $\mathbf{k}$  dependence of the interparticle interaction has to be taken into account at least to lowest order.

It is worth mentioning that Eqs. (16)–(17) reduce to the corresponding mean-field results if one neglects fluctuations of the most probable configuration  $\bar{\varphi}$ . Formally this is equivalent to setting  $d(l) \equiv 0$  [or to setting  $\Delta(\Phi_{<}) \equiv 0$  in Eq. (A8) of Appendix A]. In this case the scaling of the characteristic parameters  $M(l)$  and  $G(l)$  is governed by the trivial scaling which depends on the dimension  $D$  of the system only. Furthermore, these mean-field results also appear in the limit of zero temperature where  $\beta \rightarrow \infty$ .

All thermodynamic properties which are derivable from the grand thermodynamic potential  $\Omega(M, \beta)$  can be determined by these RG equations. Thus pressure  $P$  and number of particles  $N$ , for example, are determined by the relations

$$\frac{P\lambda_{th}^D}{kT} = -\Omega(M, \beta) \frac{\beta/\beta_\Lambda}{V\Lambda^D} (2\pi\beta/\beta_\Lambda)^{D/2}, \quad (21)$$



$$\frac{N\lambda_{th}^D}{V} = -(2\pi\beta/\beta_\Lambda)^{D/2} \frac{\partial\Omega}{\partial M}(M, \beta) \quad (22)$$

with the thermal wavelength  $\lambda_{th} \equiv \sqrt{2\pi\hbar^2/(mkT)} = \sqrt{2\pi\beta/\beta_\Lambda}/\Lambda$ . Another physical quantity which is of current experimental interest is the spatially averaged second-order coherence factor of the interacting Bose gas. In terms of the conjugate quantized field operators  $\hat{\psi}(\mathbf{x})$  and  $\hat{\psi}^\dagger(\mathbf{x})$  of the Bose gas it is defined by

$$g^{(2)}(0) = \frac{\frac{1}{V} \int d^D x \langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x}) \rangle}{\left[ \frac{1}{V} \int d^D x \langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \rangle \right]^2} \quad (23)$$

with  $\langle \cdot \rangle$  denoting averaging over the thermodynamic equilibrium distribution of the interacting Bose gas. This definition is completely analogous to the corresponding quantity in the quantum optical context. As apparent from Eq. (1) this quantity is related to the grand thermodynamic potential of Eq. (14) by

$$g^{(2)}(0) = 2(V\Lambda^D) \frac{\frac{\partial\Omega}{\partial G}(M, \beta, G)}{\left[ \frac{\partial\Omega}{\partial M}(M, \beta, G) \right]^2}. \quad (24)$$

In the partial derivatives entering Eq. (24) the grand thermodynamic potential  $\Omega$  has to be considered as a function of the fundamental parameters  $(M, \beta, G)$ . In order to keep the notation as simple as possible this dependence on  $G$  has not been indicated explicitly in all other previous equations.

Besides taking into account effects originating from confinement the RG flow equations (15)–(17) differ from the ones derived previously by Bijlma and Stoof [8] also in two other respects. First of all, separating off the zero-temperature contribution from the grand thermodynamic potential according to Eq. (13) changes the properties of the flow equations. These changes are particularly prominent in the region where  $\lambda(l) \gg 1$ , i.e., in the regions where the value of the effective momentum cutoff is still large. The RG equations (15)–(17) have the property that for  $\lambda(l) \gg 1$  the influence of the nontrivial scaling which is proportional to the characteristic level density  $d(l)$  is exponentially small. This implies that in the limit of zero temperature, i.e., for  $\beta \rightarrow \infty$ , we obtain the result  $G(l=0) = G(l \rightarrow \infty)$  which gives rise to the simple relation  $G(l=0) = 4\pi a\Lambda$  between the scattering length  $a$  and the initial value of the scaled coupling strength  $G(l=0)$ . Secondly, evaluating the particle density directly from the grand thermodynamic potential according to Eq. (22) without any further approximations yields results which are different from the ones obtainable from the approximate flow equation used in Ref. [8].

