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Generation of hyper-bunched light by single Gaussian and non-Gaussian scattering processes

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We derive theoretically that hyper-bunched light with a central normalized second-order correlation coefficient of six can be realized by a single Gaussian scattering process of parametric down conversion (PDC) light with a central normalized second-order correlation coefficient of three. The Gaussian scattering process is realized by a rotating ground-glass diffuser. We show that the photon counting probability distribution in this case obeys a Tricomi confluent hypergeometric function $U[1+n, 3/2, 1/\langle n\rangle]$ dependence. Furthermore, we also study non-Gaussian light-scattering probabilities that together with the different impinging light statistics give rise to new photon statistics accompanied by a variety of new values of the second-order correlation coefficient of the scattered light. These theoretical calculations suggest experiments using twin photons from a PDC process and characterizing their photon statistics properties before and after the scattering at the rotating diffuser. These investigations contribute to a more comprehensive understanding of the scattering process, the generated light, and new applications.

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1. INTRODUCTION

Since the first realization of the laser, there has been a perpetual interest in understanding the quantum fluctuations of light, driven both by fundamental and by application interests [1]. The generation of squeezed states of light and single photon states by means of nonlinear optics and more sophisticated source approaches has been a revolutionary, challenging step towards the generation of new tailored quantum states of light [2]. Already immediately after the advent of the laser, in 1966 Martienssen and Spiller and Arecchi [3,4] realized the so-called pseudo-thermal light source. There, scattering of laser light at a rotating diffuser transformed the Poissonian laser photon statistics into that of thermal light [5,6] exhibiting Bose–Einstein statistics with a central normalized second-order correlation coefficient [7] of two. In 1970, Bertolotti *et al.* [8,9] described this manipulation of the statistics very simply and intuitively based on the Mandel formula for the photon detection and assuming a Gaussian scattering process [10,11]. Subsequently, in 1980, a large amount of comprehensive and detailed both theoretical and experimental work focused on the double scattering of light by two subsequently following diffusers leading to the realization of light with a central normalized second-order correlation coefficient of four [12–14]. In the following, this philosophy of exploiting Gaussian scattering processes—but also considering non-Gaussian random walk scattering processes in media—led to the achievement of well-controlled states of light [8,12–16] in the framework of light with super-Poissonian statistics, i.e., bunched or even super-bunched photon counting statistics. Later on, microscopic and mesoscopic scattering concepts for the manipulation of the light statistics were comprehensively investigated and even further extended to waveguides [17–20]. In 2017 Zhao *et al.* [21] reinvestigated the double scattering process by two diffusers in the framework of having a light source with improved characteristics for ghost imaging applications, a classical photon correlation imaging technique. Unfortunately, they were not aware of either this original theoretical work from 1970 [8,10,11] or the experiments from 1980 [12–14] and therefore ignored them completely.

The concept of manipulating and tailoring of light states and exploiting these novel properties beneficially in quantum metrology applications has also been investigated by applying nonlinear optical processes onto light [22,23]. Very recently, it has been shown that disordered systems permit manipulation and tuning of the output statistics via deterministic and coherent control. Monochromatic coherent light traversing a disordered photonic medium evolved into a random field whose statistics has been dictated by the disorder level [19,24]. Deterministic control over the photon-number distribution was demonstrated by interfering two coherent beams within a disordered photonic lattice, thus enabling the generation of super-thermal and sub-thermal light [25].

In this contribution, we follow the strategy and the calculations of Bertolotti *et al.* [8] and demonstrate theoretically that by a single scattering process, being Gaussian or non-Gaussian of super-bunched twin photons [26] obtained from a parametric down conversion (PDC) process, hyper-bunched light [27] with a normalized central second-order correlation coefficient $g^{(2)}(\tau)$ larger than four and going up to six can be achieved. These investigations give new insight into the generation of super-bunched light enabling new avenues for applications in sensing.

