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Quantum Error Correction and Quantum Computation

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Abstract—In order to stabilize quantum algorithms against decoherence one has to fulfill two requirements. Firstly, one has to develop an appropriate quantum error correcting code. Secondly, one has to find a set of suitable unitary quantum transformations acting on the physical qubits which preserve the properties of this error correcting quantum code and which allow the implementation of a universal set of quantum gates. This is a challenging task in particular if we restrict ourselves to a limited class of two-particle interactions by which the physical qubits can be controlled. For the special cases of four and six physical qubits we discuss a set of such quantum gates which satisfy these two conditions for the recently developed detected-jump correcting quantum codes [1]. These quantum codes are capable of stabilizing distinguishable qubits against decoherence arising from spontaneous decay processes.

1. INTRODUCTION

Current developments in quantum information processing demonstrate in an impressive way the practical potential of quantum physics [2–4]. In quantum computation, for example, characteristic features of quantum systems, such as interference and entanglement, are exploited for solving computational tasks more efficiently than by any other known classical method. Well known examples are the celebrated factorization algorithm of Shor [5] or the fast database search algorithm of Grover [6]. However, interference and entanglement are fragile phenomena which can be destroyed easily by uncontrolled unitary transformations or interactions with an environment. In order to protect quantum information against decoherence quantum error correcting codes have been developed.

Recently, we have introduced a new family of quantum error correcting codes which requires fewer recovery operations and fewer resources for encoding quantum information than other known similar codes. These detected-jump correcting quantum codes [1] offer the possibility to stabilize quantum algorithms against decoherence originating from spontaneous decay of the physical qubits. Due to their small redundancy these codes are well suited for current experimental realizations, such as arrays of trapped ions [7] or nuclear spin systems [8]. However, in order to be able to perform quantum computation on these code spaces one has to demonstrate that it is possible to implement all possible unitary operations on these code spaces without ever leaving them during intermediate steps. Thus, a set of unitary operations is needed which generates any transformation between code words of a detected-jump correcting quantum code and which never results in a quantum state outside this code space. If the code space is d dimensional, for example, one has to demonstrate that any transformation belonging to the group $SU(d)$ can be implemented this way. In this contribution it is shown by construction that this can be achieved for the lowest dimensional cases of detected-jump correcting quantum codes which involve four and six physical qubits.

This contribution is organized as follows: In Section 2 we discuss some basic notions of quantum computation. In Section 3 we summarize briefly the theoretical description of decoherence phenomena by master equations and by quantum trajectory methods. A discussion of our recently introduced detected-jump correcting quantum codes is presented in Section 4. Finally, in Section 5 we demonstrate for the special cases of four and six physical qubits how arbitrary unitary transformations can be implemented on the associated code spaces with the help of Heisenberg-type two-body interactions between the physical qubits. Some of the more technical aspects of this construction are presented in Appendix A.

2. QUANTUM COMPUTATION AND QUANTUM ERROR CORRECTION

Quantum computation relies on the controlled and coherent manipulation of arbitrary quantum states of distinguishable physical systems which play the role of elementary data carriers. Thus, typically a theoretical model of a quantum computer consists of a register of n distinguishable d-level systems (qudits). Though in many cases one focuses on qubits with two energy levels, recently also three level systems (qutrits) and quantum systems with continuous variables have been discussed in this context. The internal state of a quantum computer consisting of n d-level systems is given by

$$|\psi\rangle = \sum_{k_1, \ldots k_n = 0}^{d-1} c_{k_n k_{n-1} \ldots k_2 k_1} |k_n k_{n-1} \ldots k_2 k_1\rangle \quad (1)$$

with the orthonormal states $|k_n k_{n-1} \ldots k_2 k_1\rangle$ denoting the computational basis.

A quantum algorithm involves the preparation of an appropriate initial state of the quantum computer, a subsequently performed sequence of unitary transformations, i.e., of quantum gates [9], and a final measurement which has to extract useful (classical) information. In any physical realization quantum gates have to be implemented by unitary transformations which are generated by suitable Hamiltonians. With the help of a universal set of quantum gates it is possible to prepare any arbitrary quantum state from any initial state of the computational basis. For unitary one-qubit transformations, for example, a universal set of quantum gates is given by

$$X(\theta) = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \quad Z(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad (2)$$

$$W(\alpha) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}.$$

Though these matrices form an infinite set due to the three continuous parameters θ, φ, and α one may also choose finite subsets with which one can approximate any unitary one-qubit transformation to any degree of accuracy. Together with the controlled-not (CNOT) gate, i.e.,

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3)$$

these one-qubit gates form a universal set of quantum gates for unitary two-qubit transformations.

