Motion tomography of a single trapped ion

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A method for the experimental reconstruction of the quantum state of motion for a single trapped ion is proposed. It is based on the measurement of the ground-state population of the trap after a sudden change of the trapping potential. In particular, we show how the $Q(\alpha)$ function and the quadrature distribution $P(x,\theta)$ can be measured directly. In an example we demonstrate the principle and analyze the sensitivity of the reconstruction process to experimental uncertainties as well as to finite grid limitations. Our method is not restricted to the Lamb-Dicke Limit and works in one or more dimensions.

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The central entity of quantum physics is the density operator ρ. It contains all measurable information about the state of a system that can be obtained according to the principles of quantum physics. Recent theoretical advances established constructive procedures to recover the full information about the state of a system from the repeated measurement of a complete set of observables. From the experimentally detected probabilities

$$Q(\alpha)=\frac{1}{\pi}\mathrm{Tr}[\rho|\alpha\rangle\langle\alpha|]=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \tag{1a}$$

$$P(x,\theta)=\mathrm{Tr}[\rho|x,\theta\rangle\langle x,\theta|]=\langle x,\theta|\rho|x,\theta\rangle, \tag{1b}$$

one can determine the state ρ uniquely. Here $|\alpha\rangle$ denotes a coherent state and $|x,\theta\rangle$ is a quadrature eigenstate. So far, the underlying theory has been developed for finite-dimensional discrete systems [1,2], such as spin- or angular-momentum states, as well as for continuous systems. This approach is generally referred to as phase-space tomography [3].

One of the most important and beautiful applications has been the tomographic measurement of the Wigner function for a single mode of the electromagnetic radiation field by Raymer and co-workers [4] following the proposal by Vogel and Risken [3]. It is based on a measurement of the quadrature distribution $P(x,\theta)$ with the help of a homodyne technique. On the other hand, the Q function has been measured recently in various experimental schemes, using well-known techniques of photodetection, together with related experiments of phase measurement [5].

Apart from cavity QED, a single trapped ion is one of the other testing grounds for the intriguing features of quantum mechanics [6] (for other applications in the area of quantum physics, see, for example, molecular spectroscopy [7] or image reconstruction for matter waves [8]). The motion of a single trapped ion can be easily modified using laser light, and decoherence in such a system can be made nearly negligible during long times. Using these properties, several proposals have emerged dealing with the preparation of nonclassical states of motion. Just recently, an observation of nonclassical states, such as Fock states, and squeezed states was reported [9]. Hence, the next step of research is to characterize these states. Given the analogy between cavity QED and a trapped ion interacting with a laser, one could imagine that some techniques developed in the framework of cavity QED can be immediately transcribed to the ion system. For example, one can characterize the motional state by measuring the evolution of the ion population inversion [10–13]. Endoscopic techniques, for example, permit a complete state detection if there is no statistical uncertainty in the state preparation process. Unfortunately, this method does not allow us to recover the whole density matrix describing the ion motion. Moreover, the mentioned analogy is only valid in the Lamb-Dicke regime, whereby the motion of the ion is restricted to a region smaller than a wavelength, which limits the applicability of these methods. Thus, it would be desirable to have a method to recover the full information about the motional state of an ion that is valid for more general situations.

In this Rapid Communication we propose a realization of a phase-space tomography to determine the motional state of an ion in a harmonic trapping potential. In contrast to a recent proposal [14] that addressed the same question, our scheme is not restricted to the Lamb-Dicke regime and can be extended easily to more than one spatial dimension. Furthermore, an implementation of this idea is feasible with present experimental setups. Specifically, we will present procedures (i) to measure the $Q(\alpha)$ function, and (ii) the quadrature distribution function $P(x,\theta)$.

Our model consists of a single ion, trapped in a harmonic potential oscillating with a frequency ν. The internal structure of the ion will be specified later in the context of the measurement of the motional state. We use a density operator ρ to describe this unknown state of the particle and represent it in the Fock basis of the harmonic oscillator, i.e.,

$$\rho=\sum_{n,m=0}^{\infty}\rho_{nm}|n\rangle\langle m|. \tag{2}$$

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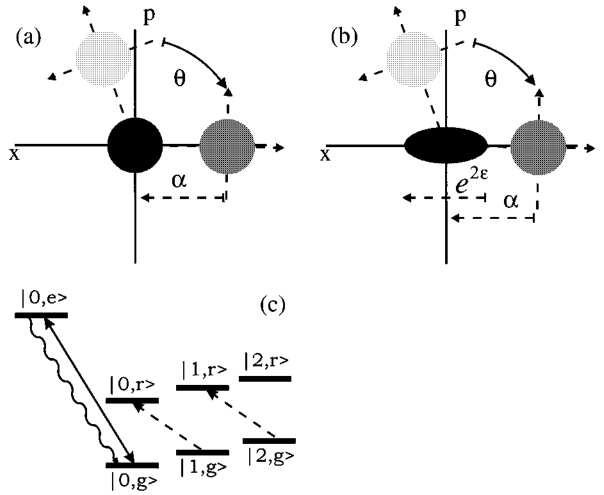

