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Quantum revivals in a periodically driven gravitational cavity

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Abstract. Quantum revivals are investigated for the dynamics of an atom in a driven gravitational cavity. It is demonstrated that the external driving field influences the revival time significantly. Analytical expressions are presented which are based on second-order perturbation theory and semiclassical secular theory. These analytical results explain the dependence of the revival time on the characteristic parameters of the problem quantitatively in a simple way. They are in excellent agreement with numerical results.

Keywords: Wavepacket dynamics, revival phenomena, periodically driven quantum systems, gravitational cavities

Periodically driven quantum systems have received considerable attention over the past few years due to the presence of Anderson-like localization [1]. Some recent numerical work indicates the presence of quantum revivals [2] in periodically driven quantum systems [3, 4] as well as in other two-degree-of-freedom systems [5]. Though many aspects of quantum localization are well understood by now, so far even the most elementary questions concerning revival effects in periodically driven quantum systems are not yet comprehensible. Under which circumstances do such revival phenomena appear and how do these revival effects depend on the frequency and on the driving amplitude of the applied periodic force? The main intention of this paper is to address these questions by considering a physical system which is of particular interest in the field of atom optics [6], namely an atom in a periodically driven gravitational cavity [7]. It will be shown that quantum revivals can be observed in this physical system and that the dominant influence of the periodic driving force results in a change of the revival time. Simple analytical results are presented which explain the quantitative dependence of the revival time on the driving frequency and the strength of the driving amplitude. These analytical results, which are based on semiclassical second-order perturbation theory and on semiclassical secular perturbation theory, are in excellent agreement with numerical results.

Let us consider an atom in a periodically driven gravitational cavity [7]. The atom moves under the influence of gravity in the positive \tilde{z} -direction and is reflected as it hits a mirror. The atomic mirror is assumed to be made up of an evanescent electromagnetic field with electric field strength $\mathbf{E}(\tilde{z}, t) = e_x E_0 e^{-\omega_L \tilde{z}/c} e^{-i\omega_L t} + \text{c.c.}$ In order to study the effect of an external driving force on this elementary quantum system the evanescent wave is assumed to be modulated, e.g., by an acousto-optical modulator [8], that is $E_0 = \mathcal{E}_0 \exp(\epsilon \sin \omega t)$. Assuming that the optical driving

frequency ω is well detuned from any atomic resonance and taking into account the symmetry of the problem in the (x, y) -plane the effective one-dimensional centre-of-mass motion of the atom is governed by the Hamiltonian [9]

$$H = \frac{p^2}{2M} + Mg\tilde{z} + \frac{\hbar\Omega_{\text{eff}}}{4} e^{-2\omega_L \tilde{z}/c + \epsilon \sin \omega t}. \quad (1)$$

Here p is the atomic centre-of-mass momentum, M is the atomic mass and g denotes the gravitational acceleration. The effective Rabi frequency Ω_{eff} characterizes the strength of the influence of the applied electric field.

The potential generated by the gravitational acceleration and by the evanescent laser field has the approximate form of a time-dependent triangular potential well as in the Fermi accelerator [10]. Thus, in the subsequent, approximate treatment we replace this potential by the idealized, simpler form of a triangular well with an infinitely high potential barrier at the position $\tilde{z} = \frac{c\epsilon}{2\omega_L} \sin \omega t$ of the mirror. In this approximation our results become independent of details of the evanescent laser field, such as the laser frequency ω_L and the value of the effective Rabi frequency Ω_{eff} .

In the moving coordinate we may write the time-dependent Schrödinger equation as [11]

$$i\hbar\dot{\chi} = \left\{ \frac{p^2}{2M} + Mz(g - \lambda\omega^2 \sin \omega t) + V_0 e^{-\kappa z} \right\} \chi, \quad (2)$$

where $z = \tilde{z} - \lambda \sin \omega t$. Here we are considering $V_0 \equiv \hbar\Omega_{\text{eff}}/4$, the steepness $\kappa \equiv 2\omega_L/c$ and the modulation strength $\lambda \equiv c\epsilon/(2\omega_L)$.

