Stabilizing Distinguishable Qubits against Spontaneous Decay by Detected-Jump Correcting Quantum Codes

G. Alber,[1] Th. Beth,[2] Ch. Charnes,[2,3] A. Delgado,[1] M. Grassl,[2] and M. Mussinger[1]

[1]*Abteilung für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany*
[2]*Institut für Algorithmen und Kognitive Systeme, Universität Karlsruhe, D-76128 Karlsruhe, Germany*
[3]*Department of Computer Science and Software Engineering, University of Melbourne, Parkville, Vic 3052, Australia*
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A new class of error-correcting quantum codes is introduced capable of stabilizing qubits against spontaneous decay arising from couplings to statistically independent reservoirs. These quantum codes are based on the idea of using an embedded quantum code and exploiting the classical information available about which qubit has been affected by the environment. They are immediately relevant for quantum computation and information processing using arrays of trapped ions or nuclear spins. Interesting relations between these quantum codes and basic notions of design theory are established.

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Much of the newly emerging field of quantum information processing is driven by the desire to push the characteristic quantum effects into the macroscopic domain as far as possible. For this purpose it is of vital interest to develop efficient methods for stabilizing the coherence of quantum systems against destructive environmental influences. Prominent examples of such environmentally induced dissipative phenomena are spontaneous decay processes originating from couplings of a quantum system to uncontrollable photonic [1] or phononic [2] degrees of freedom. So far, various efficient quantum error-correcting strategies have been developed. All of them rely on redundancy as far as the encoding of information in quantum states is concerned.

In principle, any quantum system can be stabilized against environmental influences by active quantum error-correcting codes (QECCs) with the help of repeated control measurements and appropriately conditioned recovery operations [3–8] and by exploiting the quantum Zeno effect [9,10]. But, typically these QECCs require a large number of measurement and recovery operations. In physical systems governed by collective spontaneous decay processes originating from couplings to a single reservoir it is more advantageous to use quantum error avoiding codes (QEACs) [11–13] for efficient stabilization. These QEACs rely on the existence of a sufficiently high dimensional decoherence free subsystem (DFS) which stabilizes the dynamics passively without measurements and recovery operations. However, in the opposite dynamical regime of interest in which the spacings between physical qubits are much larger than the wavelengths of spontaneously emitted photons or phonons, these qubits decay into statistically independent reservoirs [1,3]. Efficient error-correcting strategies have also been proposed for these situations. Typically they use a QECC constructed within a DFS [3,14,15]. Errors arising from the conditional time evolution between two quantum jumps

are corrected passively by the QEAC while quantum jumps are corrected actively by the QECC. In this way the total number of control measurements and recovery operations required is decreased significantly in comparison with purely active QECCs. However, so far the redundancy of these embedded codes is not satisfactory. Plenio *et al.* [15] have constructed an embedded code capable of stabilizing one logical qubit which uses eight physical ones, but have not found a shorter code. Thus, in view of present day experimental possibilities [16] it is desirable to develop alternative error-correcting strategies for these cases by which it is possible to reduce redundancy even further without losing the advantage of passive error correction between successive quantum jumps.

Motivated by this need, in this Letter a new class of embedded quantum error-correcting codes is introduced which are capable of stabilizing distinguishable qubits against spontaneous decay processes. These codes are based on the idea of embedding an active QECC within a passive QEAC and simultaneously exploiting the classical information available about which qubit has been affected by an environment. Optimal one detected-jump correcting quantum codes of even length are constructed which minimize redundancy. It turns out that their redundancy is significantly smaller than that of previously proposed embedded error-correction schemes [15]. This latter property makes these new quantum codes particularly attractive for quantum computation and information processing based on arrays of trapped ions or nuclear spins. A link to basic notions of design theory [17] is established which is expected to be useful for further explorations of this basic idea.