FIG. 1. Phase-space representation of the operations (phase shifting, displacement, and squeezing) required to measure (a) the $Q(\alpha)$ function and (b) the quadrature distribution; (c) level scheme and laser configuration for the detection of the trap ground-state population.

Let us first show how the Q function given in Eq. (1a) can be measured experimentally. For this purpose, we reexpress the Q function as

$$Q(\alpha)=\frac{1}{\pi}\langle 0|\tilde{\rho}|0\rangle, \tag{3}$$

where

$$\tilde{\rho}=U(|\alpha|,\theta)\rho U^{\dagger}(|\alpha|,\theta). \tag{4}$$

Here, $U^{\dagger}(|\alpha|,\theta)=\mathscr{R}^{\dagger}(\theta)\mathscr{D}(|\alpha|)$ is the unitary transformation that is given by the displacement operator $\mathscr{D}(\alpha)=\exp(\alpha a^{\dagger}-\alpha^{*}a)$ and the phase shifting operator $\mathscr{R}(\theta)=\exp(-i\theta a^{\dagger}a)$ that, acting on the vacuum, create a coherent state $\||\alpha|e^{i\theta}\rangle=U^{\dagger}(|\alpha|,\theta)|0\rangle$. As usual, $a^{\dagger}$ and a denote creation and annihilation operators that obey $[a,a^{\dagger}]=1$.

According to Eq. (3), one has to determine the probability of the state represented by $\tilde{\rho}$ to be in the ground state of the harmonic potential, in order to measure this Q function. Note that such a state is related to the original ρ by the unitary transformation $U(|\alpha|,\theta)$. Consequently, the identification of this transformation with a physical process would enable us to measure the Q function. In the context of an ion trapped in a harmonic potential, this identification is as follows. The operator $\mathscr{R}(\theta)$ corresponds to the free evolution, whereas the operator $\mathscr{D}(\alpha)$ corresponds to a sudden displacement of the harmonic trap. A pictoral representation of these operations in phase space is shown in Fig. 1(a). Thus, in order to measure the Q function in a trap, one simply has to perform the following steps: (i) Wait a particular time t while the ion evolves freely in the trap. This gives it the appropriate phase shift according to $\theta=\nu t$. (ii) Suddenly displace the center of the trap to the right for a distance d, so that $|\alpha|=\sqrt{m\nu/2\hbar}d$. (iii) Finally, measure the probability of the ion to be in the lowest motional state $|0\rangle$.

To achieve this last step, one may use the internal structure of the ion. Typically, it consists of three levels $|g\rangle$, $|e\rangle$, and $|r\rangle$, where $|g\rangle\rightarrow|r\rangle$ is a dipole-forbidden transition or Raman transition, whereas $|g\rangle\rightarrow|e\rangle$ is a dipole-allowed transition. Initially, the ion is in the internal ground state $|g\rangle$. After step (ii) a laser beam is tuned to the lower sideband of the $|g\rangle\rightarrow|r\rangle$ internal transition. One can then transfer completely the population of the ground states $|n,g\rangle$ (with $n=1,2,\ldots$) to the excited states $|n-1,r\rangle$ coherently as described in Ref. [15] by an adiabatic sweep of the laser frequency. After this population transfer one can switch on another laser, this time on resonance with the transition $|g\rangle\rightarrow|e\rangle$ [see Fig. 1(c)]. The appearance of fluorescence indicates the presence of population in the $|0,g\rangle$ state. One can repeat the same sequence of steps in order to determine the probability of the ion being in the ground state. An alternative (more sophisticated) way of measuring this probability may be achieved by detecting the collapses and revivals in the population inversion, since this technique provides the whole population of the Fock states [10].

