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Photon wave packets and spontaneous decay in a cavity

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A pure-state analysis of spontaneous emission of photons by a one-dimensional harmonic oscillator in a spherically symmetric cavity is presented. Using a *classical path representation* with respect to photon paths the transition between the limiting cases of coupling to a single cavity mode and the continuum limit is discussed analytically.

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Recent progress in the realization of optical cavities with high-quality factors has stimulated much experimental and theoretical work [1-9] in the field of cavity quantum electrodynamics. In this context an understanding of the spontaneous emission of photons by atoms inside a cavity is of great interest. Depending on the size of the cavity we may distinguish two limiting dynamical regimes: In the *small-cavity limit* the atomic spontaneous decay rate Γ is much smaller than the frequency difference between adjacent cavity modes $\Delta\omega$. In this case only one mode of the radiation field is excited significantly by the spontaneous decay process and, typically, energy is exchanged periodically between the atom and the field mode. This case is adequately described by the single-mode Jaynes-Cummings model [4,5]. In the *large-cavity limit,* i.e., $\Gamma \gg \Delta\omega$, the cavity is so large that many cavity modes are excited coherently by the spontaneous decay process. Thus a localized photon wave packet is generated. In the limit of an infinitely large cavity (continuum limit) this generation of a photon wave packet is connected with an irreversible exponential decay of the atom. However, if the cavity is finite, eventually the photon wave packet is reflected at the boundary of the cavity and returns to the atom with which it may then exchange energy.

In this paper we study a model problem where non-Markovian cavity-size effects, and in particular the transition between the limiting cases of the coupling to a single cavity mode and the continuum limit, can be studied analytically, namely, a one-dimensional (material) harmonic oscillator that is coupled to the quantized radiation field inside a spherical cavity [10]. Using an analogy between a harmonic oscillator and a large number of two-level atoms [11] this system might approximately describe, for example, the interaction of two-level atoms with the quantized radiation field in cases where only a few of these two-level atoms are excited. In the following it is shown that the difficulties arising from the coupling of the harmonic oscillator to a large number of cavity modes and the corresponding large eigenvalue problem for the dressed states [7] may be overcome with the help of a *classical path representation* [12]. This approach is in the spirit of the "photon wave-packet interpretation" which has been used by Parker and Stroud for the interpretation of multimode corrections to the single-mode Jaynes-Cummings model in a spherically symmetric cavity [6]. Thereby the state of the coupled oscillator-field system is represented as a sum of contributions of all classical paths of a spontaneously emitted photon in the cavity. As an example the time evolution of a spontaneously generated photon wave packet is discussed. In particular, it is shown that repeated harmonic-oscillator–photon scatterings significantly influence this time evolution.

The Hamiltonian of a one-dimensional harmonic oscillator (frequency ω_0) that is located in the center of a spherically symmetric cavity with radius R and perfectly conducting walls and which interacts with the modes of the quantized electromagnetic field (frequencies $\omega_n, n=1,\ldots$) is given by

$$H=\hbar\omega_0(b_0^\dagger b_0+\tfrac{1}{2})+\sum_{n=1,\ldots}\hbar\omega_n b_n^\dagger b_n + \sum_{n=1,\ldots}(\alpha_n b_0 b_n^\dagger+\alpha_n^* b_0^\dagger b_n) \quad (1)$$

in the dipole and rotating-wave approximation. The operators b_n and $b_n^\dagger$ are the destruction and creation operators of the harmonic oscillator ($n=0$) and of the electromagnetic field modes ($n=1,\ldots$) and the coupling constants are denoted α_n. In the following we shall be interested in cases where only the coupling to highly excited modes of the cavity is significant, i.e., $\omega_0 \approx \omega_n = \pi cn/R$ ($n \gg 1$).

The time evolution of the oscillator-field system may be described in a basis of coherent states $|\beta_0,\beta_1,\ldots\rangle$ with $b_n|\beta_0,\beta_1,\ldots\rangle=\beta_n|\beta_0,\beta_1,\ldots\rangle$ ($n=0,1,\ldots$). According to Eq. (1) the time evolution of a coherent state is given by $|\psi\rangle_t=|\beta_0(t),\beta_1(t),\ldots\rangle$ with

$$\beta_n(t)=\sum_{l,m}U_{nl}e^{-i\Lambda_l t/\hbar}U_{lm}^\dagger\beta_m(t=0)\,. \quad (2)$$