Let us consider n distinguishable qubits which are perturbed by statistically independent reservoirs inducing spontaneous decay processes. Within the Markov approximation the time evolution of the density operator ρ of these n qubits can be described by a master equation

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)]$$

$$+ \frac{1}{2}\sum_{\alpha=1}^{n}\{[L_\alpha, \rho(t)L_\alpha^\dagger] + [L_\alpha\rho(t), L_\alpha^\dagger]\}. \quad (1)$$

Thereby the Lindblad operator $L_\alpha = \sqrt{\kappa_\alpha}\,|0\rangle_\alpha\langle 1|_\alpha$ characterizes spontaneous decay of qubit α from its excited state $|1\rangle_\alpha$ into its stable state $|0\rangle_\alpha$ with rate κ_α [1]. The coherent part of the n-qubit dynamics is described by the Hamiltonian H. In the case of radiative damping of quantum optical systems the derivation of Eq. (1) involves the Born and the Markov approximations which are typically very good. These approximations rest on the assumption of weak couplings between resonantly excited two-level systems and the vacuum modes of the electromagnetic field and a sufficiently short correlation time of these vacuum modes [1,3]. However, in solid state devices where spontaneous decay processes typically originate from couplings to phononic reservoirs this Markov approximation is usually only applicable for sufficiently high temperatures of the reservoirs [2]. Within the quantum trajectory approach [1] the solution of Eq. (1) can be unraveled into a statistical ensemble of pure states. Each element of this ensemble defines a quantum trajectory which describes the n-qubit system conditioned on the observation of N quantum jumps of qubits $\alpha_1, \ldots, \alpha_N$ which take place at times $t_1 \leq \cdots \leq t_N$. The action of these quantum jumps is represented by a sequence of Lindblad operators $L_{\alpha_1}, \ldots, L_{\alpha_N}$. In this quantum trajectory representation the conditional time evolution between two successive quantum jumps is determined by the non-Hermitian effective Hamiltonian $\tilde{H} = H - i(\hbar/2)\sum_{\alpha=1}^{n}L_\alpha^\dagger L_\alpha$.

The dynamics described by Eq. (1) can be stabilized against these dissipative influences in an effective way by embedded quantum codes. For this purpose one constructs first of all a DFS which stabilizes the conditional time evolution between two successive quantum jumps passively. In a second step one inverts the occurring quantum jumps with the help of an active QECC which is constructed within this DFS [3,11,12,14,15]. Thus, for this stabilization it is necessary to observe the n-qubit system continuously. Whenever a quantum jump occurs the appropriate unitary recovery operation is applied within a time interval short in comparison with the decay times and with the coherent evolution time of the system [3]. The mean number of required recovery operations is determined by the spontaneous decay rates κ_α of the qubits. For the simplest case possible of encoding only one logical qubit Plenio et al. [15] have presented a one-error correcting embedded quantum code which applies to the important special case of equal decay rates of all the qubits. Their active QECC constructed within the DFS fulfills the conditions

$$\langle c_i|L_\alpha^\dagger L_\beta|c_j\rangle = \Lambda_{\alpha\beta}\delta_{ij} \quad (2)$$

for any qubits α, β and any logical states $|c_i\rangle$, $|c_j\rangle$ with $\langle c_i|c_j\rangle = \delta_{ij}$. These conditions are necessary and sufficient for the existence of appropriate recovery operations

[8] in all cases where an unknown qubit has been affected by a quantum jump at a known jump time. Being consistent with conditions (2) Plenio et al. [15] were not able to reduce the redundancy of their code any further.

In the subsequent treatment, however, it is demonstrated that the redundancy of embedded quantum codes can be reduced significantly by also taking into account the available information about which qubit has been affected by a quantum jump. If the qubits of a quantum computer couple to independent reservoirs then information about the jump time, say t, and about the jump "position," say α, is available. Therefore, it is natural to exploit this additional information about the position of a quantum jump for a more efficient encoding. If one can determine not only the jump time t but also the jump position α by continuously monitoring the n-qubit quantum system, one has to correct the error operator L_α only for this particular value of α. As a consequence the corresponding active QECC has to fulfill Eq. (2) only for $\alpha = \beta$. The violation of conditions (2) for $\alpha \neq \beta$ offers the possibility to construct embedded codes with a significantly smaller degree of redundancy. It should be mentioned that a similar violation of conditions (2) has also been realized previously in the treatment of the quantum erasure channel [18].

