ENTANGLEMENT AND THE LINEARITY OF QUANTUM MECHANICS

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Abstract

Optimal universal entanglement processes are discussed which entangle two quantum systems in an optimal way for all possible initial states. It is demonstrated that the linear character of quantum theory which enforces the peaceful coexistence of quantum mechanics and relativity imposes severe restrictions on the structure of the resulting optimally entangled states. Depending on the dimension of the one-particle Hilbert space such a universal process generates either a pure Bell state or mixed entangled states. In the limit of very large dimensions of the one-particle Hilbert space the von-Neumann entropy of the optimally entangled state differs from the one of the maximally mixed two-particle state by one bit only.

Introduction

Ever since its discovery by Schrödinger [1] the existence of entanglement between different quantum systems has been a major puzzling aspect of quantum theory. If a quantum mechanical many particle system is in an entangled state its characteristic physical properties are distributed over all its subsystems without being present in any one of them separately. In the newly emerging science of quantum information processing [2] these puzzling aspects of quantum theory are recognized as a potentially useful new resource which might help to perform various tasks of practical interest more efficiently than it is possible by any other classical means. Prominent examples in this respect are applications of entangled states in secret quantum key distribution (quantum cryptography) and in fast quantum algorithms (quantum computing).

In view of these developments the natural question arises whether it is possible to design universal quantum processes which entangle two or more quantum systems in an optimal way for all possible initial states of the separate subsystems. Definitely, provided the initial states of the subsystems are known it should always be possible to design a particularly tailored quantum process which produces any desired quantum state. However, this becomes less obvious if one wants to design a universal quantum process which is independent of possibly unknown input states and which performs the required task for all possible input states in the same optimal way. Which constraints are imposed by the fundamental laws of quantum mechanics on such universal, optimal processes?

Recently, similar questions have been studied extensively in the context of quantum cloning [3, 4, 5, 6, 7] where one aims at copying arbitrary quantum states by a universal quantum process. It has been known for a long time that this task cannot be performed perfectly due to constraints imposed by the linear character of quantum theory [8, 9]. According to this linear character any quantum process has to map the density operator of the initial state linearly onto the density operator of the final state. If the relation between the density operators of initial and final states were not linear, one could distinguish different unravellings of one and the same density operator physically. This would contradict the basic postulate of quantum theory that the physical state of a quantum system is described by a density operator and not by any of its possibly inequivalent unravellings [10]. This linear character of quantum theory implies, for example, that despite their nonlocal character it is not possible to use entangled states for super luminal communication [11]. This so called no-signaling constraint of quantum theory enforces the peaceful coexistence of quantum mechanics and relativity [12] and imposes severe restrictions on universal quantum processes [7]. In the context of optimal quantum cloning [3, 4, 5, 6, 7] and the universal NOT gate [13] these constraints have already been investigated. However, their influence on other universal quantum processes is still widely unknown.

Motivated by the importance which entangled states play in the context of quantum information processing in the following the question is addressed whether it is possible to design a universal quantum process which entangles quantum systems in an optimal way. How can one define such a universal, optimal entanglement process and which restrictions are imposed by the linear character of quantum theory? What is the nature of the class of resulting optimally entangled states? Answering these questions sheds new light onto the basic concept of entanglement itself and onto the question which types of entangled states can be prepared by quantum processes in a natural way.

Optimal Entanglement by Universal Quantum Processes

How can one define a universal quantum process which entangles quantum systems in an optimal way for all possible input states?

In order to put this problem into perspective let us consider the simplest possible situation, namely a quantum process which entangles two particles whose associated Hilbert spaces $\mathcal{H}_N$ have equal dimensions of magnitude N. We assume that an arbitrary, pure input state $\rho_{in}(\mathbf{m})$ is entangled with a known reference state ρ_{ref} by a general quantum process

$$\mathcal{P}: \rho_{in}(\mathbf{m}) \otimes \rho_{ref} \to \rho_{out}(\mathbf{m}) \tag{1}$$

thereby yielding the two-particle output state $\rho_{out}(\mathbf{m})$. In particular, we are looking for a universal quantum process which is independent of the input state and entangles both particles in an optimal way for all possible, pure input states $\rho_{in}(\mathbf{m})$.