A convenient way of obtaining insight into the influence of the acousto-optical external driving force on the atomic dynamics is to investigate the time evolution of an atomic centre-of-mass wavepacket. For this purpose we consider a Gaussian wavepacket $\psi(0) = (2\pi\Delta z^2)^{-1/4} \exp\{-(z -$

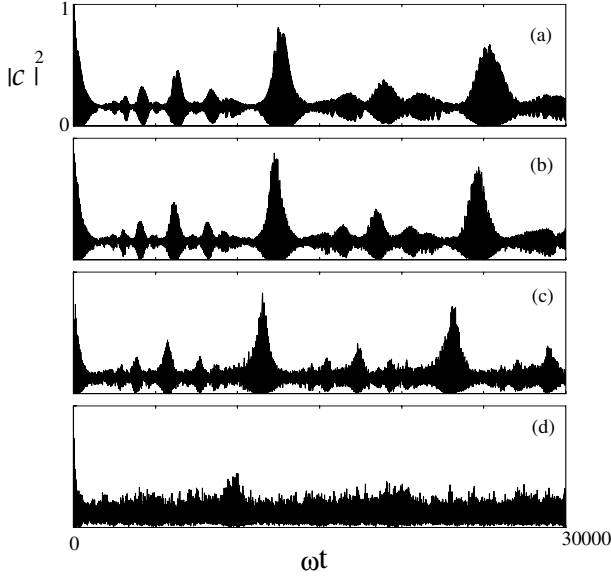


Figure 1. Revival phenomena in a gravitational cavity in the absence and in the presence of periodic modulation: autocorrelation of a Gaussian wavepacket of a caesium atom as a function of time, prepared at $t = 0$ with $\Delta z = 0.28 \mu\text{m}$, for (a) $\lambda = 0$, (b) $\lambda = 0.56 \mu\text{m}$, (c) $\lambda = 1.13 \mu\text{m}$ and (d) $\lambda = 2.26 \mu\text{m}$. The parameters are $\omega = 2\pi \times 0.93 \text{ kHz}$, $\Omega_{\text{eff}} = 23.38 \text{ kHz}$ and $\kappa = 0.57 \mu\text{m}$. The average position of the wavepacket in the gravitational cavity is $z_0 = 20.1 \mu\text{m}$, which corresponds to the mean quantum number $n_0 = 176.16$.

$z_0)^2/(2\Delta z)^2\} \exp(-ip_0(z - z_0)/\hbar)$ at $t = 0$ and propagate it in the gravitational cavity for different modulation strength λ . Here z_0 describes the average position, p_0 denotes the average momentum of the wavepacket and Δz is the spatial uncertainty. In figure 1 characteristic time dependences of the autocorrelation function $C(t) \equiv \langle \psi(0)|\psi(t) \rangle$ of the wavepacket are shown for $z_0 = 20.1 \mu\text{m}$, $\Delta z = 0.28 \mu\text{m}$ and $p_0 = 0$. Without external periodic perturbation (uppermost figure) the well known scenario of revivals and fractional revivals [12] is clearly apparent. In the presence of a sufficiently weak external periodic driving force the revivals and fractional revivals are still observable. However, the revivals decrease in magnitude and the revival time exhibits a pronounced dependence on the external driving force. We find that if the modulation strength λ exceeds 0.25 these revival phenomena disappear.

In order to obtain insight into the quantitative dependence of the revival time of an atomic wavepacket on the external driving force let us first concentrate on the case of a weak, off-resonant periodic driving which can be treated perturbatively. The revival time T of a wavepacket [12] which is centred around mean energy E_{n_0} is determined by the spectrum of the nearby discrete quasi-energies \mathcal{E}_n of the Hamiltonian appearing in equation (2), i.e.

$$T = \frac{4\pi\hbar}{\left| \frac{d^2 \mathcal{E}_n}{dn^2} \right|_{n=n_0}}. \quad (3)$$

As long as the influence of the external, periodic force can be described perturbatively these quasi-energies are approximately given by $\mathcal{E}_n = E_n + \delta E_n$ with the quadratic