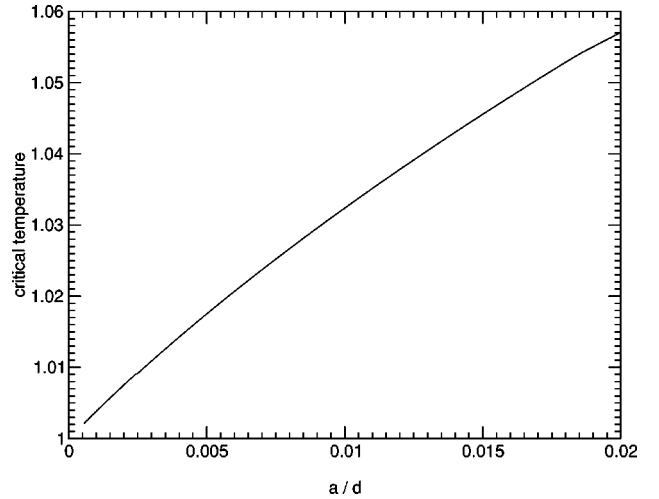


FIG. 1. Dependence of the critical temperature  $[T_c/T_o]$  of an interacting Bose gas on the scattering length  $a$ . [ $d = (V/N)^{1/3}$  denotes the mean distance between atoms and  $T_o$  is the critical temperature of an ideal Bose gas.]

### III. NUMERICAL RESULTS

In this section thermodynamic properties of a confined interacting Bose gas are investigated on the basis of the RG approach of Sec. II. We concentrate on cases in which one degree of freedom of a three-dimensional Bose gas is confined partly by changing the size of the confining potential in one degree of freedom. According to the theoretical developments of Sec. II it is assumed that the resulting physical effects of such a confinement can be described by periodic boundary conditions.

Let us first of all concentrate on the critical properties of a homogeneous interacting Bose gas in three spatial dimensions. Though the corresponding characteristic critical exponents are already well known [19], nonuniversal thermodynamic properties of a three-dimensional homogeneous Bose gas are still the subject of controversial discussions [8,10,11,20]. One example of such a thermodynamic property which is of topical interest is the critical temperature and its dependence on the scattering length  $a$  characterizing the strength of the interparticle interaction. In three spatial dimensions this scattering length is related to the characteristic interaction parameter  $G$  of Sec. II by  $G = 4\pi a\Lambda$ . The ratio between the critical temperature  $T_c$  of an interacting Bose gas and the corresponding value of an ideal Bose gas  $T_o$  can be determined easily from Eq. (22) with the help of the relation  $T_c/T_o = \{[(N/V)\lambda_{th}^3]/2.612\}^{-2/3}$ . [Thereby the relation  $(N_o/V_o)\lambda_{th}^3 = \zeta(3/2) \approx 2.612$  has been used which applies to an ideal homogeneous Bose gas at its transition temperature  $T_o$ .] For this purpose one has to evaluate the derivative  $\partial\Omega/\partial M(M, \beta)$  along the critical trajectory which merges into the unstable fixed point and whose properties will be discussed later. The resulting predictions based on the RG approach of Sec. II are depicted in Fig. 1. One realizes that in this particular case of a repulsive interparticle

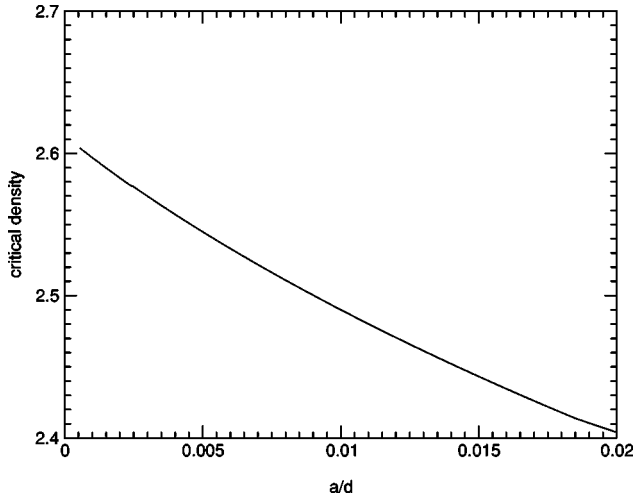


FIG. 2. Dependence of the critical density  $[(N/V)\lambda_{th}^3]$  of an interacting Bose gas on the scattering length  $a$ . ( $\lambda_{th}$  denotes the thermal wavelength.)