2. PHOTOELECTRON STATISTICS OF SCATTERED LIGHT

We start by recalling the fundamental work of Mandel and Wolf 1965 [28] expressing the probability density function (PDF) $p(n, T)$ of counting n photons in a time interval $(0, T)$ as a function of the probability distribution $P(I)$ of the intensity I or the photons n_{photon} emitted by the source, respectively:

$$p(n, T) = \frac{1}{n!}\int_0^\infty U^n e^{-U} p(U) \mathrm{d}U, \tag{1}$$

where $U = \alpha \cdot I \cdot T$ and I is the intensity of the light falling on the photodetector of quantum efficiency α within a time T. Here, the coherence time of the radiation is supposed to be much smaller than T, and thus $P(U)$ represents the light intensity I. This relationship is also often called $p(n, T)$, being the Poisson transform of $P(U)$ [29,30].

In the following, we calculate directly measured photon probability distributions via their detected photoelectron probability distributions [31–34] originating from photon probability distributions of scattered light [35]. The photoelectron probability distribution is characterized by all moments of the distributions. As a "condensed" measure of photoelectron or photon probability distributions, one can choose the Fano factor, the Mandel parameter, or the central second-order correlation coefficient $g^{(2)}(\tau = 0)$ [28]. For our work we select $g^{(2)}(\tau = 0)$, also called the central intensity correlation coefficient and measurable in a Hanbury Brown–Twiss intensity interference experiment [36], even with ultrafast time resolution for spectrally broad-band light [37]. The central second-order correlation coefficient $g^{(2)}(\tau = 0)$ [7,38] of the intensity is defined as

$$g^{(2)}(\tau = 0) = \frac{\langle I^2\rangle}{\langle I\rangle\langle I\rangle} \tag{2}$$

and can be calculated from the factorial (F) or the ordinary moments (M) of the measured intensity I, i.e., from the detected photoelectrons via

$$g^{(2)} = \frac{F^{(2)}_{\text{Photoelectrons}}}{(F^{(1)}_{\text{Photoelectrons}})^2} = \frac{M^{(2)}_{\text{Photoelectrons}} - M^{(1)}_{\text{Photoelectrons}}}{(M^{(1)}_{\text{Photoelectrons}})^2}, \tag{3}$$

where $F^{(1)}_{\text{Photoelectrons}}$ and $F^{(2)}_{\text{Photoelectrons}}$ are now the first- and second-order factorial moments of the photoelectron distribution $p(n, T)$ and $M^{(1)}_{\text{photoelectrons}}$ and $M^{(2)}_{\text{photoelectrons}}$ are its first- and second-order ordinary moments, respectively.

Due to the fact that $P(n)$ is the Poisson transform of $p(U)$ [Eq. (1)], the factorial moments $F^{(k)}_{\text{Photoelectrons}}$ of $P(n)$ are equal to the ordinary moments $M^{(k)}_{\text{lightintensity}}$ of same order k of $p(U)$, and vice versa. This facilitates the calculation of $g^{(2)}$ enormously since the calculation of the factorial moments of $P(n)$ is sometimes rather tedious or even impossible in most cases, whereas the ordinary moments of the intensity probability distribution $P(I)$ are given on a straightforward basis. In the following, we omit $(\tau = 0)$ in the second-order correlation coefficient expression, writing it simply reduced to $g^{(2)}$. Finally, this yields from Eqs (4) and (5) for $g^{(2)}$

$$\begin{aligned} g^{(2)} &= F^{(2)}_{\text{Photoelectrons}}/(F^{(1)}_{\text{photoelectrons}})^2 \\ &= M^{(2)}{}_{\text{lightintensity}}/(M^{(1)}_{\text{lightintensity}})^2. \end{aligned} \tag{4}$$

We now recall the work of Bertolotti for the calculation of the probabilities of scattered light [8]. The probability distribution of the intensity of quasi-monochromatic light $P(I_{\text{scattered light}})$ scattered by a fluctuating medium can be described as the product of two statistically independent probability distributions of two statistical variables $P_0(I_0) = P_{\text{light}}(I_0)$ and $P_1(J_1) = P_{\text{scatterer}}(J_1)$, the first representing the probability distribution of the incident light field and the second that of the scattering medium. The joint photon probability can be expressed as [13]

$$P(I)_{\text{scatteredlight}} = \int_0^\infty\int_0^\infty P_0(I_0)P_1(J_1)\delta(I - I_0 * J_1)\mathrm{d}I_0\mathrm{d}J_1. \tag{5}$$

This leads to the following expression for the Mandel formula for the measured photoelectron probability distribution $P(n, T)$ of the scattered light [8]:

$$p(n, T) = \frac{1}{n!}\int_0^\infty \gamma^n T^n I_0^n J_1^n e^{-\gamma T I_0 J_1} P_0(I_0)P_1(J_1)\mathrm{d}I_0\mathrm{d}J_1 \tag{6}$$

with $\gamma = \alpha \cdot \beta$, where β is a characteristic for the Gaussian scattering process.