Stark shift

$$\delta E_n = \frac{(M\lambda\omega^2)^2}{4} \sum_{m=-\infty}^{\infty} \frac{2|(n+m|z|n)|^2(E_n - E_{n+m})}{(E_n - E_{n+m})^2 - (\hbar\omega)^2}, \quad (4)$$

and with E_n denoting the unperturbed eigenenergies of the Hamiltonian of equation (2) with $\lambda = 0$. For large values of the quantum number n asymptotic expressions are obtainable for E_n by using semiclassical quantization, namely, $E_n \equiv \frac{1}{2}(Mg^2)^{1/3}[3\pi\hbar(n + \frac{3}{4})]^{2/3}$. Asymptotically valid expressions for the relevant matrix elements $\langle n+m|z|n \rangle$ can be derived with the help of the Bohr correspondence principle [14] in which these matrix elements are related directly to the Fourier coefficients of the classical, unperturbed motion of the atom in the triangular potential well, namely

$$\langle n+m|z|n \rangle \equiv -\frac{2E_n}{M\pi^2 m^2 g} \left(1 + \frac{m}{3(n + \frac{3}{4})} \right). \quad (5)$$

In this way all quantities appearing in equation (3) can be evaluated from the corresponding unperturbed, classical dynamics. Inserting these asymptotic expressions into equation (4) the semiclassical approximation for the Stark shift

$$\delta E_n = -\left(\frac{\omega^2 \lambda}{g} \right)^2 \frac{3E_n}{\pi^4} \sum_{m=1}^{\infty} m^{-4} \left(\frac{1 + \frac{2/3}{1 - (\Omega_n/m)^2}}{1 - (\Omega_n/m)^2} \right) \quad (6)$$

is obtained with $\Omega_n = 3(n + \frac{3}{4})\hbar\omega/(2E_n)$ and with m being integer. Thereby the quadratic Stark shift is expressed as a sum of contributions of virtual oscillators m which describes the influence of the external periodic force on the bound motion of the atom in the triangular potential well. As the effective coupling strength of these virtual oscillators to the external field decreases rapidly with increasing m the dominant contribution to the quadratic Stark shift is given by the first few terms in the sum of equation (6). Equation (6) describes the global frequency dependence of the Stark shift and is valid in the perturbative regime for small enough values of the driving strength λ . It breaks down close to resonances at which $(\Omega_n/m)^2 = 1$. Inserting equation (6) into equation (3) we obtain the revival time

$$T_\lambda = T_0 \left[1 - \left(\frac{\omega^2 \lambda}{g} \right)^2 \frac{3}{\pi^4} \sum_{m=1}^{\infty} \frac{m^{-4}}{E_{n_0}''} \times \frac{d^2}{dn^2} \left\{ E_n \left(\frac{1 + \frac{2/3}{1 - (\Omega_n/m)^2}}{1 - (\Omega_n/m)^2} \right) \Big|_{n=n_0} \right\} \right] \quad (7)$$

where T_0 is the revival time in the absence of the external modulation.

Alternatively one can calculate the revival time with the help of semiclassical secular theory, which is expected to yield a particularly good approximation close to a resonance. In the vicinity of the N th primary resonance the classical Hamiltonian corresponding to equation (2) can be expressed by action and angle variables (I, φ) [11, 13] as

$$H = \frac{H''}{2}(I - I_0)^2 + H'(I - I_0) + H_0(I_0) + \lambda V \sin(N\varphi - \omega t). \quad (8)$$

Here I_0 is the classical action associated with the initial condition and $V = (Mg)(I_0/I_N)^{2/3}$. The classical action

at the centre of the N th primary resonance is denoted by I_N . Moreover H' and H'' are the first and second derivatives of energy with respect to action calculated at I_0 . Similarly $H_0(I_0)$ is the energy of the unperturbed system for the initial action I_0 . In equation (8) we have averaged out the fast oscillating terms so that $H(I - I_0)$ represents an integrable one-degree-of-freedom physical system.