interaction, i.e.,  $a > 0$ , the critical temperature increases with increasing scattering length. The quantitative dependence shown in Fig. 1 is approximated to a high degree of accuracy by the polynomial

$$T_c = T_o [1.000 + 3.423(a/d) - 29.986(a/d)^2 + 145.183(a/d)^3]. \quad (25)$$

This polynomial dependence is indistinguishable from the RG results of Fig. 1 for scattering lengths  $a$  in the range  $0 \leq a \leq 0.02d$ . [ $d = (V/N)^{1/3}$  is the mean distance between the interacting bosons.] Therefore this polynomial dependence is not depicted in Fig. 1 explicitly. From this approximation we conclude that in the limit of a vanishing interparticle interaction, i.e.,  $a \rightarrow 0$ , one obtains  $[(T_c - T_o)/T_o] = 3.423(a/d) + O[(a/d)^2]$ . Thus Eq. (25) is in satisfactory agreement with other recent theoretical results [8,10–12]. The low-density slopes predicted by these authors differ from our low-density slope of magnitude 3.423 by less than 40%. The recently observed experimental value for this low-density slope is  $5.1 \pm 0.9$  [13] and is slightly larger. In view of the uncertainty of the experimental results the agreement between these theoretical predictions and these experimental results is satisfactory.

The critical density and its monotonically decreasing dependence on the scattering length is shown in Fig. 2. Note that in the limit  $a \rightarrow 0$  one obtains the result  $(N/V)\lambda_{th}^3 \rightarrow 2.612$  as expected for an ideal Bose gas.

The critical quantities depicted in Figs. 1 and 2 can be obtained from the RG approach described in Sec. II in a straightforward way from the knowledge of the critical trajectory. This critical trajectory is the stable manifold of the unstable fixed point for  $D=3$ . This unstable fixed point is given by  $M_c = 1/2$  and  $g_c = \pi^2/2$  with  $g(l) = G(l)/[\beta(l)/\beta_\Lambda]$ . The associated eigenvalues of the linearized RG equations for  $M(l)$  and  $g(l)$  are  $\lambda_1 = (3 - \sqrt{33})/6$  and  $\lambda_2$

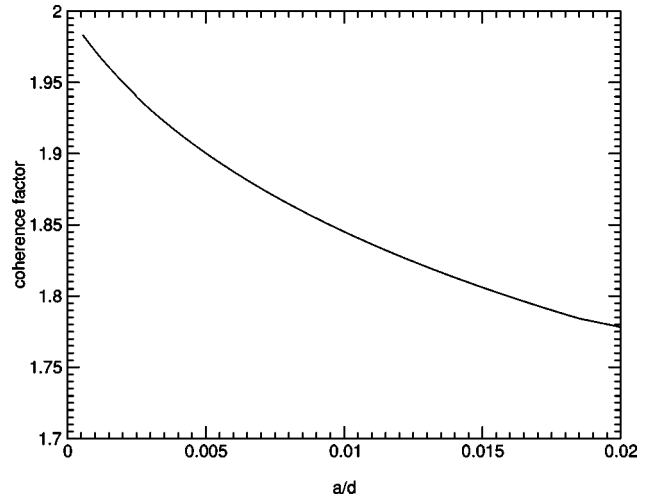


FIG. 3. Dependence of the second-order coherence factor  $g^{(2)}(0)$  of an interacting Bose gas on the scattering length  $a$  at the critical temperature.

$= (3 + \sqrt{33})/6$ . The latter eigenvalue which is associated with the unstable manifold determines the various critical exponents of the second-order phase transition. Thus the scaling of the correlation length of the interacting Bose gas, for example, is determined by the characteristic exponent  $\nu = 1/\lambda_2 = 0.686139$  which compares well with the known result of  $\nu = 0.67$  [19] which is based on a perturbative approach to the RG above the transition point. Though the implementation of the RG method presented in Sec. II involves approximations whose accuracy is difficult to access, this good agreement between the critical exponents suggests that the presented RG approach should be particularly reliable close to the transition point.

One of the quantities which is accessible experimentally and which has received attention recently [2] is the spatially averaged second-order coherence factor of an interacting Bose gas. This quantity can be determined directly from the RG equations (15)–(17) of Sec. II by using relation (24). Its critical value at the transition point and its dependence on the scattering length are depicted in Fig. 3. For vanishingly small values of the interatomic interaction this second-order coherence factor approaches the value of 2. This particular value is also known to characterize photon bunching of a chaotic electromagnetic field. With increasing scattering length  $a$  the critical value of the second-order coherence factor decreases thus indicating that with increasing repulsive interactions the bosons tend to avoid each other.