In order to protect information against uncontrollable environmental influences one has to perform error correction. In the classical context powerful methods have already been developed for protecting information against such perturbing influences [10]. A key concept underlying these developments is redundancy. A larger amount of qubits allows the encoding of information in such a way that errors can be distinguished and consequently be corrected.

A simple example demonstrating this main idea is the majority code. In this code the classical *logical* states $(0)_L$ and $(1)_L$ which constitute one bit of classical information are encoded by three *physical* bits, i.e.,

$$(0)_L = (0, 0, 0), \quad (1)_L = (1, 1, 1). \quad (4)$$

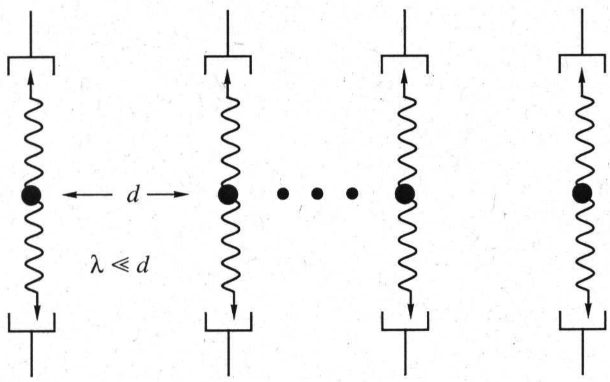
Let us assume that the only errors affecting these physical bits are bit flips which exchange the physical states 0 and 1. Definitely one bit flip affecting one of the physical qubits can be corrected with the help of this majority code. For this purpose one counts the number of zeros and ones of the physical bits. The unperturbed state is associated with the symbol which occurs more than once. Thus, according to this majority rule the perturbed state $(1, 0, 0)$, for example, has to be corrected to the state $(0, 0, 0) \equiv (0)_L$ by a recovery operation. However, if two or three errors occurred, this majority rule would lead to wrong results.

The correction of quantum states which are affected by uncontrollable environmental influences is much more complicated by several reasons. Firstly, it is not possible to copy arbitrary quantum states perfectly [11] so that a straightforward generalization of a classical majority code, for example, is not possible. Secondly, control measurements reduce a quantum state and thus may destroy quantum coherence in an irreversible way. In order to overcome these difficulties powerful quantum error correcting codes (QECCs) have been developed recently [12–19]. In order to exemplify basic ideas of these error correcting quantum codes let us assume that the influence of the uncontrollable environment can be characterized by a set of error operators, say L_α (compare with the Lindblad operators discussed in Section 3). A simple and efficient way of error correction is possible whenever these error operators posses a common eigenspace (or decoherence free subspace [16–18]) which is sufficiently highly degenerate. In this case one can use an orthonormal set of basis states of such a highly degenerate eigenspace as a computational basis. Thus, for every error L_α the basis states (or code words) $|c_i\rangle$ fulfill the equation

$$L_\alpha |c_i\rangle = \lambda_\alpha |c_i\rangle \quad (5)$$

with a complex number λ_α which can depend on the type or error but which is independent of the code words $|c_i\rangle$. Therefore, in such a basis the error operators L_α act like multiples of the unit operator and preserve quantum coherence. Whenever applicable this method of passive error correction is very efficient as it does not require any control measurements or recovery operations.

In many cases of practical interest a common, highly degenerate eigenspace of all error operators does not exist so that passive quantum error correction is not possible. In these cases one may apply methods of active quantum error correction. These latter methods may be viewed as a generalization of methods of classical error correction to the quantum domain. In particular, for this purpose one has to choose the orthonormal code words so that they are mapped onto



Schematic representation of an array of ions whose decay times and error positions are monitored by photodetectors.

distinguishable subspaces by the error operators L_α. This mapping has to be achieved in such a way that the character of the error, i.e., its syndrome, can be measured *without* learning anything about the quantum state. Afterwards a unitary recovery operation is applied which depends on the measured syndrome. Thus, this whole procedure involves the combination of a measurement process and of a unitary recovery transformation. To be able to correct a set of errors represented by the error operators L_α, the orthonormal code words $|c_i\rangle$ have to fulfill the necessary and sufficient conditions [20]

$$\langle c_i|L_\alpha^\dagger L_\beta|c_j\rangle = \Lambda_{\alpha\beta}\delta_{ij} \qquad (6)$$

for all possible values of α and β. Stated differently, these conditions express the fact that under the action of arbitrary errors orthogonal code words remain orthogonal and that all code words are affected in the same way by a given error. The efficiency of an active quantum code depends on the frequency of the control measurements and on the number of recovery operations required.