Introducing the transformation $N\varphi - \omega t = 2\theta + \frac{\pi}{2}$ and quantizing the dynamics around the N th resonance by using $I - I_0 = \frac{\hbar}{i} \frac{\partial}{\partial \varphi} = \frac{N\hbar}{2i} \frac{\partial}{\partial \theta}$ [15], the time-independent Schrödinger equation becomes

$$\left[-\frac{N^2 H'' \hbar^2}{8} \frac{\partial^2}{\partial \theta^2} + \frac{\hbar}{2i} (NH' - 1) \frac{\partial}{\partial \theta} + H_0(I_0) + \lambda V \cos 2\theta \right] \psi = \mathcal{E}_n \psi. \quad (9)$$

Thus the quasi-energies \mathcal{E}_n are determined by equation (9). Here $\psi(\theta)$ has to fulfill the periodic boundary condition. It is straightforward to write equation (9) in the form of a Mathieu equation by substituting $\psi = \phi \exp(-2i(NH' - 1)\theta/(N^2 H'' \hbar))$, namely

$$\left[\frac{\partial^2}{\partial \theta^2} + a - 2q \cos 2\theta \right] \phi = 0, \quad (10)$$

with

$$a = \frac{8}{N^2 H''(I_0) \hbar^2} \left[\mathcal{E}_n - H_0(I_0) + \frac{(NH'(I_0) - 1)^2}{2N^2 H''(I_0)} \right], \quad (11)$$

$$q = \frac{4\lambda V}{N^2 H''(I_0) \hbar^2}. \quad (12)$$

The quasi-energy eigenvalues \mathcal{E}_n are determined by the solutions of equation (10) and the requirement that $\phi(\theta + \pi) = \phi(\theta)$. The π -periodic solutions of equation (10) correspond to even functions of the Mathieu equation whose corresponding eigenvalues are real [16]. These solutions are $\phi_\nu(\theta) = e^{i\nu\theta} P_\nu(\theta)$ where $P_\nu(\theta)$ is the even-order Mathieu function. In order to obtain a 2π -periodic solution in the φ -coordinate we require the coefficient of φ to be equivalent to an integer number. This requirement provides us the opportunity to find the value for the index ν of the Mathieu functions as

$$\nu = \frac{2}{N\hbar} \left[I - 4I_0 + 3I_0 \left(\frac{I_0}{I_N} \right)^{1/3} \right], \quad (13)$$

where $I = (n + 3/4)\hbar$. The quasi-energy of the system is finally given by

$$\mathcal{E}_n = \frac{N^2 H'' \hbar^2}{8} a_{\nu(n)}(q) - \frac{(NH' - 1)^2}{2N^2 H''} + H_0(I_0), \quad (14)$$

with the Mathieu characteristic parameter $a_{\nu(n)}(q)$.

From the quasi-energies of equation (14) we can determine the revival time in the presence of the external time-dependent field. Considering $q < 1$ we expand the Mathieu characteristic parameter $a_{\nu(n)}(q)$ up to q^2 . We simplify our result by noting that $H'' = -(Mg^2)^{1/3} (\pi/9I_0^2)^{2/3}$ and $N = (\omega^3/Mg^2)^{1/3} (3I_N/\pi^2)^{1/3}$. Thus we finally obtain

$$T_\lambda = T_0 \left[1 - \frac{1}{2} \left\{ \frac{8\lambda}{\hbar^2} E_N \left(\frac{I_0}{I_N} \right)^2 \right\}^2 \frac{3\nu^2 + 1}{(\nu^2 - 1)^3} \right]. \quad (15)$$