In an N-dimensional Hilbert space an arbitrary input state can always be represented in the form

$$\rho_{in}(\mathbf{m}) = \frac{1}{N}(\mathbf{1} + m_{ij}\mathbf{A}_{ij}) \tag{2}$$

where the operators $\mathbf{A}_{ij}$ $(i,j = 1,...,N)$ form a basis for the Lie-Algebra of SU_N [14]. (We adopt the usual convention that one has to sum over all indices which appear twice .) Explicitly these operators can be represented by the $N \times N$ matrices

$$(\mathbf{A}_{ij})^{(kl)} = \delta_{ki}\delta_{jl} - \delta_{ij}\delta_{kl}/N \qquad (k,l = 1,...,N) \tag{3}$$

with δ_{ij} denoting the Kronecker delta-function. These operators might be viewed as generalizations of the Pauli spin operators σ_x, σ_y and σ_z to cases with $N > 2$ and they fulfill the relations $\mathrm{Tr}\{\mathbf{A}_{ij}\} = 0$, $\mathbf{A}_{ij}^{\dagger} = \mathbf{A}_{ji}$. The characteristic quantity $\mathbf{m}$ whose components are denoted m_{ij} $(i,j = 1,..,N)$ might be viewed as a generalized Bloch vector. For $N = 2$ the operators $\mathbf{A}_{ij}$ are related to the Pauli spin operators by $\sigma_z \equiv 2\mathbf{A}_{11} \equiv -2\mathbf{A}_{22}$, $\sigma_x + i\sigma_y \equiv 2\mathbf{A}_{12}$ and $\sigma_x - i\sigma_y \equiv 2\mathbf{A}_{21}$. The self-adjointness of the density operator $\rho_{in}(\mathbf{m})$ implies that $m_{ij} = m_{ji}^*$. Furthermore, $\rho_{in}(\mathbf{m})$ represents a pure state only if $\mathrm{Tr}[\rho_{in}^2(\mathbf{m})] = \mathrm{Tr}[\rho_{in}(\mathbf{m})] = 1$ which implies the relation $[m_{ij}m_{ji} - (m_{ii})^2/N] = N^2(1 - 1/N)$. In a similar way also the two-particle output state of Eq.(1) can be expressed in terms of these generators of SU_N according to

$$\begin{aligned}\rho_{out}(\mathbf{m}) \quad = \quad & \frac{\mathbf{1}\otimes\mathbf{1}}{N^2} + \alpha_{ij}^{(1)}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{1} + \\ & \alpha_{ij}^{(2)}(\mathbf{m})\mathbf{1}\otimes\mathbf{A}_{ij} + K_{ijrs}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{A}_{rs}.\end{aligned} \tag{4}$$

What are the basic requirements which an optimal, universal entanglement process $\mathcal{P}$ of the general form of Eq.(1) should fulfill? Definitely the notion of optimal entanglement is not well defined in particular for mixed states due to the lack of a unique measure of entanglement [15, 16, 17]. Despite these difficulties it appears natural to regard the following two conditions as a minimal set of requirements for an optimal, universal entanglement process for two particles, namely

$$\mathrm{Tr}_2\{\rho_{out}(\mathbf{m})\} \quad = \quad \mathrm{Tr}_1\{\rho_{out}(\mathbf{m})\} = \frac{\mathbf{1}}{N}, \tag{5}$$

$$S[\rho_{out}(\mathbf{m})] \quad = \quad -\mathrm{Tr}\{\rho_{out}(\mathbf{m})\ln[\rho_{out}(\mathbf{m})]\} \to \text{minimum} \tag{6}$$

for all possible input states $\rho_{in}(\mathbf{m})$. The first condition expresses the well know property of pure, two-particle entangled states that they behave as maximally mixed states as far as all one-particle properties are concerned. ($\mathrm{Tr}_{1(2)}$ denotes the trace over the state space of particle 1(2).) The second condition states that the entangled two-particle output state $\rho_{out}(\mathbf{m})$ should be as pure as possible so that the associated von-Neumann entropy $S[\rho_{out}(\mathbf{m})]$ is minimal. (In Eq.(6) this entropy is measured in units of Boltzmann's constant.) Together these two requirements imply that in the resulting quantum state $\rho_{out}(\mathbf{m})$ the quantum information is distributed over both particles without being present in each one of the separate particles alone. If one does not consider both particles together one looses a maximum amount of information. In this sense the requirements (5) and (6) characterize optimal entanglement between both particles. In the subsequent treatment it is demonstrated that these two conditions which concentrate on the information theoretic aspects of entanglement characterize uniquely a universal quantum process which yields entangled two-particle output states for arbitrary dimensions of the one-particle Hilbert space $\mathcal{H}_N$.