The unitary matrix U_{nl} ($n,l=0,1,\ldots$) characterizes the transformation of the Hamiltonian of Eq. (1) to normal modes and Λ_l are the dressed energies of the coupled harmonic-oscillator-field system. From Eq. (1) we find the explicit expressions ($m=0,1,\ldots$)

$$U_{0m}=\left[1+\sum_{n=1,\ldots}\frac{|\alpha_n|^2}{|\hbar\omega_n-\Lambda_m|^2}\right]^{-1/2} = \left[-\frac{df}{d\Lambda}\right]_{\Lambda=\Lambda_m}^{-1/2}, \quad (3)$$

$$U_{n,m}=-\frac{\alpha_n}{\hbar\omega_n-\Lambda_m}U_{0m}\quad(n=1,\ldots)\,,$$

with Λ_m being determined by the relation $f(\Lambda_m)=0$ with

$$f(\Lambda_m)=\hbar\omega_0-\Lambda_m-\sum_{n=1,\ldots}\frac{|\alpha_n|^2}{\hbar\omega_n-\Lambda_m}. \quad (4)$$

If the harmonic oscillator couples only to highly excited modes of the cavity, we may perform the sum in Eq. (4) with the help of the Poisson sum formula [13] thus obtaining

$$f(\Lambda)=\hbar\omega_0-\hbar\delta\omega-\Lambda-i\hbar\Gamma/2 +2i\left(\frac{\hbar\Gamma}{2}\right)\frac{e^{iS_{\rm ph}(\Lambda)}}{1-e^{iS_{\rm ph}(\Lambda)}}, \quad (5)$$

with the Lamb shift $\delta\omega$ and the spontaneous decay rate $\Gamma=2|\alpha_n|^2R/(\hbar^2c)|_{\omega_n=\omega_0}$ of the first excited state of the harmonic oscillator. These quantities characterize the coupling of the harmonic oscillator to the electromagnetic field in the absence of conducting walls, i.e., in the continuum limit. The influence of the perfectly conducting walls on the dressed energies of the oscillator-field system is described by the terms which involve the classical action $S_{\rm ph}(\Lambda)=2\Lambda R/\hbar c$ which a photon of energy Λ accumulates on a purely radial path from the center of the cavity to the wall and back again.

Equations (2), (3), and (5) give a practical description of the dynamics of the coupled oscillator-field system as long as only a few normal modes (dressed states) are excited. In the *small-cavity limit*, i.e., $\Gamma\ll\pi c/R$, the harmonic oscillator couples to only one mode of the electromagnetic field (with frequency $\omega_{\bar n}$) significantly. From Eq. (5) we find that in this case the (dressed) energies which contribute significantly to $U_{n,m}$ are approximately given by $\Lambda_\pm=\Delta\Lambda_\pm+\hbar\omega_{\bar n}$ with

$$\Delta\Lambda_\pm=\frac{\hbar(\omega_0-\delta\omega-\omega_{\bar n})}{2}\pm\left[\left(\frac{\hbar(\omega_0-\delta\omega-\omega_{\bar n})}{2}\right)^2+(\hbar\Omega)^2\right]^{1/2} \quad (6)$$

and the "vacuum Rabi frequency" $\Omega=(\Gamma c/2R)^{1/2}$ [9].

In the *large-cavity limit*, i.e., $\Gamma\gg\pi c/R$, many dressed states contribute to the sum of Eq. (2). In these cases a simple description of the dynamics may be obtained by expressing Eq. (2) by a contour integral with the help of Eq. (5), i.e.,

$$\beta_n(t)=\sum_m\frac{-1}{2i\pi}\int_{-\infty+i0}^{\infty+i0}d\Lambda\,U_n(\Lambda)[f(\Lambda)]^{-1}\times\frac{df}{d\Lambda}U_m^*(\Lambda)\beta_m(t=0)e^{-i\Lambda t/\hbar}, \quad (7)$$

with $U_n(\Lambda_m)=U_{nm}$ given in Eq. (3). Inserting Eq. (5) into Eq. (7) and expanding the denominator into a geometric series we obtain a *classical path representation.* Thereby the complex amplitudes of the coherent state, i.e., $\beta_n(t)$, are expressed as a sum of contributions of all radial paths of a photon which originate at the position of the harmonic oscillator in the center of the spherically symmetric cavity.