For the calculation of $g^{(2)}$ of the photoelectron probability distribution in the case of scattered light, we recall that the probability distribution of the scattered light $P(I)_{\text{scatteredlight}}$ is obtained as the product of two independent probability distributions of two statistical variables $P_0(I_0) = P_{\text{light}}(I_0)$ and $P_1(J_1) = P_{\text{scatterer}}(J1)$. This means that the ordinary moments M of $P(I)_{\text{scatteredlight}}$ are the products of the ordinary moments M of $P_0(I_0) = P_{\text{light}}(I_0)$ and $P_1(J_1) = P_{\text{scatterer}}(J_1)$, respectively. This results in $g^{(2)}$ of the scattered light according to Eq. (5):

$$g^{(2)}_{\text{scattered light}} = (M^{(2)}_{P1} * M^{(2)}_{P0})/(M^{(1)}_{P1} * M^{(1)}_{P0})^2. \tag{7}$$

The equivalence of this relation for calculating $g^{(2)}$ according to Eq. (7) or by calculating $g^{(2)}$ according to the moments of Eq. (6) will be exploited in the following if numerically possible. In the subsequently following investigations, we use the nomenclature of $XXYY$ for the scattered light with the impinging light statistics XX and the scattering statistics YY.

3. GAUSSIAN SCATTERING PROCESSES

At first, we assume now a Gaussian scattering process considering that the susceptibility fluctuations of the scattering medium are produced by random motions of particles. Its probability distribution of $P_1(J_1)$ is given by an exponential distribution (E) [10]

$$P_{\text{scatterer}} = P_1(J_1) = \frac{1}{\langle J_1\rangle} e^{\frac{-J}{\langle J_1\rangle}} \quad (E) \tag{8}$$

with $\langle J_1\rangle$ being the mean intensity of $P_1(J_1)$. We note that in fact there are also non-Gaussian scattering approaches [13,15] being relevant in a lot of cases and which we shall investigate in Section 4. However, here we start with Gaussian scattering processes, e.g., experimentally realized by a rotating ground-glass diffuser representing the ideal Gaussian case, supported by the clear Bose–Einstein distribution of the detected photoelectrons when laser light is scattered (see, e.g., [6,39]). Two simple cases for the detected photoelectron probability distribution result immediately.

A. Dirac Delta

1. With the probability distribution of the light $P_0(I_0)$ being a Dirac delta (DD) function,

$$P_0(I) = \delta(I - \langle I_0\rangle) \quad (DD), \tag{9}$$

which is equivalent to the case of laser light impinging on a Gaussian medium with exponential distribution [Eq. (8)], one obviously obtains through Eqs. (5), (8), and (9) the following expression for the PDF of the scattered light:

$$P(I)_{\text{scatteredlight}} = \frac{1}{\langle n\rangle} e^{\frac{-I}{\langle n\rangle}} \quad (DDE), \tag{10}$$

where $\langle n\rangle = \gamma T I_0 J_1$ represents the average number of counts recorded in the time interval T.

The Poisson transform of Eq. (10) results in

$$p(n, T) = \frac{1}{\langle n\rangle} \frac{1}{\left(1 + \frac{1}{\langle n\rangle}\right)^{n+1}} \quad (DDE), \tag{11}$$

which represents the usual Bose–Einstein distributions (also called geometrical distribution) for $P(n, T)$.

B. Exponential Distribution

2. The case with $P_0(I_0)$ being an exponential distribution (E)

$$P_0(I) = \frac{1}{\langle I_0\rangle} e^{\frac{-I}{\langle I_0\rangle}} \quad (E) \tag{12}$$

is equivalent to the case of a Gaussian thermal light beam impinging on a Gaussian medium and scattered by it. Equations (5), (8), and (12) immediately deliver

$$P(I)_{\text{scatteredlight}} = \frac{2}{\langle n\rangle} K_0\left(\frac{2\sqrt{I}}{\sqrt{\langle n\rangle}}\right) \quad (EE), \tag{13}$$

where $\langle n\rangle = \gamma T I_0 J_1$ with K_0 being the Bessel function of first order [29].