Let us now investigate the influence of confining this homogeneous interacting Bose gas with respect to one degree of freedom, say the  $z$  direction. Furthermore, let us assume that the influence of this confinement can be described quantitatively by periodic boundary conditions. In such a case the resulting density of states  $d(l)$  which enters the RG equations (15)–(17) is given by Eq. (20). In Fig. 4 isotherms of the pressure of the interacting Bose gas are depicted for various values of the characteristic confinement length  $L_z$  in the  $z$  direction. Thereby above the phase transition, i.e., for negative values of the scaled chemical potential  $M(l)$ ,

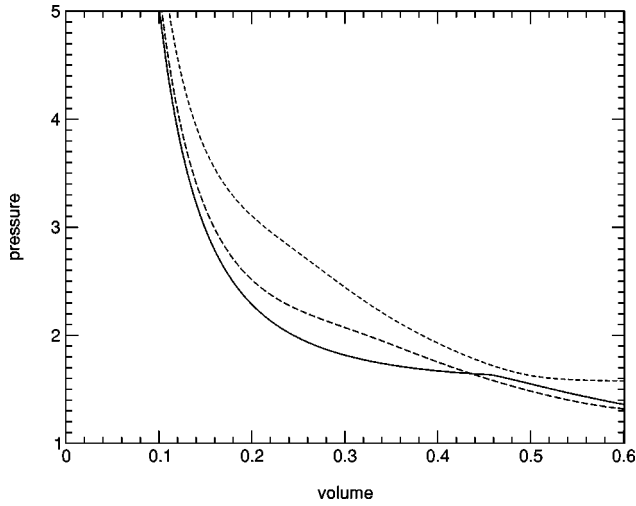


FIG. 4. Isotherm of an interacting Bose gas for various values of the characteristic length of confinement  $L_z$  (solid line,  $L_z/a=10^{12}$ ; dashed line,  $L_z/a=30$ ; dotted line,  $L_z/a=20$ ). The scaled pressure  $[p\lambda_{th}^3/(kT)]$  is depicted as a function of the scaled volume  $[(V/N)\lambda_{th}^{-3}]$ . The thermal wavelength is  $\lambda_{th}=10\sqrt{2\pi}a$  which corresponds to a temperature  $T=1.98\times 10^{-6}$  K for rubidium atoms with a scattering length  $a=5.3$  nm.

renormalization-group equations have been used which apply to a vanishing order parameter [8].

The temperature of these isotherms is chosen so that  $\lambda_{th}=10\sqrt{2\pi}a\approx 25a$ . For rubidium atoms, for example, with a scattering length of magnitude  $a=5.3$  nm this condition corresponds to a temperature of  $T=1.98\times 10^{-6}$  K. If the length of confinement  $L_z$  is large in comparison with both the thermal wavelength  $\lambda_{th}$  and the scattering length  $a$  (compare with full curve), the characteristic signatures of a well pronounced second-order phase transition are realized at a critical volume of magnitude  $(V/N)\lambda_{th}^{-3}=0.456$  in agreement with the result of Fig. 2. As soon as the length of confinement  $L_z$  becomes comparable to the thermal wavelength  $\lambda_{th}$  the pressure dependence is modified significantly. The most prominent feature of the depicted pressure dependence is the disappearance of the characteristic signature of the second-order phase transition and the smoothing of this pressure dependence.

In view of the recent interest in the behavior of the second-order coherence factors of interacting Bose gases its isothermal dependence on the density of the interacting Bose gas is depicted in Fig. 5. The parameters chosen are the same as the ones used in Fig. 4. Again for very large values of the confinement length  $L_z$  one notices a well pronounced second-order phase transition at the same critical volume as in Fig. 4. Far below the transition point, i.e., for very small densities, the second-order coherence factor approaches a value of 1 which is the expected value for a Bose condensate at zero temperature. Lowering the length of confinement  $L_z$  in the  $z$  direction the characteristic signature of the second-order phase transition disappears.