In many cases it is advantageous to combine active and passive methods [1, 19, 21] of quantum error correction by constructing an active quantum code within a decoherence free subspace. These embedded quantum codes are more efficient than purely active codes provided as much as possible is corrected passively. However, typically the construction of an active quantum code within a decoherence free subspace implies also an increase of redundancy in comparison with comparable active quantum codes which do not involve any decoherence free subspaces. Thus, for the development of embedded quantum codes with a small degree of redundancy it is important to exploit as much information as possible about the errors affecting the physical data carriers. In Section 4, an optimal embedded quantum code is discussed which is capable of protecting distinguishable qubits against spontaneous decay processes and which exploits not only information about error times but also about error positions.

3. DECOHERENCE, QUANTUM JUMPS, AND QUANTUM TRAJECTORIES

The coupling of a quantum system to an environment whose degrees of freedom are unaccessible to observation leads to decoherence. Typically, if the coupling to this environment is weak and also its correlation time is sufficiently small, the resulting time evolution of the reduced density operator $\rho(t)$ of the quantum system can be described by a quantum master equation of the form [22]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger) + \sum_\alpha L_\alpha \rho L_\alpha^\dagger, \qquad (7)$$

with the non-hermitian effective Hamiltonian

$$H_{\text{eff}} = H - \frac{i\hbar}{2}\sum_\alpha L_\alpha^\dagger L_\alpha. \qquad (8)$$

Thereby the system Hamiltonian H describes the coherent dynamics of the quantum system in the absence of any coupling to the environment. The Lindblad operators L_α characterize the influence of the environment on the quantum system. Typically, the conditions for application of the Born–Markov approximation underlying the derivation of Eq. (7) are well fulfilled for quantum optical systems.

For the subsequent development it is of interest to specialize this master equation to the case of N distinguishable qubits which may decay spontaneously from their excited states $|1\rangle_i$ to their ground states $|0\rangle_i$ by photon (or phonon) emission. Thereby the index $i = 1, \ldots, N$ denotes the positions of these distinguishable qubits whose mean nearest neighbor spacing d is assumed to be large in comparison with the wave lengths λ of the spontaneously emitted particles (compare with the figure). Under these conditions these N qubits may be viewed as decaying into N statistically independent reservoirs. Thus, in the corresponding quantum master equation the spontaneous decay of qubit i from its excited state $|1\rangle_i$ into its ground state $|0\rangle_i$ with decay rate κ_i is characterized by the Lindblad operator [22]

$$L_i = \sqrt{\kappa_i}|0\rangle_{ii}\langle 1|. \qquad (9)$$

If the initial state of the quantum system is pure a formal solution of the quantum master equation (7) is given by

$$\rho(t) = \sum_{n=0}^{\infty}\sum_{i_1,\ldots,i_n}\int_0^t dt_n\int_0^{t_n} dt_{n-1}\ldots\int_0^{t_2} dt_1 \qquad (10)$$
$$\times |t; t_n, i_n; \ldots; t_1, i_1\rangle\langle t; t_n, i_n; \ldots; t_1, i_1|$$

with

$$|t; t_n, i_n; \ldots; t_1, i_1\rangle = e^{-\frac{i}{\hbar}H_{\text{eff}}(t-t_n)} L_{i_n} \ldots L_{i_2} e^{-\frac{i}{\hbar}H_{\text{eff}}(t_2-t_1)} \times L_{i_1} e^{-\frac{i}{\hbar}iH_{\text{eff}}t_1} |t = 0\rangle. \quad (11)$$

This formal solution is the starting point for simulating solutions of the quantum master equation (7) by the quantum jump method [23, 24]. The physical interpretation of this method of simulation is based on the observation that for the Lindblad operators of Eq. (9) the unnormalized pure state $|t; t_n, i_n; \ldots; t_1, i_1\rangle$ of Eq. (11) can be interpreted as the pure quantum state of the N-qubit quantum system which results from spontaneous decay processes taking place at $t_1, \ldots, t_n$ and affecting qubits $i_1, \ldots, i_n$ [25]. These decay processes are characterized by the Lindblad operators L_i and the evolution between two successive quantum jumps is described by the effective Hamiltonian H_{eff} of Eq. (8). Within this theoretical framework the norm of the quantum state of Eq. (11) yields the probability with which a particular quantum trajectory $(t_n, i_n; \ldots; t_1, i_1)$ contributes to the density operator $\rho(t)$. The method of quantum trajectories provides an appropriate framework for simulating the stabilization of quantum algorithms with the help of quantum error correction.