Let us concentrate on the important special case of equal spontaneous decay rates of all the qubits, i.e., $\kappa_\alpha = \kappa_\beta \equiv \kappa$. If the number of physical qubits n is even, the DFS of maximal dimension with respect to the conditional time evolution between successive quantum jumps is formed by all n-particle quantum states with $(n/2)$ excited and $(n/2)$ unexcited qubits. This DFS is the eigenspace of the operator $\sum_{\alpha=1}^{n}L_\alpha^\dagger L_\alpha$ with eigenvalue $\kappa(n/2)$ and with dimension $d = \binom{n}{n/2} \equiv n!/[(n/2)!]^2$. Thus, the conditional time evolution between successive quantum jumps is not perturbed by the reservoirs. Furthermore, for a given number of physical qubits n the dimension of this DFS is maximal so that the degree of redundancy is minimal. For the correction of quantum jumps we have to develop an active QECC within this DFS. Thereby we want to exploit the fact that we have to correct quantum jumps only which take place at a known position, say α. Let us start with the simplest case possible, namely, the encoding of a single logical qubit. We propose the following four-qubit encoding (omitting normalization)

$$|c_0\rangle = |1100\rangle + |0011\rangle, \quad |c_1\rangle = |0110\rangle + |1001\rangle \quad (3)$$

formed by complementary pairs within this DFS. Thereby $|c_0\rangle$ and $|c_1\rangle$ encode the logical states $|0\rangle_L$ and $|1\rangle_L$. The complementary pairs appearing in Eq. (3) involve an excited state at any position. Provided the error position α is known this encoding represents an active QECC correcting $t = 1$ detected-jump error and is formed by superpositions of basis states of the DFS with dimension $d = 6$. This encoding violates Eq. (2) for $\alpha \neq \beta$ as $\langle c_0|L_4^\dagger L_2|c_1\rangle = \langle c_0|L_3^\dagger L_1|c_1\rangle \neq 0$, for example. Provided a quantum jump L_α has occurred at position α the immediate application

of the unitary recovery operator $R_\alpha = \pi_\alpha (\prod_{\alpha \neq \beta} C_{\alpha\beta}) \times \pi_\alpha H_\alpha$ restores the unperturbed quantum state again. Here π_α, H_α, and $C_{\alpha\beta}$ represent a π rotation, a Hadamard transformation of qubit α, and a conditional CNOT operation with control and target qubits α and β. On the code space formed by all linear combinations of the logical states of Eq. (3) R_α is the left inverse of the quantum jump operator L_α. For the construction of such a left-inverse unitary recovery operator R_α [8] the code words have to fulfill the necessary and sufficient conditions

$$\langle c_i | L_\alpha^\dagger L_\alpha | c_j \rangle = \Lambda_\alpha \delta_{ij}. \qquad (4)$$

These conditions reflect the fact that the invertibility conditions of Eq. (2) have to be fulfilled only for $\alpha = \beta$. It should also be mentioned that it is also possible to encode a third logical quantum state $|2\rangle_L$ within the above mentioned DFS by the state $|c_2\rangle = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)$. Thus, the three logical quantum states $|c_0\rangle, |c_1\rangle, |c_2\rangle$ represent a three-dimensional one detected-jump correcting quantum code formed by four physical qubits, two of which are excited. Correspondingly we denote this code by 1-JC$(4, 2, 3)$.

It is straightforward to generalize this construction to arbitrary large numbers of logical states. In analogy to Eq. (3) one starts from an even number n of physical qubits and from the corresponding DFS of dimension $d = \binom{n}{n/2}$. A basis of this DFS consists of all n-qubit states with $(n/2)$ excited and $(n/2)$ unexcited qubits. Within this DFS one forms the logical states of the active QECC from all equally weighted complementary pairs of states. The resulting embedded quantum code can correct $t = 1$ detected-jump error. It is optimal in the sense that for a given number n of physical qubits the number of logical states $l = \frac{1}{2}\binom{n}{n/2}$ is maximal. Thus, for a large number of physical qubits n the associated number of logical qubits that can be encoded is given by $\log_2 l = n - \frac{1}{2}\log_2 n + O(1)$.