The Poisson transform of Eq. (13) results in a photoelectron distribution

$$P(n) = \frac{\Gamma(1+n)}{\langle n\rangle} U\left(1+n,\, 1,\, \frac{1}{\langle n\rangle}\right) \quad (EE) \tag{14}$$

with $U(1+n, 1, \frac{1}{\langle n\rangle})$ being the Tricomi confluent hypergeometric function of second kind.

The Tricomi hyperconfluent function of second kind can also be expressed in terms of the Whittaker function [40] resulting in the following equivalent expression for the PDF $P(n)$ [8]:

$$P(n, T) = \frac{n!}{\langle n\rangle^{\frac{1}{2}}} \exp\left(\frac{1}{2\langle n\rangle}\right) \text{Whittaker}\, W_{-(n+\frac{1}{2}),0}\left(\frac{1}{2\langle n\rangle}\right) \quad (EE) \tag{15}$$

with $\langle n\rangle = \gamma \cdot T \cdot I_0 \cdot J_1$, and $W_{k,m}(x)$ being the Whittaker function [40].

These two derived fundamental and exemplaric photoelectron distributions yield the following values for $g^{(2)}$:

$$\text{From Eq. (11)}: g^{(2)}_{\text{scatteredlight}} = g^{(2)}_{\text{light}} \cdot g^{(2)}_{\text{scatterer}} = g^{(2)}_{DD} \cdot g^{(2)}_{E} = 2.0, \tag{16}$$

$$\text{From Eq. (14)}: g^{(2)}_{\text{scatteredlight}} = g^{(2)}_{\text{light}} \cdot g^{(2)}_{\text{scatterer}} = g^{(2)}_{E} \cdot g^{(2)}_{E} = 4.0. \tag{17}$$

This means that the effect of bunching and super-bunching by scattering of light is realized. Scattering Dirac delta light (DD) at a Gaussian scatterer (E) transforms $g^{(2)}$ from one to two, and scattering thermal Gaussian light (E) transforms $g^{(2)}$ from two to four.

C. DeGiorgio

With this basis, we extend now the formalism depicted above towards scattering of super-bunched light [26]. Degiorgio [9,41,42] has derived that the probability distribution for this superbunched light, sometimes also called Gaussian square light or chi_1^2 light [29] or light having an exponential square law probability distribution, can be described by

$$P(I)_{\text{DeGiorgio}} = \sqrt{\frac{1}{\pi I_0 I}} \text{Exp}(-I/I_0) \quad (DG). \tag{18}$$

The first- and second-ordinary moments of this super-bunched twin photon light distribution $P(I)_{\text{light}}$ [Eq. (18)] amount to $\langle I\rangle = \frac{I_0}{2}$ and $\langle I^2\rangle = \frac{3\langle I_0\rangle^2}{4}$, respectively, and the second-order correlation coefficient $g^{(2)}$ thus to three [41], experimentally confirmed by [26] by exploiting two-photon absorption in multiplicative semiconductor detectors [26], which even enables measurements of photon statistics properties with ultrafast time resolution on a 10 fs time scale. The corresponding photon statistics, i.e., the Poisson transform of Eq. (18), is found to be [41,42]

$$P(n)_{\text{DeGiorgio}} = \frac{n!!}{n!} \frac{I_0^n}{2^{n+1}(1+I_0)^{n+1/2}} \quad (DG) \tag{19}$$

with a second-order correlation coefficient

$$g^{(2)} = 3 \quad (DG). \tag{20}$$

Now, we take this type of light and perform a single Gaussian scattering process represented by an exponential probability distribution [8]. The intensity distribution of the light to be scattered is assumed to have an exponential square law probability distribution (DG) [Eq. (18)] [29,41]. Thus, the PDF $P(I)$ of the scattered light is the product or the convolution [according to Eq. (5) of an exponential (Eq. (8)] for the scatterer and an exponential square law PDF [Eq. (18)] for the light, respectively, resulting in