For a physical discussion let us consider the spontaneous decay of an energy eigenstate of the (material) harmonic oscillator with energy $E_k=\hbar\omega_0(k+\frac{1}{2})$ in more detail. The probability of finding this harmonic oscillator in an energy eigenstate $|E_l\rangle$ at a later time is given by the binomial distribution $P_l(t)=\binom{k}{l}|f_0(t)|^{2l}[1-|f_0(t)|^2]^{k-l}$ $(0\le l\le k)$ with

$$f_0(t)=e^{-i(\omega_0-\delta\omega-i\Gamma/2)t}+\frac{i\hbar\Gamma}{2i\pi}\sum_{M=1}^{\infty}\int_{-\infty+i0}^{\infty+i0}d\Lambda e^{-i\Lambda t/\hbar}\times e^{iS(\Lambda)_{\rm ph}}(\tilde\chi e^{iS(\Lambda)_{\rm ph}})^{M-1}[\Lambda-\hbar(\omega_0-\delta\omega)+i\hbar\Gamma/2]^{-2}. \quad (8)$$

The first term of Eq. (8) describes the spontaneous decay of the harmonic oscillator due to spontaneous emission of photons. This decay process is characterized by the decay rate Γ and the Lamb shift $\delta\omega$ and is therefore not influenced by the walls of the cavity. In the continuum limit, i.e., $t<2R/c, R\to\infty$, this is the only contribution to $f_0(t)$ [10] [compare also with Eq. (10)]. At time $t\approx R/c$ the spontaneously emitted photons are reflected at the wall of the spherical cavity and return to the harmonic oscillator at multiples of the photon return time $T=2R/c$. The Mth term in the sum of Eq. (8) may be interpreted as the contribution which originates from the Mth return of the spontaneously emitted photons to the center of the cavity where some of them are absorbed by the harmonic oscillator. Due to the spherical symmetry of the cavity all these emitted photons return at the *same time* and *in phase.* During each of the $(M-1)$ intermediate returns to the harmonic oscillator photons may be absorbed and reemitted again in different directions. This elastic scattering process is described by the scattering matrix element

$$\tilde\chi=1-\frac{i\hbar\Gamma}{\Lambda-\hbar(\omega_0-\delta\omega)+i\hbar\Gamma/2}. \quad (9)$$

Evaluating Eq. (8) with contour integration we finally find the equivalent result

$$f_0(t)=e^{-i(\omega_0-\delta\omega-i\Gamma/2)t}+\sum_{M=1}^{\infty}\Theta(t-2RM/c)e^{-i(\omega_0-\delta\omega-i\Gamma/2)[t-2RM/c]}\times\frac{-\Gamma(t-2RM/c)}{M}L_{M-1}^{(1)}(\Gamma(t-2RM/c)), \quad (10)$$

with the Laguerre polynomial

$$L_{M-1}^{(1)}(x)=\sum_{r=0}^{M-1}\binom{M}{M-1-r}\frac{(-x)^r}{r!}$$

and the unit step function $\Theta(x)$. The classical path representations as given in Eqs. (7) and (10) are the main results of this paper. According to Eq. (10) the topologically different classes of photon paths in the cavity are characterized by the total number of returns to the center of the cavity M and the number of intermediate scatterings r $(0 \leq r \leq M-1)$ in the time between the initial emission and the final absorption by the harmonic oscillator.

In Fig. 1 the time evolution of $P_1(t)=|f_0(t)|^2$ $(k=l=1)$ is shown. In this special case the results of the spontaneously decaying harmonic oscillator reduce to the corresponding results of a two-level system. In Fig. 1(a) the spontaneous decay time $1/\Gamma$ is comparable to the photon return time $T=2R/c$. This implies that only a few modes of the radiation field are exited significantly which causes an approximately periodic exchange of energy between the oscillator and the few resonantly coupled cavity modes. The time scale of this energy exchange is determined by the "vacuum Rabi frequency" $\Omega=(\Gamma c/2R)^{1/2}$. This case can be described conveniently by the dressed-state representation of Eq. (2). Such Rabi-type oscillations are a typical quantum phenomenon [5] and appear similarly in the Jaynes-Cummings model [4].