#### IV. SUMMARY

In summary, a theoretical description of thermodynamic properties of an interacting confined Bose gas has been de-

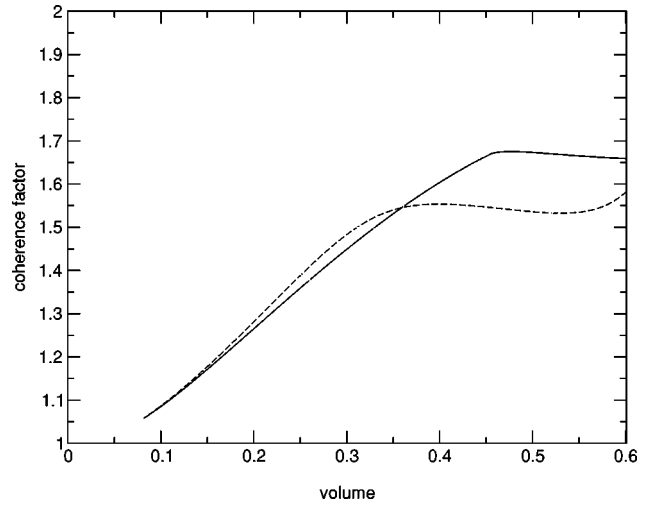


FIG. 5. Isotherm of the second-order coherence factor  $g^{(2)}(0)$  of an interacting Bose gas for  $L_z/a=10^{12}$  (solid line) and  $L_z/a=30$  (dashed line) and for the same temperature as in Fig. 4.

veloped which is based on a RG approach. Thereby physical effects which originate from the presence of confinement have been taken into account by periodic boundary conditions. Thus this approach yields a proper description of thermodynamic properties also in cases in which the characteristic length of confinement becomes comparable to the thermal wavelength of a Bose gas and in which local-density approximations break down. A more realistic and more complete treatment of boundary conditions will be postponed to subsequent work. It has been demonstrated that this approach gives a reliable description of the critical behavior of an interacting Bose gas which is consistent with recent theoretical and experimental work. Furthermore, concentrating on the bunching properties of an interacting Bose gas the influence of the confinement length on the characteristic aspects of the phase transition has been worked out.

#### ACKNOWLEDGMENT

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#### APPENDIX A

In this appendix the derivation of the RG transformation of Sec. II is outlined in which an infinitesimal momentum shell of the partition function  $Z(z,\beta)$  is integrated out. These equations are valid in the dynamical regime below the transition temperature which is characterized by a nonzero order parameter and a positive renormalized chemical potential  $M(l)$ .

The starting point is the path-integral representation of Eq. (9) with the (scaled) action functional of Eq. (6). Introducing a symmetry-breaking, real-valued uniform field configuration  $\bar{\varphi}$  this action functional can be written in the form

$$\begin{aligned}
S(\Phi, \Phi^*) = & \left\{ -M\bar{\varphi}^2 + \frac{1}{2}g\bar{\varphi}^4 \right\} + [\varphi_{00} + \varphi_{00}^*][ -M\bar{\varphi} + g\bar{\varphi}^3 ] + \sum_{\mathbf{m}, n} [ -i\omega_n + \epsilon_{\mathbf{m}} - M + 2g\bar{\varphi}^2 ] |\varphi_{\mathbf{m}n}|^2 \\
& + \sum_{\mathbf{m}, n} \frac{1}{2}g\bar{\varphi}^2 [\varphi_{-\mathbf{m}-n}\varphi_{\mathbf{m}n} + \varphi_{-\mathbf{m}-n}^*\varphi_{\mathbf{m}n}^*] + \sum_{\mathbf{m}_1, n_1 \dots \mathbf{m}_3, n_3} \bar{\varphi}g [\varphi_{-\mathbf{m}_1-n_1}^*\varphi_{\mathbf{m}_2 n_2}\varphi_{\mathbf{m}_3 n_3} + \varphi_{-\mathbf{m}_1-n_1}^*\varphi_{-\mathbf{m}_2-n_2}^*\varphi_{\mathbf{m}_3 n_3}] \\
& \times \delta(n_1+n_2+n_3)\delta(\mathbf{m}_1+\mathbf{m}_2+\mathbf{m}_3) + \frac{1}{2} \sum_{\mathbf{m}_1, n_1 \dots \mathbf{m}_4 n_4} g\varphi_{-\mathbf{m}_1-n_1}^*\varphi_{-\mathbf{m}_2-n_2}^*\varphi_{\mathbf{m}_3 n_3}\varphi_{\mathbf{m}_4 n_4} \\
& \times \delta(n_1+n_2+n_3+n_4)\delta(\mathbf{m}_1+\mathbf{m}_2+\mathbf{m}_3+\mathbf{m}_4)
\end{aligned} \tag{A1}$$