$$P(I)_{\text{scatteredlight}} = \frac{1}{\sqrt{I\langle n\rangle}} e^{-2\sqrt{\frac{I}{\langle n\rangle}}} \quad (DGE) \tag{21}$$

with $\langle n\rangle = I_0 \cdot I_1$ containing the characteristic parameters of the light intensity [Eq. (18)] and the scatterer [Eq. (8)], respectively. The ordinary moments of Eq. (21) amount to $\langle I\rangle = 1/2 \cdot I_0 \cdot I_1$ and $\langle I^2\rangle = 3/2(I_0 \cdot I_1)^2$ with $g^{(2)} = 6$. Applying the Poisson transform onto the PDF of Eq. (21) yields the photon electron distribution according to Eq. (1):

$$P(n) = \frac{\Gamma[\frac{1}{2}+n]}{\langle n\rangle\sqrt{\pi}} U\left(1+n, \frac{3}{2}, \frac{1}{\langle n\rangle}\right) \quad (DGE) \tag{22}$$

with $U(1+n, \frac{3}{2}, \frac{1}{\langle n\rangle})$ being the Tricomi confluent hypergeometric function of second kind and $\Gamma[\,]$ being the Gamma function [43]. The same results can also be achieved by Poisson transforming Eqs. (8) and (18) according to Eq. (6).

For the calculation of the mean $\langle n\rangle$, the second moment $\langle n^2\rangle$, and the central second-order correlation coefficient $g^{(2)}$ of the photoelectron distribution $P(n)$ of this scattered light [Eq. (22)], we rely on both Eqs. (3) and (7).

The first- and second-ordinary moments of the scattering distribution $P(I)_{\text{scatterer}}$ [Eq. (8)] amount to I_1 and $2(I_1)^2$, respectively, with a $g^{(2)}$ value of two. This results in a normalized second-order correlation coefficient $g^{(2)}$ of the scattered light [Eq. (22)] according to Eq. (7):

$$g^{(2)} = 6 \quad (DGE). \tag{23}$$

Thus, we achieved the generation of hyper-bunched light [27] with a Tricomi confluent hypergeometric function of second kind for the photoelectrons and with a second-order correlation coefficient $g^{(2)}$ of six by scattering superbunched light with a $g^{(2)}$ of three at a Gaussian scatterer.

4. NON-GAUSSIAN SCATTERING PROCESSES

We now extend our calculations of the photoelectron and photon probability distribution to the case of a non-Gaussian scatterer represented by a Rayleigh scattering process.

Because the resulting Meijers G function and the generalized hypergeometric function [43–46] have never been discussed so far in the context of photoelectron statistics—as we shall see in the following—we plot in Section 5 in Figs. 1 and 2 all the photoelectron probability distributions for the different cases for the same mean. This depiction should visualize for comparison the different $g^{(2)}$ behavior of the Meijers G function, the Whittaker function, the geometric distribution, and the Poisson distribution. The first represents the thermal light (the Rayleigh scattering case), the second the thermal light (the Gaussian scattering case), the third the thermal light reference, and the fourth the laser light reference, the fourth being quasi the reference for the double scattering process of laser light and the last two representing the reference of the distribution function for Bose–Einstein statistics and laser light. Furthermore we also summarize in Section 5 in Table 1 the $g^{(2)}$ values for the various light and scatterer probability distributions and the resulting $g^{(2)}$ for the scattered light.

We would like to complement our theoretical calculation investigations now for the case in which the scattering media are described by a non-Gaussian scattering function. An interesting approach is assuming a Rayleigh-type function for the scattering medium, which can be experimentally realized by a volume scatterer element [47]. The intensity probability distribution (PDF) with the characteristic parameter σ is written as

$$P(I)_{\text{scatter}} = P(I)_{\text{Rayleigh}} = \frac{2I}{\sigma^2}\exp\left(\frac{-I^2}{\sigma^2}\right) \quad (R), \tag{24}$$

where for the Rayleigh distribution, values for $\langle I\rangle$, $\langle I^2\rangle$, and $g^{(2)}$ are given by

$$\langle I\rangle = \frac{\sqrt{\pi}\sigma}{2} \quad \langle I^2\rangle = \sigma^2 \quad g^{(2)} = \frac{4}{\pi} \quad (R). \tag{25}$$

With the three selected representative light distributions, Dirac delta-like laser [DD; Eq. (9)], exponential thermal-like [E; Eq. (12)], and χ^2 [DG; Eq. (18)], we calculate now the following scattered light distributions $P(I)_{\text{scatteredlight}}$ and the photoelectron distributions $P(n)$ after Poisson transformation, concluding in the calculation of $g^{(2)}$.