Up to now, we have not addressed the question of the final reconstruction of our state from the experimental data of the Q function that is obtained in this manner. In principle, one could use the method [16] that relates this function to the matrix elements $\rho_{n,m}$. This is, however, impractical, since it requires the nth and mth derivatives of the Q function, i.e., knowledge of them over a continuous interval. Another possibility would be to assume that $\rho_{n,m}=0$ for $n,m>n_{\max}$, for a given $n_{\max}$. The measurement of $Q(\alpha_i)$ for $n_{\max}^2$ (independent) values of α_i would allow us to find $\rho_{n,m}$ by simple matrix inversion. This procedure is also of limited usefulness since small deviations from the exact values of the Q function (such as experimental uncertainties) give large errors in the reconstruction. This is due to the fact that Q is the smoothest function of all s-parametrized quasidistributions.

An alternative way of reconstructing the state of a quantum system is by means of quantum tomography. Tomography is an experimental tool used in several areas of research that allows us to reconstruct an unknown object from measured data. In the context of quantum physics, the data we will have to measure are the so-called quadrature distribution functions given by Eq. (1b), where $|x,\theta\rangle=\mathscr{R}^{\dagger}(\theta)|x\rangle$ is the eigenstate of the operator $\hat{x}(\theta)=\mathscr{R}^{\dagger}(\theta)\hat{x}\mathscr{R}(\theta)$, with eigenvalue x ($\hat{x}$ is the dimensionless position operator of the harmonic oscillator $m\nu/\hbar\rightarrow1$). This distribution is equivalent to that given by the marginal distribution for quadrature components using the Wigner-function description of the state [5].

Our scheme for the measurement of $P(x,\theta)$ is based on the well-known property of the squeezed states,

$$|x,\theta\rangle=\lim_{|\epsilon|\rightarrow\infty}\mathscr{N}_{\epsilon}|\alpha,\epsilon\rangle, \tag{5}$$

where $|\alpha,\epsilon\rangle=\mathscr{D}(\alpha)\mathscr{S}(\epsilon)|0\rangle$, $\mathscr{S}(\epsilon)=\exp[(\epsilon^{*}a^{2}-\epsilon a^{\dagger 2})/2]$ is the "squeeze" operator, $\epsilon=|\epsilon|e^{2i\theta}$, and $\alpha=xe^{i\theta}/\sqrt{2}$ ($0\leqslant\theta<\pi$). As proper position eigenstates are not normalizable, there is a constant of proportionality $\mathscr{N}_{\epsilon}=[\exp(2|\epsilon|)/(4\pi)]^{1/4}$ that increases with the degree of squeezing. As before, we can use these states to reexpress the quadrature distribution in the form

$$P(x,\theta)=\lim_{|\epsilon|\to\infty}|\mathcal{N}_\epsilon|^2\langle 0|\tilde{\rho}|0\rangle. \tag{6}$$

Here

$$\tilde{\rho}=U(|\epsilon|,|\alpha|,\theta)\rho U^\dagger(|\epsilon|,|\alpha|,\theta), \tag{7}$$

where $U^\dagger(|\epsilon|,|\alpha|,\theta)=\mathcal{R}^\dagger(\theta)\mathcal{D}(|\alpha|)\mathcal{S}(|\epsilon|)$ denotes the operation that creates a squeezed state, and we furthermore use the property $\mathcal{S}(\epsilon)=\mathcal{R}^\dagger(\theta)\mathcal{S}(|\epsilon|)\mathcal{R}(\theta)$. Thus, to measure the quadrature distribution one has to find the physical processes that correspond to the unitary operators $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{S}$. The first two are the same as those needed for the experimental determination of the Q function. On the other hand, it is well known that sudden changes in the frequency of a harmonic oscillator lead to squeezed states, a process that can be readily achieved in a trap, just opening or closing the harmonic potential [17]. In particular, changing the trap frequency from ν to ν' leads to a squeezing parameter $|\epsilon|=\frac{1}{2}\ln(\nu/\nu')$.

Thus, in order to measure the quadrature distribution in our trap one has to follow these four steps. (i) Wait for a time t, such as $\theta=\nu t$. (ii) Perform a sudden displacement of the center of the trap to the right a distance d, so that $|\alpha|=\sqrt{m\nu/2\hbar}d$. (iii) Change the trap frequency instantaneously from ν to ν'. (iv) Determine the population of the motional ground state. Note that the steps (i), (ii), and (iv) are the same as before.

We are now in position to extract the full information about the unknown quantum state starting from the quadrature distribution. This can be done as in the case where one measures the quantum state of light by means of balanced homodyne detection. As has been shown by Vogel and Risken [3], one can reconstruct the Wigner function $W(x,p)$ by means of the so-called inverse Radon transformation. Alternatively, one can use one of the algorithms that have been developed for reconstructing the density matrix directly from discrete measured data [18].