4. DETECTED-JUMP CORRECTING QUANTUM CODES

Recently, we have introduced a new class of embedded quantum codes which is capable of stabilizing N distinguishable qubits against spontaneous decay processes provided these decay processes can be described by a master equation of the form of Eq. (7) with Lindblad operators as given by Eq. (9). These quantum codes work ideally provided the mean nearest neighbor spacing between the qubits is larger than the wave lengths of the spontaneously emitted photons (or phonons) and the spontaneous decay rates of all the qubits are equal. This family of quantum codes is based on the idea of correcting the modified time evolution between successive quantum jumps passively and inverting the quantum jumps originating from the Lindblad operators of Eq. (9) actively. Thereby the passive error correction between successive quantum jumps guarantees that the number of control measurements and recovery operations required is reduced significantly in comparison with purely active methods of error correction. In addition, this encoding also takes advantage of all the information available about the errors, namely the times and positions at which they take place. As the mean distance between adjacent qubits is assumed to be large in comparison with the wave length of the spontaneously emitted particles, it is possible to determine the position i of the qubit which has decayed to its ground state. In practice this determination of error times and error positions can be achieved by monitoring the spontaneous emission of the qubits continuously with photodetectors (compare with the figure). Provided information about both the error time, say t_1, and error position, say i_1, is available an active error correcting code has to fulfill the conditions of Eq. (6) for $\alpha = \beta \equiv i_1$ only. Thus, a unitary recovery operation can be constructed even if conditions (6) are violated for $\alpha \neq \beta$. This reduced number of constraints which an active error correcting quantum code has to fulfill offers the possibility of lowering redundancy. Typically, disregarding information about error positions increases redundancy significantly [1, 21].

The simplest example of this new class of embedded quantum codes which is capable of correcting one error at a time can be constructed with the help of four physical qubits. The (unnormalized) code words of this particular detected-jump correcting quantum code represent a logical qutrit and are given by

$$\begin{aligned} |c_0\rangle &= |0011\rangle + |1100\rangle, \\ |c_1\rangle &= |0101\rangle + |1010\rangle, \\ |c_2\rangle &= |0110\rangle + |1001\rangle. \end{aligned} \quad (12)$$

Thus this code consists of four-qubit states only in which half of the qubits are excited. As this one-error correcting detected-jump correcting quantum code involves three logical states and four physical qubits two of which are excited we call it a $1-JC(4, 2, 3)$ code. Thereby the equal number of excited qubits guarantees that the effective time evolution between successive quantum jumps is corrected passively. Another characteristic feature of this quantum code is the complementary pairing of states with equal amplitudes, e.g., of the states $|1100\rangle$ and $|0011\rangle$. This latter property guarantees the validity of the necessary and sufficient conditions of Eq. (6) as long as $\alpha = \beta$.

This construction of a one-error correcting embedded, quantum code can be generalized easily to any even number of physical qubits. Thus, a $1-JC\left(N, N/2, \binom{N}{N/2-1}\right)$ quantum code is constructed by an analogous complementary paring of N-qubit states with half of these qubits being excited. This way it is possible to construct $\binom{N}{N/2-1}$ orthogonal code words which form a one-error correcting embedded quantum code for spontaneous decay processes. The lowest dimensional representative of this infinite family of quantum codes is the already discussed $1-JC(4, 2, 3)$ quantum code. The next higher dimensional quantum code is the $1-JC(6, 3, 10)$ code which involves 10 logical states. It can be shown that this family of one-error correcting quantum codes is optimal in the sense that their redundancy cannot be reduced any further [1].

Action of the state swapping operators $E_{i,j}$ of Eq. (14) on the modified basis states of the $1 - JC$ (4, 2, 3) code

$E_{i,j}$	$\|c_0\rangle$	$\|c_1\rangle$	$\|c_2\rangle$
$E_{1,2}$	$\|c_0\rangle$	$-\|c_2\rangle$	$-\|c_1\rangle$
$E_{1,3}$	$-\|c_2\rangle$	$\|c_1\rangle$	$-\|c_0\rangle$
$E_{1,4}$	$-\|c_1\rangle$	$-\|c_0\rangle$	$\|c_2\rangle$
$E_{2,3}$	$\|c_1\rangle$	$\|c_0\rangle$	$\|c_2\rangle$
$E_{2,4}$	$\|c_2\rangle$	$\|c_1\rangle$	$\|c_0\rangle$
$E_{3,4}$	$\|c_0\rangle$	$\|c_2\rangle$	$\|c_1\rangle$

Finally, it should also be mentioned that detected-jump correcting quantum codes can also be constructed which are capable of correcting more than one error at a time. An example of such a two-error correcting code is presented in [1].

5. IMPLEMENTING THE UNITARY GROUP ON DETECTED-JUMP CORRECTING QUANTUM CODES

Any of the previously discussed $1 - JC\left(N, N/2, \binom{N}{N/2}\right)$ quantum codes offers interesting perspectives for stabilizing qubits against spontaneous decay processes. However, in order to be useful for purposes of quantum computation one has to find universal sets of quantum gates which guarantee that any unitary operation can be implemented entirely within these code spaces. In the subsequent sections it is demonstrated for the special cases of the $1 - JC(4, 2, 3)$ and the $1 - JC(6, 3, 10)$ quantum codes that such desirable universal quantum gates can be implemented on the basis of Heisenberg Hamiltonians acting on the physical qubits.