A. Dirac Delta–Rayleigh *DDR*

By assuming the Dirac-delta light intensity distribution (DD) Eq. (9) we obtain the following distribution of the scattered light:

$$P(I)_{\text{scatteredlight}} = 2 \cdot I \cdot \exp\left(\frac{-I^2}{I_0{}^2\sigma_2{}^2}\right) / I_0{}^2\sigma_2{}^2 \quad (DDR), \tag{26}$$

yielding the following expression for the photoelectron PDF:

$$P(n) = 2^{(-1-n)}(1+n)I_0\sigma^n U\left(1+\frac{n}{2}, \frac{1}{2}, \frac{I_0^2\sigma^2)}{4}\right) \quad (DDR) \tag{27}$$

with the characteristic parameters of $P(n)$ by using the formalism of Eq. (7):

$$\langle n\rangle = \frac{1}{2} I_0\sigma\sqrt{\pi} \quad \langle n^2\rangle = I_0{}^2\sigma^2 \quad g^{(2)} = 4/\pi \quad (DDR). \tag{28}$$

B. Exponential–Rayleigh

By assuming the exponential light intensity distribution [(E), Eq. (12)] we obtain the following distribution of the scattered light:

$$P(I)_{\text{scatteredlight}} = \left(I \cdot \text{MeijerG}\left[((),()), \left(\left(-\frac{1}{2}, 0, 0\right), ()\right), \frac{I^2}{4I_0{}^2\sigma^2}\right]\right) / [2I_0{}^2\sigma^2\sqrt{\pi}] \quad (ER), \tag{29}$$

$$P(n) = \frac{1}{3\sqrt{\pi}(I_0\sigma)^{\frac{5}{2}} n!}\Bigg(3{I_0}^2\sigma^2\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}+n\right)$$
$$\times \text{HypergeometricPFQ}\left[\left(\frac{1}{4}+\frac{n}{2},\frac{3}{4}+\frac{n}{2}\right),\left(\frac{1}{4},\frac{1}{2}\right),-\frac{1}{{I_0}^2\sigma^2}\right] - 12I_0\sigma\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{2}+n\right)$$
$$\times \text{HypergeometricPFQ}\left[\left(\frac{3}{4}+\frac{n}{2},\frac{5}{4}+\frac{n}{2}\right),\left(\frac{3}{4},\frac{3}{2}\right),-\frac{1}{{I_0}^2\sigma^2}\right] + 8(1+n)\sqrt{\pi I_0\sigma}\, n!$$
$$\times \text{HypergeometricPFQ}\left[\left(1+\frac{n}{2},\frac{3}{2}+\frac{n}{2}\right),\left(\frac{5}{4},\frac{7}{4}\right),-\frac{1}{{I_0}^2\sigma^2}\right]\Bigg) \quad (DGR), \tag{33}$$

yielding the following expression for the photoelectron PDF after Poisson transformation:

$$P(n) = \Bigg(2^n \cdot \text{MeijerG}\Bigg[\left(\left(\frac{1}{2}(-n-1), -\frac{n}{2}\right), ()\right),$$
$$\left(\left(-\frac{1}{2}, 0, 0\right), ()\right), \frac{1}{{I_0}^2\sigma^2}\Bigg]\Bigg)/({I_0}^2\sigma^2\pi n!) \quad (ER). \tag{30}$$

For the mean $\langle n\rangle$, the second moment $\langle n^2\rangle$ of n, and $g^{(2)}$ we obtain via the formalism of Eq. (7)

$$\langle n\rangle = \text{mean}_E \cdot \text{mean}_R = \frac{\sqrt{\pi}I_0\sigma}{2}$$
$$\langle n^2\rangle = 2{I_0}^2\sigma^2$$
$$g^{(2)} = 8/\pi \quad (ER). \tag{31}$$