The optimality of this encoding can be shown by the following estimate of dimension. For a given number n of physical qubits with k excited states and a given number t of errors at known positions $\alpha_1, \ldots, \alpha_t$ the number of logical states l is bounded by the inequality $l \leq \binom{n-t}{k-t}$. This upper bound originates from the fact that after t quantum jumps t qubits are in state $|0\rangle$ at known positions. As the logical states have to be recovered from these latter states by a unitary transformation the dimension of this latter Hilbert space also determines the maximum possible number of orthogonal logical states. By the basic symmetry property of the binomial coefficients the maximum number of logical states is achieved for $k = [n/2]$. ($[x]$ denotes the largest integer smaller or equal to x.) Thus we arrive at the final result that for $t = 1$ the maximum number of logical quantum states is given by $l = \binom{n-1}{n/2-1} \equiv \frac{1}{2}\binom{n}{n/2}$.

These one detected-jump correcting quantum codes can be generalized so that they are capable of correcting an arbitrary number t of errors of an arbitrary number of qubits. Correspondingly, we define a t detected-jump correcting

quantum code, denoted by t-JC(n, k, l), by a set of l code words $\{|c_i\rangle, i = 1, \ldots, l\}$ formed by the linear superpositions of n-qubit states each of which involves k excited and $n - k$ unexcited states. Analogous to Eq. (4) these code words have to fulfill the conditions

$$\langle c_i | L_\mathbf{e}^\dagger L_\mathbf{e} | c_j \rangle = \Lambda_\mathbf{e} \delta_{ij} \qquad (5)$$

which are necessary and sufficient for the existence of a unitary recovery operation. Thereby the error operator $L_\mathbf{e}$ denotes an arbitrary product of Lindblad operators, say $L_{\alpha_m} \ldots L_{\alpha_1}$, corresponding to a jump pattern $\mathbf{e} \equiv (\alpha_1, \ldots, \alpha_m)$ of length m. Equation (5) has to be fulfilled for all jump patterns $\mathbf{e}$ of lengths m not greater than t. According to this terminology the previously constructed optimal one detected-jump correcting quantum codes are of the type 1-JC$[n, n/2, \frac{1}{2}\binom{n}{n/2}]$ with n being even. Furthermore, the above dimension estimate implies that t detected-jump correcting quantum codes of the type t-JC$[n, n/2, \binom{n-t}{n/2-t}]$ would be optimal.

The constructed one detected-jump correcting quantum codes are particularly well suited for stabilizing quantum algorithms against spontaneous decay of the qubits into statistically independent reservoirs. Thus, for example, they may be applied for stabilizing trapped-ion systems [19] against radiative or for stabilizing nuclear spin arrays [20] against phononic damping provided the mean distance between the ions or spins representing the qubits is larger than the wavelengths of the spontaneously emitted photons or phonons. For this purpose one has to determine which qubit has been affected by the spontaneous decay process. For spontaneously emitted photons, for example, this may be achieved by photodetection techniques or by measuring the recoil of the affected particle. This latter method may also be used in phononic decay processes. Furthermore, one has to ensure that in the absence of errors the quantum system remains within the appropriate DFS throughout the entire computation. Recent investigations by Bacon et al. [21] demonstrate that this latter requirement may be achieved with the help of suitably chosen universal quantum gates which do not leave this DFS during their application. In solid state implementations such gates may be realized by appropriately tuning the coefficients of the Heisenberg-type exchange terms [21] by externally applied electric or magnetic fields. Similarly, such a tuning appears also feasible for ions in arrays of microtraps [19] by applying appropriate laser pulses which push the ions out of their equilibrium positions in a state dependent way. At the time of writing this Letter the controlled manipulation of four qubits in ion traps seems to be in reach [16]. Therefore, already the most simple example of the presented optimal one detected-jump correcting codes, namely, the 1-JC$(4, 2, 3)$ code, might give rise to interesting experimental settings. We also want to point out that strictly speaking all the presented detected-jump correcting quantum codes stabilize qubit systems only with identical spontaneous decay rates for all qubits. However, recent
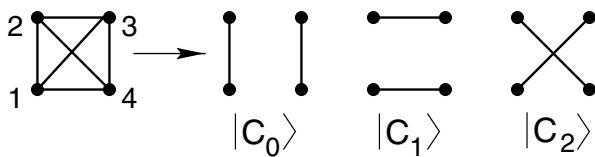
FIG. 1. Graphical representation of the affine plane of four points and six lines. The partition into three disjoint parallel classes of lines defines the states of the 1-JC(4, 2, 3).

investigations on stability properties of concatenated quantum codes indicate [22] that these codes are expected to stabilize also other qubit systems to a satisfactory degree as long as all relative differences between spontaneous decay rates remain small.