In order to put the problem into perspective let us assume that it is possible to control the dynamics of the physical qubits to such a degree that the time evolution resulting from any Heisenberg Hamiltonian of the type

$$H = \sum_{ij\alpha} J_{ij}^{(\alpha)} \mathbf{S}_i^{(\alpha)} \mathbf{S}_j^{(\alpha)} \tag{13}$$

can be realized with arbitrary tunings of the coupling constants $J_{ij}^{(\alpha)}$. Thereby $\mathbf{S}_i = (\sigma_x^i, \sigma_y^i, \sigma_z^i)$ denotes the Pauli spin vector with components $\mathbf{S}_i^{(\alpha)}$ acting on qubit i. Thus, with one particular tuning of the coupling constants, for example, it is possible to realize the Hamiltonian

$$E_{ij} = \frac{1}{2}(\mathbf{1} + \mathbf{S}_i \mathbf{S}_j) \tag{14}$$

which exchanges the states of the physical qubits i and j. This particular swapping Hamiltonian has been discussed in detail by Bacon *et al.* [26] in connection with quantum computation on decoherence free subspaces. With another choice of the coupling constants, for example one can realize the Hamiltonian

$$F_{ij} = \frac{1}{2}(\mathbf{1} + \sigma_z^i \sigma_z^j) \tag{15}$$

which will be useful for our subsequent discussion.

5.1. The 1 – JC(4, 2, 3) Quantum Code and Unitary Transformations of the Group SU(3)

With the help of Heisenberg Hamiltonians of the form of Eq.(14) it is possible to generate a universal set of quantum gates acting entirely within the code space of a $1 - JC(4, 2, 3)$ quantum code provided we introduce an additional phase in the code words of Eq. (12). This can be achieved by applying a σ_z on the first qubit, for example, which preserves our decoherence free subspace and which allows the same active error correction as before. The resulting modified (unnormalized) code words of the $1 - JC(4, 2, 3)$ quantum code are

$$|c_0\rangle = |0011\rangle - |1100\rangle, \quad |c_1\rangle = |0101\rangle - |1010\rangle,$$
$$|c_2\rangle = |0110\rangle - |1001\rangle.$$

It is apparent from the table that the swap operators of Eq. (14) preserve this codespace.

Therefore, any sequence of unitary transformations of the form $\exp(it\sum_{ij} a_{ij}E_{ij})$ with real-valued coefficients a_{ij} also leaves this code space invariant. Thus, it is straightforward to demonstrate that any unitary transformation of the group $SU(3)$ acting within the code space can be implemented with an appropriate sequence of state swapping transformations. In particular, the eight operators

$$A_{11} = \frac{1}{3}(E_{12} + E_{34}) - \frac{1}{6}(E_{13} + E_{24}) - \frac{1}{6}(E_{14} + E_{23}),$$

$$A_{22} = -\frac{1}{6}(E_{12} + E_{34}) + \frac{1}{3}(E_{13} + E_{24}) - \frac{1}{6}(E_{14} + E_{23}),$$

$$B_{12}^{+} = \frac{1}{2}(E_{23} - E_{14}), \quad B_{13}^{+} = \frac{1}{2}(E_{24} - E_{13}), \tag{16}$$

$$B_{23}^{+} = \frac{1}{2}(E_{34} - E_{12}), \quad B_{12}^{-} = i[B_{13}^{+}, B_{23}^{+}],$$

$$B_{13}^{-} = i[B_{12}^{+}, B_{23}^{+}], \quad B_{23}^{-} = i[B_{12}^{+}, B_{13}^{+}]$$

are linearly independent generators of the group $SU(3)$. In the code space the action of the state swapping transformations E_{ij} can be represented by the matrices

$$E_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ldots. \qquad (17)$$

Correspondingly, in the code space the action of the eight generators is represented by the matrices

$$A_{11} = \frac{1}{6}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_{22} = \frac{1}{6}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$B_{12}^{+} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{13}^{+} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$B_{23}^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_{12}^{-} = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_{13}^{-} = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad B_{23}^{-} = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

From the additional relations

$$e^{i(\alpha A + \beta B)} = \lim_{n \to \infty}\left(e^{i\frac{\alpha}{n}A} e^{i\frac{\beta}{n}B}\right)^n \qquad (18)$$

and

$$e^{i(i[\alpha A, \beta B])} = \lim_{n \to \infty}\left(e^{i\frac{\alpha}{\sqrt{n}}A} e^{i\frac{\beta}{\sqrt{n}}B} e^{-i\frac{\alpha}{\sqrt{n}}A} e^{-i\frac{\beta}{\sqrt{n}}B}\right)^n \qquad (19)$$

which are valid for arbitrary hermitian operators A and B it becomes apparent that any unitary transformation of the group $SU(3)$ can be approximated with the help of the state swapping Hamitonians E_{ij}.