Conditions (5) and (6) imply that an optimal $\rho_{out}(\mathbf{m})$ can always be found by the covariant ansatz

$$\rho_{out}(\mathbf{Rm}) \quad = \quad U(\mathbf{R})\otimes U(\mathbf{R})\rho_{out}(\mathbf{m})U^{\dagger}(\mathbf{R})\otimes U^{\dagger}(\mathbf{R}) \tag{7}$$

for all possible $\mathbf{R} \in SU_N$. Eq.(7) states that the set of possible two-particle output states $\rho_{out}(\mathbf{m})$ forms a representation of the group $SU_N \times SU_N$ thus ensuring that the von-Neumann entropy is the same for all possible input states $\rho_{in}(\mathbf{m})$. Thereby $\mathbf{R} \in SU_N$ represents the particular unitary transformation with matrix elements R_{ijkl} which transforms a given input state $\rho_{in}(\mathbf{m})$ into an arbitrary other input state $\rho_{in}(\mathbf{m}')$ according to the transformation law $m'_{ij} = R_{ijkl}m_{kl}$. In fact, as conditions (5) and (6) are independent of the input state, the optimal output state can even be found by the more restrictive invariant ansatz $\rho_{out}(\mathbf{Rm}) = \rho_{out}(\mathbf{m})$ for all possible $\mathbf{R} \in SU_N$. This ansatz is a special case of the covariant relation of Eq.(7). However, as we want to investigate universal quantum processes also in a more general context we do not want to impose this more restrictive invariance condition already from the very beginning. Furthermore, as conditions (5) and (6) are also invariant under permutations of the particles an optimal $\rho_{out}(\mathbf{m})$ also has to be permutation invariant.

Apart from the covariance condition of Eq.(7) any quantum process also has to be compatible with the linearity of quantum mechanics. This linearity implies that $\rho_{out}(\mathbf{m})$ has to be a linear function of the generalized Bloch vector $\mathbf{m}$ which characterizes the input state. Thus the characteristic quantities $\alpha_{ij}^{(1)}(\mathbf{m}), \alpha_{ij}^{(2)}(\mathbf{m})$ and $K_{ijrs}(\mathbf{m})$ of Eq.(4) all have to be linear functions of $\mathbf{m}$. This linear dependence guarantees that different unravellings of the same input state yield the same output state after application of the universal quantum process so that this process cannot distinguish between different unravellings of $\rho_{in}(\mathbf{m})$.

Both the covariance condition of Eq.(7) and the linearity constraint impose severe restrictions on general universal quantum processes of the form of Eq.(4).

Covariant and linear universal quantum processes

What is the structure of general covariant and linear quantum processes which result in a two-particle output state which is invariant under permutations of both particles? Answering this question will yield a unified theoretical description for a general class of universal quantum processes which include both universal optimal quantum cloning and universal optimal entanglement as special cases.

The covariance condition of Eq.(7) can be implemented easily by observing that only a tensor product of the form $\mathbf{S} = \mathbf{A}_{ij} \otimes \mathbf{A}_{ji}$ transforms as a scalar under $SU_N \times SU_N$, i.e. $\mathbf{U}(\mathbf{R}) \otimes \mathbf{U}(\mathbf{R}) \mathbf{S} \mathbf{U}^\dagger(\mathbf{R}) \otimes \mathbf{U}^\dagger(\mathbf{R}) = \mathbf{S}$. Similarly, it is straightforward to demonstrate that only tensor products of the form $\mathbf{V}_{il} = \mathbf{A}_{ij} \otimes \mathbf{A}_{jl}$ or $\mathbf{V}_{il}^\dagger$ transform like generalized vectors under $SU_N \times SU_N$, i.e. $\mathbf{U}(\mathbf{R}) \otimes \mathbf{U}(\mathbf{R}) \mathbf{V}_{il} \mathbf{U}^\dagger(\mathbf{R}) \otimes \mathbf{U}^\dagger(\mathbf{R}) = \mathbf{V}_{km} R_{kmil}$. Thus the most general two-particle quantum process which is covariant, linear in $\mathbf{m}$ and invariant under permutations of both particles is of the form