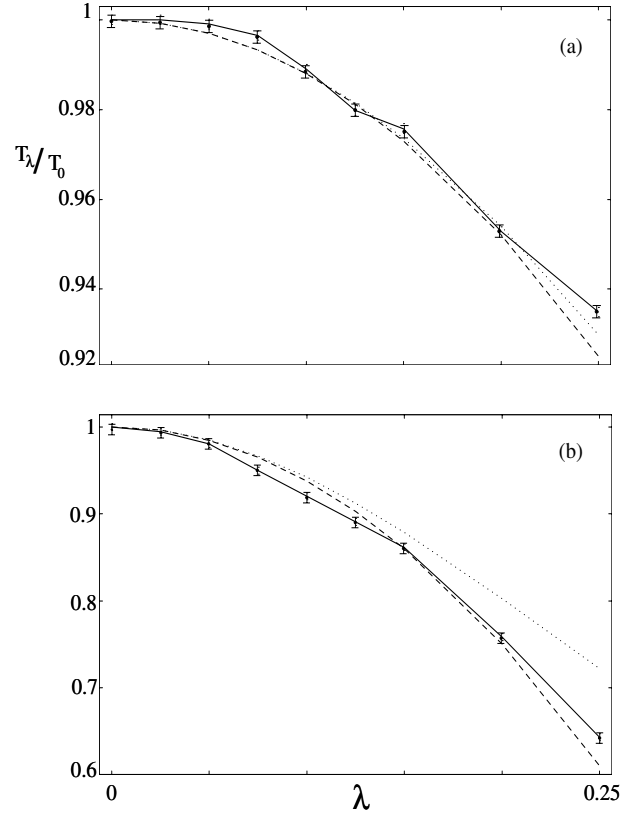


Figure 2. Revival times T_λ as obtained from exact numerical calculations of the time-dependent Schrödinger equation (2) (solid curves), and from analytical results based on equations (7) (dotted curves). Initial conditions are (a) $z_0 = 29.8 \mu\text{m}$, $r = 0.87$ and (b) $z_0 = 20.1 \mu\text{m}$, $r = 0.77$. The other parameters are the same as in figure 1. Here we have $\tilde{\lambda} = \lambda\omega^2/g$. The error bars indicate the uncertainty in determining the revival times from the numerical data. In all cases this uncertainty is less than 2%.

On substituting the value of ν , calculated at $I = I_0$, into equation (15) we obtain

$$T_\lambda = T_0 \left[1 - \frac{1}{8} \left\{ \frac{M\lambda g}{E_{n_0}} \right\}^2 \frac{3(1-r)^2 + a^2}{((1-r)^2 - a^2)^3} \right] \quad (16)$$

where $r \equiv (E_N/E_{n_0})^{1/2}$ and $a \equiv r^2 \hbar \omega / 4E_{n_0}$. If the initial energy is large, i.e. $E_{n_0} \gg \hbar \omega$, we may consider a^2 much smaller than $(1-r)^2$, which leads to

$$T_\lambda = T_0 \left[1 - \frac{3}{8} \left\{ \frac{M\lambda g}{E_{n_0}} \right\}^2 \frac{1}{(1-r)^4} \right]. \quad (17)$$

The analytical results of equations (7) and (17) are main results of this paper. They explain in a simple way the quantitative dependence of the revival time T_λ on the characteristic parameters of the problem, namely the driving frequency ω and the driving amplitude λ . In order to access the accuracy of these perturbative results we calculate the revival time T_λ by integrating the Schrödinger equation equation (2) numerically, and compare it with the analytically obtained results of equations (7) and (17). For this comparison we have considered two different initial conditions of the atomic centre-of-mass wavepacket above

the surface of the atomic mirror. In figure 2(a) $z_0 = 29.8 \mu\text{m}$, which corresponds to a state with mean principal quantum number $n_0 = 322.51$. In figure 2(b) $z_0 = 20.1 \mu\text{m}$, which implies $n_0 = 176.16$. In both cases $p_0 = 0$. The first initial condition lies further away from the centre of the corresponding primary resonance as compared with the second one. As expected, in figure 2(a) the revival time obtained by means of second-order perturbation theory agrees well with the numerical results whereas close to resonance it deviates from the numerical result in figure 2(b). However, the revival time obtained with the help of semiclassical secular theory agrees well with the numerical results in both cases.

Numerically we find that the change in revival time depends quadratically on the strength of external modulation λ as predicted from equations (7) and (17) and displayed in figure 2. The change of the revival time is smaller in the case depicted in figure 2(a) than in the case shown in figure 2(b). This can be understood from our analytical result: in case (a) of figure 2 our chosen initial condition $z_0 = 29.8 \mu\text{m}$ has a higher energy E_{n_0} than in case (b) where $z_0 = 20.1 \mu\text{m}$. Since the change in revival time has inverse dependence on the square of the energy E_{n_0} , as a result, we observe a smaller change in revival time for case (a) as compared with case (b).

We have demonstrated that the dynamics of a material wavepacket in a periodically driven gravitational cavity exhibits quantum mechanical revivals as long as the driving strength of the periodic force does not exceed a critical value. First results on the quantitative dependence of the revival time on the characteristic parameters of the problem, namely the driving frequency and the driving strength, have been presented. It has been shown that this dependence can be understood quantitatively in a satisfactory way by using semiclassical perturbation theory. In view of recent experimental [8] developments the presented quantitative predictions should be accessible to experimental observation.

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