5.2. The 1 – JC(6, 3, 10) Quantum Code and Unitary Transformations of the Group SU(10)

The previously discussed procedure can be extended also to the $1 - JC(6, 3, 10)$ quantum code. This detected-jump correcting quantum code consists of 10 code words, say $|c_0\rangle, \ldots, |c_9\rangle$. Therefore, in this case the action of the state swapping transformations of Eq. (14) in the logical state space is represented by 10×10 matrices, such as

$$E_{12} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \end{pmatrix}.$$

In order to obtain a universal set of quantum gates acting entirely within the $1 - JC(6, 3, 10)$ code space we also need the Heisenberg Hamiltonians F_{ij} of Eq. (15) with $i, j \in \{1, \ldots, 6\}$. In the 10-dimensional code space their matrix representations are diagonal with ones at all those positions k ($k = 0, 1, \ldots, 9$) for which qubits i and j are both in the same state in the corresponding code words $|c_k\rangle$. By linear combinations of these operators, all possible diagonal matrices D_k ($k = 0, \ldots, 9$) can be constructed which have zeroes everywhere except at the kth diagonal position. Thus, the matrix representation of $D_0 = 1/2(-\mathbf{1} + F_{12} + F_{13} + F_{23})$, for example, is given by

$$D_0 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

The full set of equations which relate all operators D_k to the diagonal matrices F_{ij} are given in Appendix A. By linear combination of operators E_{ij} and D_k one can construct exchange operators G_{ij} which act on parts of the code words thereby leaving the remaining code words invariant. An example is the operator $G_{12} = E_{12} - (D_0 +$

$D_1 + D_2 + D_3$) with the matrix representation

$$G_{12} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \end{pmatrix}.$$

This way 15 non-diagonal hermitian operators can be constructed. Commutators of these operators generate another set of 15 non-diagonal hermitian operators K_{ij}, such as

$$K_{12} = i[G_{13}, G_{23}] = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&-i \\ 0&0&0&0&0&0&0&0&-i&0 \\ 0&0&0&0&0&0&0&-i&0&0 \\ 0&0&0&0&0&0&i&0&0&0 \\ 0&0&0&0&0&i&0&0&0&0 \\ 0&0&0&0&i&0&0&0&0&0 \end{pmatrix}.$$

Therefore, another set of 15 linearly independent generators can be constructed. In summary, commutators of all these operators generate a total amount of 90 non-diagonal, linearly independent generators. Together with the 10 linearly independent diagonal operators D_k they form a complete set of generators by which it is possible to implement arbitrary unitary operations on the $1 - JC(6, 3, 10)$ code space.

6. CONCLUSIONS

We have discussed main ideas underlying the recently introduced detected-jump correcting quantum codes by which it is possible to protect distinguishable qubits against spontaneous decay processes. Due to their low redundancy these quantum codes are attractive for stabilizing quantum algorithms against this particular source of decoherence. However, in order to be useful for these purposes one has to investigate the question whether it is possible to implement arbitrary quantum gates on these code spaces without leaving these spaces at any time. As a first step towards this final goal we have presented first results which apply to detected-jump correcting quantum codes involving four and six physical qubits. For these cases we have shown by construction that it is possible to realize any unitary transformation within these codes spaces with the help of Heisenberg Hamiltonians which act on the physical qubits.

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APPENDIX A: EQUATIONS FOR THE $SU(10)$ OPERATORS

In this appendix explicit expressions for all linear independent generators of the group $SU(10)$ are presented in terms of the Heisenberg-type Hamiltonians E_{ij} and F_{ij}. Note that for the physical operators E_{ij}, F_{ij}, and G_{ij} the indices indicate the physical qubit on which these operators act. For all other operators defined in this appendix the indices indicate the code words on which they act in the code space. The operators D_k ($k = 0, \ldots, 9$) with a single nonzero matrix element on the diagonal are defined by

$$\begin{aligned} D_0 &= 1/2(-\mathbf{1} + F_{12} + F_{13} + F_{23}), \\ D_1 &= 1/2(-\mathbf{1} + F_{12} + F_{14} + F_{24}), \\ D_2 &= 1/2(-\mathbf{1} + F_{12} + F_{15} + F_{25}), \\ D_3 &= 1/2(-\mathbf{1} + F_{12} + F_{16} + F_{26}), \\ D_4 &= 1/2(-\mathbf{1} + F_{13} + F_{14} + F_{34}), \\ D_5 &= 1/2(-\mathbf{1} + F_{13} + F_{15} + F_{35}), \\ D_6 &= 1/2(-\mathbf{1} + F_{13} + F_{16} + F_{36}), \\ D_7 &= 1/2(-\mathbf{1} + F_{14} + F_{15} + F_{45}), \\ D_8 &= 1/2(-\mathbf{1} + F_{14} + F_{16} + F_{46}), \\ D_9 &= 1/2(-\mathbf{1} + F_{15} + F_{16} + F_{56}). \end{aligned} \tag{A1}$$