with  $g = (V\Lambda^D)^{-1}(\beta/\beta_\Lambda)^{-1}G$ . In order to eliminate the infinitesimal momentum shell around the maximum momentum ( $\hbar\Lambda$ ), for which  $e^{-l} < |\mathbf{k}_{\mathbf{m}}| < 1$  ( $0 < l \ll 1$ ), one expands the (scaled) action functional of Eq. (6) up to second order in the large-momentum field components  $\delta\varphi_{\mathbf{m}n}$ , i.e.,

$$S(\Phi, \Phi^*) = S(\Phi_{<}, \Phi_{<}^*) + \frac{1}{2}\delta\varphi^T \mathbf{M} \delta\varphi \tag{A2}$$

with

$$\delta\varphi^T = (\dots, \delta\varphi_{\mathbf{m}n}, (\delta\varphi_{\mathbf{m}n})^*, \delta\varphi_{-\mathbf{m}-n}, (\delta\varphi_{-\mathbf{m}-n})^*, \dots). \tag{A3}$$

Thereby one chooses the symmetry-breaking field  $\bar{\varphi}$  identical to the most probable uniform field configuration of Eq. (12) so that terms linear in  $\varphi_{00}$  vanish in Eq. (A1). The small-momentum field components  $\varphi_{\mathbf{m}n}$  which are not integrated out and which are kept constant constitute the field  $\Phi_{<}$ . The symmetric matrix  $\mathbf{M}$  is given by

$$\mathbf{M} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & A+i\omega_n & B & 0 \\ A+i\omega_n & 0 & 0 & B^* \\ B & 0 & 0 & A-i\omega_n \\ 0 & B^* & A-i\omega_n & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{A4}$$

with

$$\begin{aligned}
A = & \epsilon_{>} - M + 2g \left[ \bar{\varphi}^2 + \bar{\varphi} \sum_{\mathbf{m}n}^< \varphi_{\mathbf{m}n} + \bar{\varphi} \sum_{\mathbf{m}n}^< (\varphi_{\mathbf{m}n})^* \right. \\
& \left. + \sum_{\mathbf{m}n}^< (\varphi_{\mathbf{m}n})^* \varphi_{\mathbf{m}n} \right], \\
B = & g \left[ \bar{\varphi}^2 + 2\bar{\varphi} \sum_{\mathbf{m}n}^< (\varphi_{\mathbf{m}n})^* + \sum_{\mathbf{m}n}^< (\varphi_{\mathbf{m}n})^* (\varphi_{-\mathbf{m}-n})^* \right].
\end{aligned} \tag{A5}$$

The quantity  $\epsilon_{>} = 1/2$  denotes the scaled energy of the eliminated momentum shell. According to the RPA [14], in Eq. (A2) only products of large-momentum field components

$\delta\varphi_{\mathbf{m}n}$  are kept for which the sum of the momenta and of the Matsubara frequencies vanishes. Performing the Gaussian integrations over the fields  $\delta\varphi_{\mathbf{m}n}$  one obtains

$$Z(z, \beta) = \left[ \prod_{\mathbf{m}n}^< \int \frac{d^2\varphi_{\mathbf{m}n}}{N_{\mathbf{m}n}} \right] e^{-S_{\text{new}}(\Phi_{<}, \Phi_{<}^*)} \tag{A6}$$

with the new effective action

$$\begin{aligned}
S_{\text{new}}(\Phi_{<}, \Phi_{<}^*) = & S(\Phi_{<}, \Phi_{<}^*) \\
& - \frac{1}{2} \sum_{\mathbf{m}}' \sum_{n=-\infty}^{\infty} \ln \frac{(\zeta_{\mathbf{m}n}\beta_\Lambda/\beta)^2 + \omega_n^2}{A^2 + \omega_n^2 - |B|^2} \\
= & S(\Phi_{<}, \Phi_{<}^*) \\
& - \frac{1}{2} \sum_{\mathbf{m}}' \ln \frac{e^{(\beta/\beta_\Lambda)(\epsilon_{>} - M)}}{4 \sinh^2[\lambda(\Phi_{<})/2]}.
\end{aligned} \tag{A7}$$