$$\begin{aligned}\rho_{out}(\mathbf{m}) &= \frac{\mathbf{1} \otimes \mathbf{1}}{N^2} + \alpha m_{ij} \mathbf{A}_{ij} \otimes \mathbf{1} + \\ &\alpha m_{ij} \mathbf{1} \otimes \mathbf{A}_{ij} + C \mathbf{A}_{ij} \otimes \mathbf{A}_{ji} + \\ &\beta m_{il} \mathbf{A}_{ij} \otimes \mathbf{A}_{jl} + \beta m_{li} \mathbf{A}_{ji} \otimes \mathbf{A}_{lj} \quad (8)\end{aligned}$$

and is characterized uniquely by the real-valued parameters α, β and C. These characteristic parameters have to be restricted to their physical domain which is defined by the requirement that $\rho_{out}(\mathbf{m})$ is a density operator and must have non-negative eigen values with $\mathrm{Tr}[\rho_{out}(\mathbf{m})] = 1$.

In order to obtain insight into the physical contents of the class of universal, covariant and linear quantum processes which is described by Eq.(8) let us investigate the structure of $\rho_{out}(\mathbf{m})$ more explicitly. Due to the covariance condition we can restrict ourselves to a pure input state with $m_{ij} = N\delta_{i1}\delta_{j1}$ without loss of generality. Introducing an orthogonal basis $\{|1\rangle, ..., |N\rangle\}$ in the N-dimensional one-particle Hilbert space $\mathcal{H}_N$ in which state $|1\rangle$ denotes the input state, i.e. $\rho_{in}(\mathbf{m} = m_{11}\mathbf{e}_{11}) = |1\rangle\langle 1|$, one obtains from Eq.(8) the expression

$$\begin{aligned}\rho_{out}(\mathbf{m} &= m_{11}\mathbf{e}_{11}) = M_{11}|11\rangle\langle 11| + \\ &\sum_{j=2}^{N} |jj\rangle\langle jj|(M_{23} + C) + \\ &\sum_{j=2}^{N} \{|1j\rangle\langle 1j| M_{12} + |1j\rangle\langle j1|(C + \beta m_{11}) + \\ &|j1\rangle\langle 1j|(C + \beta m_{11}) + |j1\rangle\langle j1| M_{12}\} + \\ &\sum_{i<j=2}^{N} \{|ij\rangle\langle ij| M_{23} + |ij\rangle\langle ji| C + \\ &|ji\rangle\langle ij| C + |ji\rangle\langle ji| M_{23}\} \quad (9)\end{aligned}$$

with

$$\begin{aligned}M_{23} &= 1/N^2 - 2\alpha m_{11}/N - C/N + 2\beta m_{11}/N^2, \\ M_{12} &= M_{23} + \alpha m_{11} - 2\beta m_{11}/N, \\ M_{11} &= 1/N^2 + 2\alpha m_{11}(1 - 1/N) + C(1 - 1/N) + \\ &2\beta m_{11}(1 - 1/N)^2.\end{aligned}$$

The non-negativity of $\rho_{out}(\mathbf{m})$ implies the constraints $M_{23} \geq |C|$, $M_{12} \geq |C + \beta m_{11}|$, $M_{23} + C \geq 0$ and $M_{11} \geq 0$.

The two-particle output states of Eqs.(8) or (9) characterize all possible permutation invariant, covariant, linear mappings. They describe in a unified way the restrictions which are imposed by the linearity of quantum mechanics on universal quantum processes which treat both particles in a symmetric way. The universality of these processes guarantees that they fulfill any additional conditions for all possible input states. The general covariance condition of Eq.(7) implies that these additional conditions need not be invariant under unitary transformations. They may very well depend on properties of the initial input state.

As a special case of such a universal quantum process let us consider optimal cloning of pure states [3, 4, 5, 6, 7]. In this case one is looking for a mapping $\mathcal{P}$ of the form of Eq.(1) which fulfills the additional constraint

$$\mathrm{Tr}\{\rho_{in}(\mathbf{m}) \otimes \rho_{in}(\mathbf{m}) \rho_{out}(\mathbf{m})\} \to \text{maximum} \quad (10)$$