The exchange operators are given by

$$\begin{aligned} G_{12} &= E_{12} - (D_0 + D_1 + D_2 + D_3), \\ G_{13} &= E_{13} - (D_0 + D_4 + D_5 + D_6), \\ G_{14} &= E_{14} - (D_1 + D_4 + D_7 + D_9), \\ G_{15} &= E_{15} - (D_2 + D_5 + D_7 + D_9), \\ G_{16} &= E_{16} - (D_3 + D_6 + D_8 + D_9), \\ G_{23} &= E_{23} - (D_0 + D_7 + D_8 + D_9), \\ G_{24} &= E_{24} - (D_1 + D_5 + D_6 + D_9), \end{aligned}$$

$$\begin{aligned}
G_{25} &= E_{25} - (D_2 + D_4 + D_6 + D_8),\\
G_{26} &= E_{26} - (D_3 + D_4 + D_5 + D_7),\\
G_{34} &= E_{34} - (D_2 + D_3 + D_4 + D_9),\\
G_{35} &= E_{35} - (D_1 + D_3 + D_6 + D_7),\\
G_{36} &= E_{36} - (D_1 + D_2 + D_6 + D_7),\\
G_{45} &= E_{45} - (D_0 + D_3 + D_6 + D_7),\\
G_{46} &= E_{46} - (D_0 + D_2 + D_5 + D_8),\\
G_{56} &= E_{56} - (D_0 + D_1 + D_4 + D_9).
\end{aligned} \tag{A2}$$

The hermitian operators constructed by commutators of these generators are defined by

$$\begin{aligned}
K_{12} &= i[G_{13}, G_{23}] & K_{13} &= i[G_{12}, G_{23}]\\
K_{14} &= i[G_{12}, G_{24}] & K_{15} &= i[G_{12}, G_{25}]\\
K_{16} &= i[G_{12}, G_{26}] & K_{23} &= i[G_{12}, G_{13}]\\
K_{24} &= i[G_{12}, G_{14}] & K_{25} &= i[G_{12}, G_{15}]\\
K_{26} &= i[G_{12}, G_{16}] & K_{34} &= i[G_{13}, G_{14}]\\
K_{35} &= i[G_{13}, G_{15}] & K_{36} &= i[G_{13}, G_{16}]\\
K_{45} &= i[G_{14}, G_{15}] & K_{46} &= i[G_{14}, G_{16}]\\
& & K_{56} &= i[G_{15}, G_{16}].
\end{aligned} \tag{A3}$$