Thereby the relation

$$\sinh^2(x)/x^2 = \prod_{n=-\infty, n \neq 0}^{\infty} \left[ 1 + \left( \frac{x}{n\pi} \right)^2 \right]$$

has been used to perform the summation over all Matsubara frequencies. The quantity  $\lambda(\Phi_{<})$  is given by

$$\lambda(\Phi_{<}) = (\beta/\beta_\Lambda) \sqrt{\epsilon_{>} + 2M - \Delta(\Phi_{<})} \tag{A8}$$

with

$$\begin{aligned}
\Delta(\Phi_{<}) = & C \sum_{\mathbf{m}}^< \sum_{n=-\infty}^{\infty} [\varphi_{\mathbf{m}n} + (\varphi_{\mathbf{m}n})^*] \\
& + D \sum_{\mathbf{m}}^< \sum_{n=-\infty}^{\infty} [\varphi_{\mathbf{m}n}\varphi_{-\mathbf{m}-n} + (\varphi_{\mathbf{m}n})^*(\varphi_{-\mathbf{m}-n})^*] \\
& + (2D + C/\bar{\varphi}) \sum_{\mathbf{m}}^< \sum_{n=-\infty}^{\infty} |\varphi_{\mathbf{m}n}|^2,
\end{aligned} \tag{A9}$$

$$C = (-4\epsilon_{>} - 2M)M/\bar{\varphi},$$

$$D = -3M^2/\bar{\varphi}^2.$$



The symbol  $\langle$  in the summations of Eq. (A9) indicates that only momenta have to be taken into account which have not yet been integrated out. The prime in the sum of Eq. (A7) indicates summation over all the eliminated momentum components. In the continuum limit this latter summation reduces to

$$\sum_{\mathbf{m}}' = (V\Lambda^D) \frac{\Omega_D}{(2\pi)^D} l \quad (0 < l \ll 1). \quad (\text{A10})$$

For the derivation of the RG transformation we first of all split off the zero-temperature contribution according to

$$S_{new}(\Phi_{<}, \Phi_{<}^*) = S(\Phi_{<}, \Phi_{<}^*) + (\beta/\beta_\Lambda) \delta\Omega(M, \beta \rightarrow \infty) + \delta S(M, \beta). \quad (\text{A11})$$

The remaining temperature-dependent contribution  $\delta S(M, \beta)$  is then expanded up to second order in the small-momentum field components which constitute the field  $\Phi_{<}$ , i.e.,

$$\begin{aligned} \delta S(M, \beta) &= - \sum_{\mathbf{m}}' \frac{1}{2} \ln \frac{e^\lambda}{4 \sinh^2(\lambda/2)} \\ &+ \frac{1}{2} \frac{(\beta/\beta_\Lambda)^2 \Delta(\Phi_{<})}{2\lambda} \sum_{\mathbf{m}}' \frac{d}{d\lambda} \ln \frac{e^\lambda}{4 \sinh^2(\lambda/2)} \\ &- \frac{1}{2} \left[ \frac{(\beta/\beta_\Lambda)^2 \Delta(\Phi_{<})}{2\lambda} \right]^2 \sum_{\mathbf{m}}' \frac{1}{2} \frac{d^2}{d\lambda^2} \\ &\times \ln \frac{e^\lambda}{4 \sinh^2(\lambda/2)} \end{aligned} \quad (\text{A12})$$

with  $\lambda = \lambda(\Phi_{<} \equiv 0)$  and with  $\Delta(\Phi_{<})$  as defined by Eq. (A9). Note that the terms involving  $e^\lambda$  result from the separation of the zero-temperature contribution according to Eq. (13). This expanded effective action involves terms linear in the field  $\Phi_{<}$ . These linear terms can be absorbed in a redefinition of the most probable configuration, i.e.,

$$\bar{\varphi}_{new} = \bar{\varphi} + \Delta\bar{\varphi} \quad (\text{A13})$$

with

$$\Delta\bar{\varphi} = \frac{1}{4M} \sum_{\mathbf{m}}' (\beta/\beta_\Lambda)^2 \frac{1}{2\lambda} [\coth(\lambda/2) - 1] C. \quad (\text{A14})$$

Equation (A12) implies a change of the chemical potential, i.e.,

$$M_{new} = M + \Delta M \quad (\text{A15})$$

with

$$\begin{aligned} \Delta M &= - \sum_{\mathbf{m}}' (\beta/\beta_\Lambda)^2 \frac{1}{2\lambda} [\coth(\lambda/2) - 1] D \\ &+ 2 \frac{\Delta\bar{\varphi}}{\bar{\varphi}} M - \sum_{\mathbf{m}}' \frac{1}{2} (\beta/\beta_\Lambda)^4 \frac{1}{4\lambda^2} \\ &\times \left\{ \frac{1}{2 \sinh^2(\lambda/2)} + \frac{1}{\lambda} [\coth(\lambda/2) - 1] \right\} C^2. \end{aligned} \quad (\text{A16})$$