for all possible input states $\rho_{in}(\mathbf{m})$. This constraint involves the input state explicitly and it is equivalent to maximizing M_{11} in Eq.(9). Physically speaking this condition imposes the constraint that the output state $\rho_{out}(\mathbf{m})$ should be as close as possible to the ideally cloned state $\rho_{in}(\mathbf{m}) \otimes \rho_{in}(\mathbf{m})$. It is straightforward to work out the optimal parameters which satisfy Eq.(10), namely $2\alpha m_{11} = (N+2)/[N(N+1)], \beta m_{11} = 1/[2N+2], C = 0$. Inserting these parameters into Eq.(9) one realizes that optimal cloning can be achieved only imperfectly with a probability of $P_{11} \equiv M_{11} = 2/(N+1) < 1$. With a probability of $1 - P_{11} = (N-1)/(N+1)$ in this process also an unavoidable maximally mixed state is generated which involves all possible Bell states of the form $|\psi_{1j}\rangle^{(+)} = (|1j\rangle + |j1\rangle)/\sqrt{2}$ with equal probabilities. Thereby state $|j\rangle$ can be any of the $(N-1)$

basis states which are orthogonal to the pure input state $\rho_{in}(\mathbf{m} = m_{11}\mathbf{e}_{11})$. Thus the two-particle output state of the universal, optimal quantum cloning process is given by

$$\rho_{out}(\mathbf{m} = m_{11}\mathbf{e}_{11}) = P_{11}|11\rangle\langle 11| + \frac{(1-P_{11})}{N-1}\sum_{j=2}^{N}|\psi_{1j}\rangle^{(+)}\ {}^{(+)}\langle\psi_{1j}|. \quad (11)$$

Nature of the universal, optimally entangled two-particle states

What is the nature of the entangled states which are produced by the optimal entanglement process characterized by the covariant and linear map of Eq.(9) and by conditions (5) and (6)?

Let us first of all determine the values of the characteristic parameters α, β and C of this universal, optimal entanglement process. Condition (5) implies that $\alpha = 0$. Minimizing the von-Neumann entropy $S[\rho_{out}(\mathbf{m})]$ implies that we have to determine the remaining parameters β and C in such a way that the number of eigen values of magnitude zero is as large as possible. The physical region of the two remaining parameters C and βm_{11} is indicated in Fig. 1 by the black area.

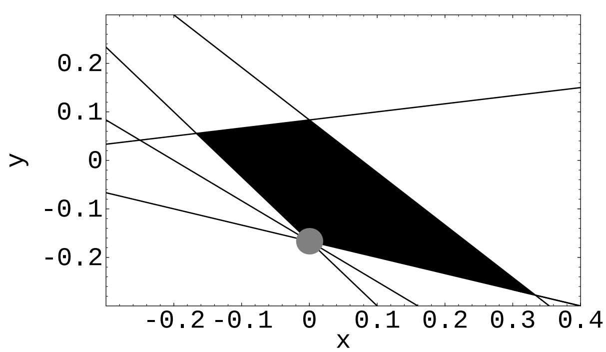

Figure 1: Schematic representation of the physical region of the parameters $y = C$ and $x = \beta m_{11}$ (black) for $\alpha = 0$ and $N = 3$. It is determined by the requirement that $\rho_{out}(\mathbf{m})$ has to be non-negative. Each straight line indicates the parameter values for which one of the eigen values of $\rho_{out}(\mathbf{m})$ is zero. The grey dot indicates the condition for optimal, universal entanglement. It is the only point in which three types of eigen values of $\rho_{out}(\mathbf{m})$ are zero simultaneously.

From Fig. 1 it is straightforward to show that the condition of minimal entropy is fulfilled for $\beta = 0$ and $C = -1/[N(N-1)]$. This implies that $\rho_{out}(\mathbf{m})$ transforms indeed as a scalar under $SU_N \times SU_N$ as we have already anticipated earlier. Thus the two-particle output state which is produced by the optimal, universal entanglement process is independent of the input state $\rho_{in}(\mathbf{m})$ and is given by

$$\rho_{out}(\mathbf{m}) = \frac{2!}{N(N-1)}\sum_{i<j=1}^{N}|\psi_{ij}\rangle^{(-)}\ {}^{(-)}\langle\psi_{ij}|. \quad (12)$$