Finally, the exchange operations between two code words are given by

$$\begin{aligned}
V_{01} &= i[-K_{34}, (K_{12} + D_0)] & W_{01} &= i[G_{34}, G_{12} + D_0]\\
V_{02} &= i[-K_{35}, (K_{12} + D_0)] & W_{02} &= i[G_{35}, G_{12} + D_0]\\
V_{03} &= i[-K_{36}, (K_{12} + D_0)] & W_{02} &= i[G_{36}, G_{12} + D_0]\\
V_{04} &= i[-K_{24}, (K_{13} + D_0)] & W_{04} &= i[G_{24}, G_{13} + D_0]\\
V_{05} &= i[-K_{25}, (K_{13} + D_0)] & W_{05} &= i[G_{25}, G_{13} + D_0]\\
V_{06} &= i[-K_{26}, (K_{13} + D_0)] & W_{06} &= i[G_{26}, G_{13} + D_0]\\
V_{07} &= i[-K_{16}, (K_{23} + D_0)] & W_{07} &= i[G_{16}, G_{23} + D_0]\\
V_{08} &= i[-K_{15}, (K_{23} + D_0)] & W_{08} &= i[G_{15}, G_{23} + D_0]\\
V_{09} &= i[-K_{14}, (K_{23} + D_0)] & W_{09} &= i[G_{14}, G_{23} + D_0]\\
V_{12} &= i[-K_{45}, (K_{36} + D_1)] & W_{12} &= i[G_{45}, G_{36} + D_1]\\
V_{13} &= i[-K_{46}, (K_{35} + D_1)] & W_{13} &= i[G_{46}, G_{35} + D_1]\\
V_{14} &= i[-K_{23}, (K_{14} + D_1)] & W_{14} &= i[G_{23}, G_{14} + D_1]\\
V_{15} &= i[-K_{16}, (K_{24} + D_1)] & W_{15} &= i[G_{16}, G_{24} + D_1]\\
V_{16} &= i[-K_{15}, (K_{24} + D_1)] & W_{16} &= i[G_{15}, G_{24} + D_1]\\
V_{17} &= i[-K_{25}, (K_{14} + D_1)] & W_{17} &= i[G_{25}, G_{14} + D_1]\\
V_{18} &= i[-K_{26}, (K_{14} + D_1)] & W_{18} &= i[G_{26}, G_{14} + D_1]\\
V_{19} &= i[-K_{13}, (K_{24} + D_1)] & W_{19} &= i[G_{13}, G_{24} + D_1]\\
V_{23} &= i[-K_{56}, (K_{34} + D_2)] & W_{23} &= i[G_{56}, G_{24} + D_2]\\
V_{24} &= i[-K_{16}, (K_{24} + D_2)] & W_{24} &= i[G_{16}, G_{24} + D_2]\\
V_{25} &= i[-K_{23}, (K_{15} + D_2)] & W_{25} &= i[G_{23}, G_{15} + D_2]\\
V_{26} &= i[-K_{14}, (K_{25} + D_2)] & W_{26} &= i[G_{14}, G_{25} + D_2]\\
V_{27} &= i[-K_{24}, (K_{15} + D_2)] & W_{27} &= i[G_{24}, G_{15} + D_2]\\
V_{28} &= i[-K_{13}, (K_{25} + D_2)] & W_{28} &= i[G_{13}, G_{25} + D_2]\\
V_{29} &= i[-K_{24}, (K_{15} + D_2)] & W_{29} &= i[G_{24}, G_{15} + D_2]\\
V_{34} &= i[-K_{15}, (K_{26} + D_3)] & W_{34} &= i[G_{15}, G_{26} + D_3]\\
V_{35} &= i[-K_{14}, (K_{26} + D_3)] & W_{35} &= i[G_{14}, G_{26} + D_3]\\
V_{36} &= i[-K_{23}, (K_{16} + D_3)] & W_{36} &= i[G_{23}, G_{16} + D_3]\\
V_{37} &= i[-K_{13}, (K_{26} + D_3)] & W_{37} &= i[G_{13}, G_{26} + D_3]\\
V_{38} &= i[-K_{24}, (K_{16} + D_3)] & W_{38} &= i[G_{24}, G_{16} + D_3]\\
V_{39} &= i[-K_{25}, (K_{16} + D_3)] & W_{39} &= i[G_{25}, G_{16} + D_3]\\
V_{45} &= i[-K_{45}, (K_{26} + D_4)] & W_{45} &= i[G_{45}, G_{26} + D_4]\\
V_{46} &= i[-K_{46}, (K_{25} + D_4)] & W_{46} &= i[G_{46}, G_{25} + D_4]\\
V_{47} &= i[-K_{35}, (K_{26} + D_4)] & W_{47} &= i[G_{35}, G_{26} + D_4]\\
V_{48} &= i[-K_{36}, (K_{25} + D_4)] & W_{48} &= i[G_{36}, G_{25} + D_4]\\
V_{49} &= i[-K_{12}, (K_{34} + D_4)] & W_{49} &= i[G_{12}, G_{34} + D_4]\\
V_{56} &= i[-K_{56}, (K_{24} + D_5)] & W_{56} &= i[G_{56}, G_{24} + D_5]\\
V_{57} &= i[-K_{34}, (K_{26} + D_5)] & W_{57} &= i[G_{34}, G_{26} + D_5]\\
V_{58} &= i[-K_{12}, (K_{35} + D_5)] & W_{58} &= i[G_{12}, G_{35} + D_5]\\
V_{59} &= i[-K_{36}, (K_{15} + D_5)] & W_{57} &= i[G_{36}, G_{15} + D_5]\\
V_{67} &= i[-K_{12}, (K_{36} + D_6)] & W_{67} &= i[G_{12}, G_{36} + D_6]\\
V_{68} &= i[-K_{34}, (K_{25} + D_6)] & W_{68} &= i[G_{34}, G_{25} + D_6]\\
V_{69} &= i[-K_{35}, (K_{24} + D_6)] & W_{69} &= i[G_{35}, G_{24} + D_6]\\
V_{78} &= i[-K_{56}, (K_{23} + D_7)] & W_{78} &= i[G_{56}, G_{23} + D_7]\\
V_{79} &= i[-K_{46}, (K_{23} + D_7)] & W_{79} &= i[G_{46}, G_{23} + D_7]\\
V_{89} &= i[-K_{45}, (K_{23} + D_8)] & W_{89} &= i[G_{45}, G_{23} + D_8]
\end{aligned} \tag{A4}$$

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