The quantities  $C$  and  $D$  are defined by Eq. (A9). The corresponding change of the scaled two-body coupling strength  $G$  is determined by the requirement that

$$\bar{\varphi}_{new} = \sqrt{(V\Lambda^D)(\beta/\beta_\Lambda)M_{new}/G_{new}} \quad (\text{A17})$$

in accordance with Eq. (12). This implies that

$$G_{new} = G + \Delta G \quad (\text{A18})$$

with

$$\Delta G = G \left[ \frac{\Delta M}{M} - 2 \frac{\Delta\bar{\varphi}}{\bar{\varphi}} \right]. \quad (\text{A19})$$

Equations (A13)–(A16) and (A18) and (A19) characterize the Kadanoff transformation [4], i.e., the elimination of the infinitesimal momentum shell for which  $e^{-l} < |\mathbf{k}_{\mathbf{m}}| < 1$  with  $l \ll 1$ . The RG transformation is completed by performing the scaling transformations which restore the original momentum cut-off ( $\hbar\Lambda$ ), namely,

$$\begin{aligned} \mathbf{k}_{\mathbf{m}'} &= \mathbf{k}_{\mathbf{m}} e^l, \\ \omega_{n'} &= \omega_n e^{2l}, \\ V' &= V e^{-Dl}, \\ \beta' &= \beta e^{-2l}, \\ \varphi'_{n', \mathbf{m}'} &= \varphi_{nm} e^{-l}. \end{aligned} \quad (\text{A20})$$

With  $M(l) = (M + \Delta M)e^{2l}$ ,  $G(l) = (G + \Delta G)e^{(2-D)l}$ ,  $\beta(l) = \beta e^{-2l}$  these scaling relations together with the Kadanoff transformations of Eqs. (A13)–(A16), (A18), and (A19) imply the RG equations of Eqs. (15)–(17).

## APPENDIX B

For the sake of completeness in this appendix the RG equations are summarized which apply to the case of a vanishing order parameter above the transition point. In this dynamical regime the scaled chemical potential  $M(l)$  might become negative. The RG equations which apply to this regime of negative values of  $M(l)$  can be obtained in a straightforward way with the help of second-order perturbation theory by assuming a vanishing order parameter [8,14]. Thus, for  $M(l) < 0$  one obtains the RG flow equations

$$\begin{aligned}
\frac{d\omega(l)}{dl} &= \frac{(V\Lambda^D)}{(\beta/\beta_\Lambda)} d(l) e^{-Dl} \ln(1 - e^{-(\beta/\beta_\Lambda)(\epsilon_{>} - M(l))}), \\
\frac{dM(l)}{dl} &= 2M(l) - d(l)G(l) \left\{ \frac{1}{\tanh\{(\beta/\beta_\Lambda)[\epsilon_{>} - M(l)]/2\}} - 1 \right\}, \\
\frac{dG(l)}{dl} &= (2-D)G(l) - d(l)G^2(l) \left\{ \frac{1}{\tanh\{(\beta/\beta_\Lambda)[\epsilon_{>} - M(l)]/2\}} \frac{1}{2(\beta/\beta_\Lambda)^2[\epsilon_{>} - M(l)]} \right. \\
&\quad \left. + (\beta/\beta_\Lambda) \frac{1}{\sinh^2\{(\beta/\beta_\Lambda)[\epsilon_{>} - M(l)]/2\}} \right\}.
\end{aligned} \tag{B1}$$

Starting from a positive chemical potential, i.e.,  $M(l=0) \equiv M > 0$ , one has to switch from Eqs. (16) and (17) to Eq. (B1) as soon as  $M(l)$  becomes negative in the process of renormalization. If one starts from a negative chemical potential, i.e.,  $M(l=0) \equiv M < 0$ , one has to solve Eq. (B1) with the initial conditions

$$\begin{aligned}
\omega(l=0) &= 0, \\
M(l=0) &= M < 0, \\
G(l=0) &= G.
\end{aligned} \tag{B2}$$

The corresponding grand thermodynamic potential is given by  $\Omega(M, \beta) \equiv \omega(l \rightarrow \infty)$ .

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