Increasing the photon return time T by increasing the radius of the spherical cavity the spontaneous emission process eventually becomes localized in time in the sense that $1/\Gamma \ll T$ [Fig. 1(b)]. Thereby a photon wave packet is generated. This photon wave packet represents a state of the electromagnetic field which is localized with respect to the radial coordinate in comparison with the extension of the cavity. Its angular coordinates are delocalized and in the dipole approximation their probability distribution is determined by the vector spherical harmonic with angular momentum quantum numbers $l=1$, $m=0$ [14]. Whenever this wave packet returns to the harmonic oscillator some of the photons may be absorbed. Alternatively these photons may also be scattered into different directions and finally be absorbed again at one of the wave packets subsequent returns to the oscillator. Thus observing transition probabilities of the harmonic oscillator as a function of time we obtain a picture of the time evolution of this photon wave packet in the cavity. In particular, from Eq. (10) and Fig. 1(b) we notice that the contribution which is due to the Mth return of the photon wave packet consists of M maxima. They arise from the $r=0,\ldots,(M-1)$ possible numbers of intermediate scatterings of the photon wave packet during the time of its initial preparation at $t=0$ and its final absorption at t. According to Eq. (10) the contribution to $f_0(t)$ which is due to the Mth return with r intermediate scatterings is maximal at time $t=2MR/c+(r+1)/(\Gamma/2)$. This indicates that each scattering process is associated with a time delay $\tau=2/\Gamma$. Furthermore, according to Eq. (10) the factor $(-1)^{r+1}$ indicates that each scattering process leads to a resonant phase shift of magnitude π. Thus after r scatterings the original photon wave packet has developed r maxima.

To observe the kind of effects discussed, for example, in Fig. 1(b) with a spontaneous decay rate of the order of $\Gamma=10^9\ \mathrm{s}^{-1}$ would require typical cavity sizes of the order of $R=1$ m which is rather large in comparison with typical cavities used in present experiments. However, any collective spontaneous emission process which involves many atoms increases the relevant spontaneous decay rate significantly thus decreasing the required cavity size. In the future classical path representations like the one presented here might become a valuable tool for studying the time evolution of coherent matter-photon excitations in cavity-quantum-optical problems in particular in cases where the mode structure of the cavity is complicated.

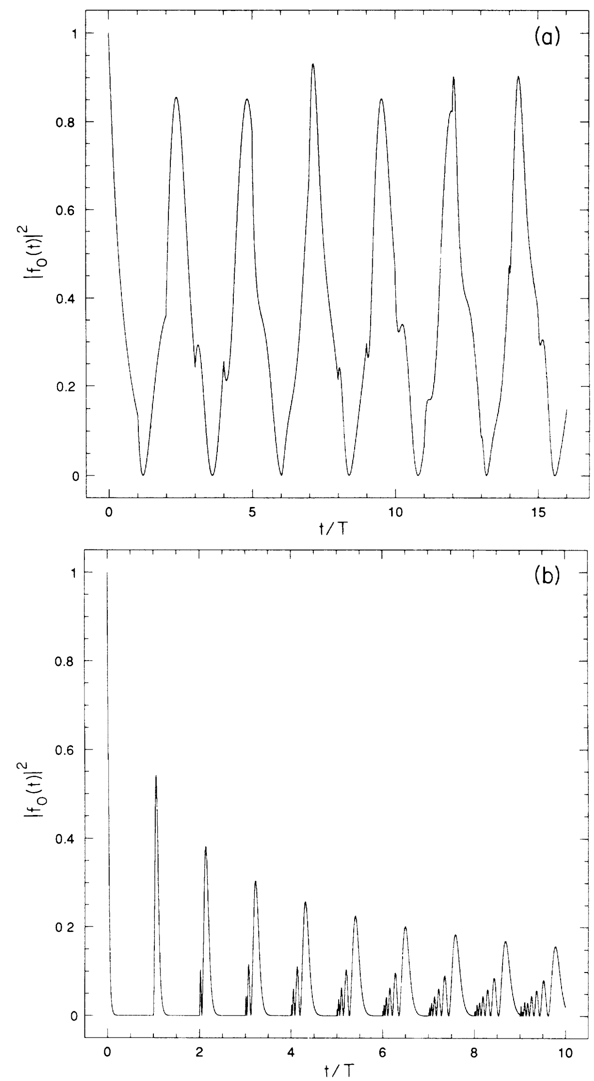

FIG. 1. $|f_0(t)|^2$ as a function of time (in units of $T=2R/c$) for resonant coupling to the cavity modes $(\omega_0-\delta\omega=n\pi c/R$, $n\gg 1)$ and different sizes of the cavity: (a) $\Gamma R/c=1.0$; (b) $\Gamma R/c=20.0$.

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