C. DeGiorgio–Rayleigh

Combining a Rayleigh scattering process [R, (Eq. (24)] and the DeGiorgio light intensity PDF [DG, (Eq. (18)] according to Eq. (5) we obtain the following intensity PDF of the scattered light:

$$P(I) = \frac{1}{3{I_0}^2\sigma^2}\Bigg(3\sqrt{\frac{I_0\sigma}{I\pi}}\Bigg(I_0\sigma\Gamma\left(\frac{3}{4}\right)$$
$$\times \text{HypergeometricPFQ}\left[(),\left(\frac{1}{4},\frac{1}{2}\right),-\frac{I^2}{4{I_0}^2\sigma^2}\right]$$
$$-4I\Gamma\left(\frac{5}{4}\right)$$
$$\times \text{HypergeometricPFQ}\left[(),\left(\frac{3}{4},\frac{3}{2}\right),-\frac{I^2}{4{I_0}^2\sigma^2}\right]\Bigg)$$
$$+8I$$
$$\times \text{HypergeometricPFQ}\left[(),\left(\frac{5}{4},\frac{7}{4}\right),-\frac{I^2}{4{I_0}^2\sigma^2}\right]\Bigg)$$
$$(DGR) \tag{32}$$

which after Poisson transformation leads to the photoelectron PDF of the scattered light according to

where HypergeometricPFQ[(,),(,),( )] denotes the generalized hypergeometric function also written as $_pF_q[;\,;\,;\,;\,;\,]$.

Here, for the calculation of the moments and of $g^{(2)}$ of $P(n)_{\text{DeGiorgio–Rayleigh}}$ we again use the formalism as described above by Eq. (7) resulting in

$$\langle n\rangle_{\text{DeGiorgio-Rayleigh}} = \langle n\rangle_{\text{DeGiorgio}} \cdot \langle n\rangle_{\text{Rayleigh}} = \frac{1}{4}I_0\sigma\sqrt{\pi},$$
$$\langle n^2\rangle_{\text{DeGiorgio-Rayleigh}} = \langle n^2\rangle_{\text{DeGiorgio}} \cdot \langle n^2\rangle_{\text{Rayleigh}} = \frac{3}{4}{I_0}^2\sigma^2,$$
$$g^{(2)}_{\text{DeGiorgio-Rayleigh}} = 12/\pi \quad (DGR). \tag{34}$$

5. VISUALIZATION OF THE PROBABILITY DENSITY FUNCTIONS AND THE CORRESPONDING SECOND-ORDER CORRELATION COEFFICIENT

In Figs. 1 and 2 we depict the PDF of the scattered light distribution for the six cases as calculated in Sections 3 and 4. For the calculation we considered a value of $\gamma = 1.0$ and of $\sigma = 2.0$. In addition as a reference we depict the Poissonian and the geometrical (Bose–Einstein) distribution. The super- and hyper-bunching is qualitatively visible from the steepening of the PDF towards zero photon number developing from the Poissonian distribution. This is in more detail and quantitatively illustrated in Table 1 where according to the chosen nomenclature of the light and scattering statistics the achieved $g^{(2)}$ is listed. We see that the highest hyper bunching value of $g^{(2)} = 6.0$ is realized for an EE process; however, still for an ER process a hyper-bunching value with a $g^{(2)} = 12/\pi$ is achieved.

6. VIEW ON EXPERIMENTS

The necessary experimental setup is described straightforward in the literature. We refer to the original Martienssen experiment from 1966 [3,4], including the laser light source and the rotating scatterer for the realization of the pseudothermal light source, subsequently exploited, e.g., by Ferri *et al.* [6,48–50] and Bondani *et al.* [22,23] and even discussed under didactical student lab course aspects [51,52]. The aspects of the realization and properties of a PDC source can be found in [53–55]. We note here that an interesting approach would be exploiting bunched light originating from amplified spontaneous emission