In general, $\rho_{out}(\mathbf{m})$ is a maximally disordered mixture of all possible anti-symmetric Bell states

$$|\psi_{ij}\rangle^{(-)} = \frac{1}{\sqrt{2}}(|ij\rangle - |ji\rangle) \quad (13)$$

which can be formed by two possible basis states $|i\rangle$ and $|j\rangle$ of the N-dimensional one-particle Hilbert space $\mathcal{H}_N$. The number of these anti-symmetric Bell states is given by $[N(N-1)/2!] = \binom{N}{2}$. It is interesting to realize that it is only the anti-symmetric Bell states which appear in this optimal, universal entanglement process. This is understandable from the fact that these Bell states are the only ones which are invariant under arbitrary unitary transformations. This invariance property guarantees that one obtains entangled output states for all possible input states so that the resulting entanglement process is universal. The other three two-particle Bell states, namely

$$|\psi_{ij}\rangle^{(+)} = \frac{1}{\sqrt{2}}(|ij\rangle + |ji\rangle),$$
$$|\Phi_{ij}\rangle^{(\pm)} = \frac{1}{\sqrt{2}}(|ii\rangle \pm |jj\rangle), \quad (14)$$

do not have this invariance property. If they appeared in the two-particle output state, it would always be possible to find a particular input state which produces a separable, non-entangled output state. Thus such a quantum process would not fulfill the universality requirement.

For the case of universal optimal entanglement of a qubit, i.e. for $N = 2$, there is only one possible anti-symmetric Bell state, namely $|\psi_{12}\rangle^{(-)}$. Thus in this particular case the universal entanglement process of Eq.(12) produces the pure two-particle output state $\rho_{out}(\mathbf{m}) = |\psi_{12}\rangle^{(-)}\ {}^{(-)}\langle\psi_{12}|$ which is known to violate Bell inequalities maximally [10]. For all higher values of the dimension of the Hilbert space $\mathcal{H}_N$ the two-particle output state is mixed. Nevertheless according to condition (6) the von-Neumann entropy of all possible output states is always as small as possible within the linearity constraints imposed by quantum theory. Furthermore, it is straightforward to show that all output states are not separable as their partial transposes have at least one negative eigen value [18] of magnitude $\lambda = -1/[N(N-1)] < 0$.

How do these optimal, universal two-particle output states behave for high values of the dimension of the one-particle Hilbert space $\mathcal{H}_N$? In general the von-Neumann entropy of the two-particle output state is given by

$$S[\rho_{out}(\mathbf{m})] = \ln\binom{N}{2} = \ln[N(N-1)] - \ln[2!]. \quad (15)$$

For $N \gg 1$ this entropy approaches the value $S[\rho_{out}(\mathbf{m})] \to \ln[N^2] - \ln[2!]$. Thereby $\ln[N^2]$ is the entropy of the maximally disordered two-particle state $\rho_{max} = \mathbf{1} \otimes \mathbf{1}/N^2$. Thus, in the limit of large dimensions of the Hilbert space $\mathcal{H}_N$ the entropies of ρ_{max} and of $\rho_{out}(\mathbf{m})$ differ by one bit only . This shows that in the limit of large N these universal, optimally entangled states are located very close to the maximally mixed state ρ_{max} from the information theoretic point of view. They are very fragile with respect to any disturbances. Loosing one bit of information only changes them to a maximally mixed state ρ_{max}. Nevertheless, it is worth pointing out that this does not necessarily imply that these states are also close to ρ_{max} in state space. In order to characterize the distance of a mixed quantum state ρ from the maximally mixed one in state space one usually decomposes ρ according to

$$\rho = (1-\epsilon)\mathbf{1}/d + \epsilon\rho_1 \tag{16}$$

with a suitably chosen density operator ρ_1 (with $\mathrm{Tr}[\rho_1] = 1$) and with d denoting the dimension of the relevant Hilbert space. The quantity ϵ might be considered as characterizing the separation of ρ from the maximally mixed state. Mixed states which are close to the maximally mixed one in the sense that $0 \leq \epsilon \ll 1$ are of particular interest for quantum information processing in high-temperature nuclear magnetic resonance [19, 20, 21]. In this context Braunstein et al. [22] have shown recently that in systems consisting of n-qubits with $d = 2^n$ one can always find a sufficiently small neighborhood around the maximally mixed state with $\epsilon = O(4^{-n})$ within which all states are separable. In view of this result it is of interest to work out also the distance of the universal, optimally entangled states of Eq.(12) from the maximally mixed state ρ_{max} in state space. As many of the possible N^2 eigen values of $\rho_{out}(\mathbf{m})$ are zero these states are characterized by $\epsilon = 1$. Thus, despite their closeness to ρ_{max} as far as the von-Neumann entropy is concerned, these latter states are well separated from ρ_{max} in state space for all possible values of the dimension of the Hilbert space $\mathcal{H}_N$.

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