For code words consisting of linear superpositions of quantum states with identical amplitudes we can establish a surprising and far reaching connection with the area of combinatorial design theory [17]. This link seems to be particularly fruitful for further explorations of general t-JC(n, k, l) codes. In order to exhibit basic ideas of this connection let us finally reconsider the previously introduced optimal 1-JC(4, 2, 3) code. Its three code words $|c_0\rangle, |c_1\rangle, |c_2\rangle$ can be represented graphically by the connected diagram depicted in Fig. 1. Each point of this diagram is associated with a qubit. Two connected points, i.e., a block, indicate that these two qubits are in the excited state $|1\rangle$. Within the framework of finite geometry [17] this connected diagram forms an affine finite plane over the binary field. In this context the six blocks of Fig. 1 represent lines, i.e., one-dimensional subspaces of this geometry. The three code words $|c_0\rangle, |c_1\rangle, |c_2\rangle$ correspond to the three disjoint pairs of lines. We call this combinatorial structure given by the partition of the set of lines of Fig. 1 a $t = 1$ spontaneous-emission-error design, 1-SEED(4, 2, 3), on $n = 4$ points of block size $k = 2$ with $l = 3$ disjoint classes. Generalizing this notion to arbitrary values of (t, n, k, l) we arrive at the notion of

a t-SEED(n, k, l). As an example, let us consider the 2-SEED(9, 3, 3) depicted in Fig. 2. Here the lines connecting three points indicate states of nine qubits in which three are excited. The sets of these points are called blocks. Superposition of the nine blocks of size three contained in the three parallel classes depicted by any of the three rows of diagrams in Fig. 2 gives the three code words $|c_0\rangle$, $|c_1\rangle, |c_2\rangle$ of a 2-JC(9, 3, 3), e.g., $|c_0\rangle = |111\,000\,000\rangle +$ $|000\,111\,000\rangle + |000\,000\,111\rangle + |100\,001\,010\rangle +$ $|010\,100\,001\rangle + |001\,010\,100\rangle + |100\,010\,001\rangle +$ $|010\,001\,100\rangle + |001\,100\,010\rangle$. For the construction of an arbitrary t-SEED(n, k, l) design theory [17] offers powerful combinatorial methods which will be described in a subsequent article.

In summary, a new class or error-correcting quantum codes have been introduced for stabilizing qubits against spontaneous decay into independent reservoirs. It is based on the idea of using embedded quantum codes and simultaneously exploiting classical information about the error position. Thus, redundancy can be reduced significantly. The systematic construction and classification of t-JC(n, k, l) codes with $t \geq 2$ which minimize redundancy is still a challenging task which is currently under active investigation. Here the newly discovered relation to design theory seems to play a key role, especially for the construction of t-SEEDs with large t.
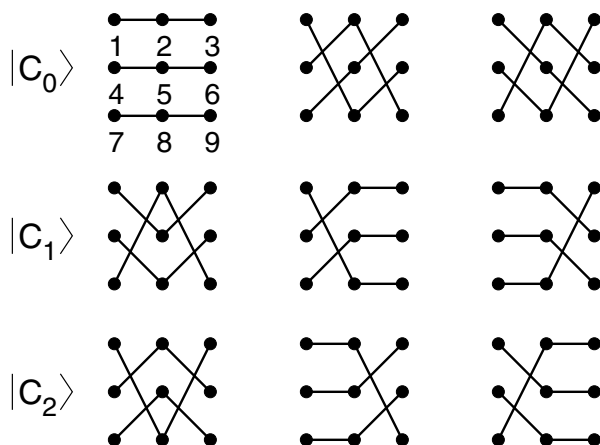
FIG. 2. This arrangement of 27 blocks of size three into disjoint rows of three parallel classes forms a 2-SEED(9, 3, 3). Superposition of the nine blocks in each row of diagrams yields the states $|c_0\rangle$, $|c_1\rangle$, and $|c_2\rangle$ of a 2-JC(9, 3, 3).

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