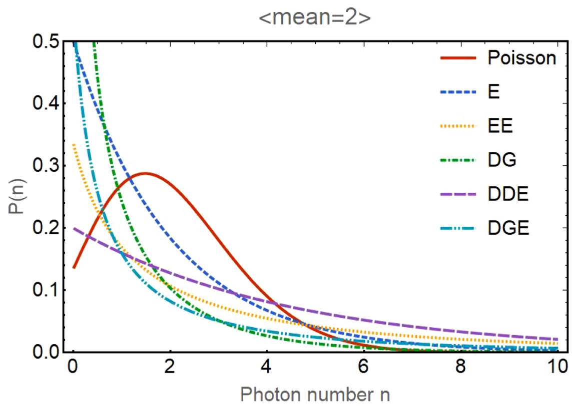

Fig. 1. Depiction of photoelectron distributions $P(n)$ as a function of the photon number n calculated for laser light [Dirac-delta (DD)] scattered at a Gaussian Medium (E) DDE, for thermal light (exponential) (E) scattered at a Gaussian medium (E) EE, for Chi squared light with super-Poissonian statistics (DG) and for Chi squared light with super-Poissonian statistics (DG) scattered at a Gaussian medium (E) DGE, all for the same mean $\langle n \rangle = 2.0$. For comparison, a Poisson photon distribution P and the geometrical (or Bose–Einstein or exponential) disribution (E) for the same mean are also shown.

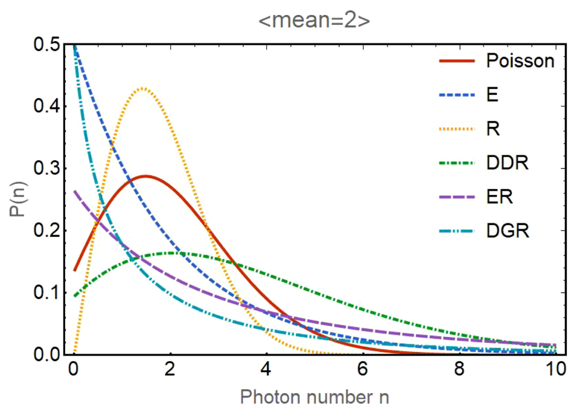

Fig. 2. Depiction of photonelectron distributions $P(n)$ as a function of the photon number n calculated for laser light [Dirac-delta (DD)] scattered at a Raleigh medium (R) DDR, for thermal light (exponential E) scattered at a Rayleigh medium (R) ER, and for Chi squared light (DG) with super-Poissonian statistics (DG) scattered at a Raleigh medium (R) DGR, all for the same mean $\langle n \rangle = 2.0$. For comparison a simple Rayleigh distribution (R), a Poisson photon (P) distribution, and a geometrical (or Bose–Einstein or exponential) distribution (E) are shown, all for the same mean.

(ASE) light generated by semiconductor-based super luminescent diodes, which exhibit an ideal thermal emitter character with a $g^{(2)}$ value equal to 2.0 thus being real thermal light [56], instead of the pseudo-thermal light.

7. CONCLUSION

In conclusion, we have demonstrated that a single scattering process of superbunched light with a second correlation coefficient of three within a Gaussian scattering media leads to light with a second correlation coefficient of six and within a Rayleigh scattering process to light with a second correlation coefficient of up to $12/\pi$. These theoretical investigations suggest directly straightforward experimental investigations as proof for the generation of super-bunched light. Required appropriate experimental concepts have been shortly outlined. The quantum aspects of these comprehensively characterized novel states of super- or even hyper-bunched light in respect of spatial, spectral, and polarization correlations, tailored and optimized with respect to real-world metrology applications, should then be exploited in quantum metrology applications [57–60] based on a correlated photon approach as ghost imaging [56], ghost spectroscopy [61], and ghost polarimetry [62], thus paving the avenue for demonstrating their superior metrology performance.

Table 1. Overview of the Statistics of the Impinging Light and the Scatterer Statistics Underlying the Scattering Process and the Resulting Second-Order Correlation Coefficient $g^{(2)}$ of the Scattered Light

Light-Scatterer $XXYY$	$g^{(2)}_{light}$	$g^{(2)}_{scatterer}$	$g^{(2)}_{scatteredlight}$
Poisson	1.0	–	–
DDE	1.0	2.0	2.0
EE	2.0	2.0	4.0
DGE	3.0	2.0	6.0
DDR	1.0	4/Pi	4/Pi
ER	2.0	4/Pi	8/Pi
DGR	3.0	4/Pi	12/Pi

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the author upon reasonable request.

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