To illustrate this procedure, we have numerically simulated the reconstruction of a quantum state. We assumed that the system is prepared initially in a ''Schrödinger-cat'' state of the form

$$|\Psi\rangle=\frac{1}{\sqrt{2(1+e^{-2|\alpha|^2})}}(|\alpha\rangle+|-\alpha\rangle), \tag{8}$$

where $|\alpha\rangle$ is a coherent state. This is a highly nonclassical state, and can be easily produced in the trapped ion system [19]. In Fig. 2 we plot the real part of the reconstructed matrix elements $\rho_{n,m}$ corresponding to the initial state Eq. (8) with $\alpha=1.5$. We have taken the values of $P(x,\theta)$ for a set of points (x_i,θ_j) and with a finite ϵ. Starting from these data we have reconstructed the state $\rho_{n,m}$ using the algorithm of Leonhardt *et al.* [20]. We have selected a uniform grid of N_x points corresponding to values of x ranging between $\pm 4[\hbar/(m\nu)]^{1/2}$, and a uniform grid of N_θ points for $0\leqslant\theta<\pi$. Figure 2(b) corresponds to a squeezing parameter $|\epsilon|=2$ and a grid $N_x\times N_\theta=30\times 30$ points. The reconstructed state is indistinguishable from the original one. We have checked that even for $|\epsilon|=1$ the obtained state is remarkably similar to the original one. We have also tested the dependence of the reconstruction on the number of grid points. In Fig. 2(b) we have taken a grid of $N_x\times N_\theta=30\times 15$ points, keeping the squeezing parameter $|\epsilon|=2$. In this case, the reconstruction is also quite faithful. Reducing the number of grid points causes small residual background structures. On the other hand, reconstructing density operators that involve higher Fock states (increasing α) requires an increased range of x values and a larger number of grid points, since it is necessary to resolve the oscillatory behavior of these states. Finally, in a real experiment one cannot measure the probability distribution $P(x_i,\theta_j)$ with arbitrary precision, due to the fact that the number of measurements is always finite. We have simulated the statistical error caused by the finite sampling number by truncating the values of $P(x_i,\theta_j)$ to one decimal digit. That is, we have approximated each of the exact values by one of the following numbers 0.0, 0.1, . . . ,1.0. The results of this simulation are shown in Fig. 2(c). In this case, the grid size is again $N_x\times N_\theta=30\times 30$ points, and $|\epsilon|=2$. The reconstruction still resembles the original one even in the presence of these uncertainties. Therefore, it would be enough to perform about 100 measurements per grid point to obtain the density matrix. Obviously, states with a larger phonon number will require more measurements.

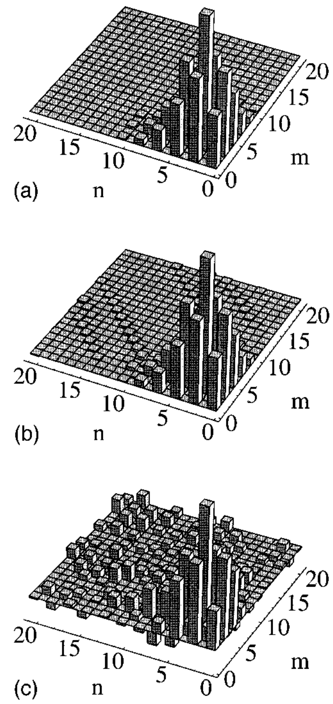

FIG. 2. Real part of the reconstructed density-matrix elements $\rho^R_{n,m}$ for a ''Schrödinger-cat'' state with $\alpha=1.5$: (a) $N_x\times N_\theta=30\times 30$ and $|\epsilon|=2$; (b) $N_x\times N_\theta=30\times 15$ and $|\epsilon|=2$; (c) same parameters as in (a), but with $P(x,\theta)$ rounded to one decimal digit.

In summary, we have presented a scheme to measure the quantum state of motion for a single ion confined in a harmonic potential (statistical mixtures as well as pure states). It is based on the detection of the ground-state population of the trap after a sudden change of the trapping potential. We wish to emphasize that the effect of a sudden displacement of the trap center and a sudden opening of the trap can be obtained (in principle) by a single noninstantaneous process that yields the same symplectic phase-space transformation. Moreover, these two operations can be mimicked using Raman pulses with two lasers of frequencies differing by ν and 2ν, respectively [9,21]. Note, however, that in this case the scheme only works in the Lamb-Dicke limit. Finally, it is straightforward to generalize the schemes presented here to measure the quantum state of motion in two and three spatial dimensions. This can be done by moving and opening the trapping potential along different directions.

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