Polarisationsanalyse von Licht mittels Stokes-Formalismus

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Inhaltsverzeichnis

1. Grundlagen

In diesem Praktikumsversuch soll der Polarisationszustand von Licht mittels Stokes-Formalismus vollständig bestimmt werden. Polarisation ist eine Eigenschaft von transversalen Wellen, die unter anderem in der Optik das Interferenz-, Reflexions- und Transmissionsverhalten von Licht beeinflusst und damit sowohl in der Theorie als auch in der Praxis eine wichtige Rolle spielt. Für eine vollständige Analyse wird ein Formalismus benötigt, der in der Lage ist, nicht nur vollständig polarisiertes Licht, sondern ebenso teilweise polarisiertes Licht zu beschreiben. Ein entsprechender Formalismus wurde 1852 von G.G. Stokes [10] entwickelt. Er charakterisiert den Polarisationszustand durch vier direkt messbare Parameter.

1.1. Polarisation

Anfang des 19. Jahrhunderts stellten Fresnel und Arago durch Interferenzexperimente fest, dass es sich bei Licht um rein transversale Wellen handeln muss. Aus den Maxwellgleichungen folgt in Vakuum die Wellengleichung:

$$\vec{\Delta}\vec{E} = \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Aus der Wellengleichung folgt, dass sich ebene in z-Richtung propagierende Wellen als Überlagerung zweier orthogonaler Komponenten darstellen lassen:

$$\vec{E}(z,t) = E_x(z,t)\vec{e}_x + E_y(z,t)\vec{e}_y \quad \text{mit}$$

$$E_x(z,t) = E_{0x}\cos(\omega t - kz) \tag{1a}$$

$$E_y(z,t) = E_{0y}\cos(\omega t - kz + \delta) \tag{1b}$$

Für die Phasenverschiebung $\delta = 0$ zwischen der x- und der y-Komponente oszilliert der E-Feldvektor in einer Ebene. Man spricht in diesem Fall von linearer Polarisation. Für $\delta = \frac{\pi}{2}$ und $E_x = E_y$ beschreibt der E-Feldvektor eine Kreisschraube um die z-Achse, deren Projektion auf die x-y-Ebene einen Kreis ergibt. Dieses wird als zirkular polarisiertes Licht bezeichnet. Je nach Phasenverschiebung ergibt sich eine Rechts- oder Linksschraube. Nach allgemeiner Konvention wird der Drehsinn aus Richtung des Detektors betrachtet. Bei $\delta \neq \frac{\pi}{2}$ oder $E_x \neq E_y$ wird die Polarisation als elliptisch bezeichnet, denn die Projektion des E-Feldvektors beschreibt dabei eine Ellipse.

1.1.1. Polarisationsellipse

Durch Elimination der Zeit- und Ortsabhängigkeit aus den Gleichungen (1) ergibt sich eine Ellipsengleichung:

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - \frac{2E_xE_y\cos(\delta)}{E_{0x}E_{0y}} = \sin^2(\delta) \tag{2}$$

Eine ausführliche Diskussion der Eigenschaften dieser Polarisationsellipse findet sich bei Collet [3]. Eine Reihe der wichtigen Eigenschaften werden im Folgenden zusammengetragen. Für $\delta = 0$ ergibt sich eine Linie mit einer von den Amplituden der Schwingungen der E-Feldkomponenten bestimmten Steigung:

$$E_y = \pm\frac{E_{0y}}{E_{0x}}E_x$$

Für $\delta = \frac{\pi}{2}$ und $E_{0x} = E_{0y} = E_0$ ergibt sich ein Kreis:

$$\frac{E_x^2}{E_0^2} + \frac{E_y^2}{E_0^2} = 1$$

Eine Ellipse mit den Halbachsen a und b ist durch ihre Orientierung ψ, ihre Elliptizität χ und den Winkel α charakterisiert. Die Elliptizität gibt hierbei das Verhältnis der beiden Halbachsen der Ellipse an. Die Definition der Winkel ist in Abb. 1 dargestellt.

Es gelten für $(-\pi/4 < \chi, \psi < \pi/4)$ die folgenden Beziehungen:

$$\tan(2\psi) = \frac{2E_{0x}E_{0y}\cos(\delta)}{E_{0x}^2 - E_{0y}^2} \tag{3a}$$

$$\tan(\chi) = \frac{b}{a} \tag{3b}$$

Zur experimentellen Bestimmung des Polarisationszustandes auf Basis der Parameter der Polarisationsellipse kann das Licht mit einer Kombination aus einem Polarisator und einem $\frac{\lambda}{4}$-Plättchen gemäß Tabelle 1 analysiert werden.

1.1.2. Stokes-Formalismus

Um die charakteristischen Parameter der Polarisationsellipse zu bestimmen, ist es notwendig, die Zeitabhängigkeit des Feldvektors, der mit einer Frequenz von ca. 10^{15} Hz oszilliert, zu eliminieren. Mit der Intensität wird ein zeitliches Mittel über die Quadrate der Amplituden gemessen. Hierdurch kann zwar zwischen linear- und zirkular polarisiertem Licht unterschieden werden (siehe Tabelle 1). Der Drehsinn und der Grad der Polarisation bleiben allerdings verborgen.

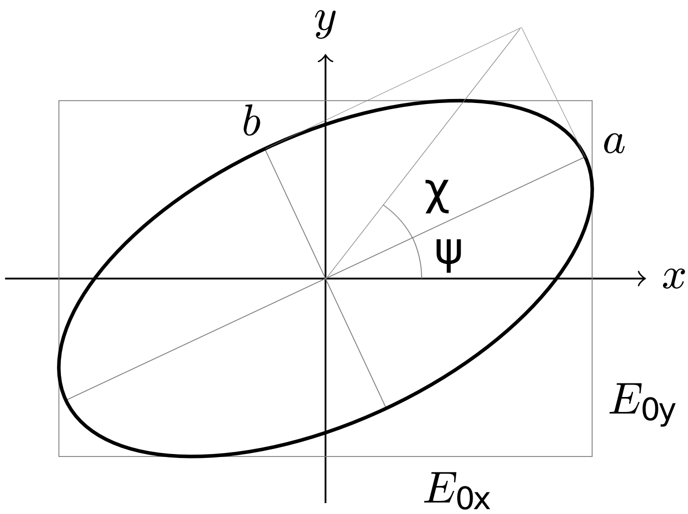

Abbildung 1: Polarisationsellipse für $E_{0x} = 3$,$E_{0y} = 2$ und $\delta = 60°$

Table 27A ANALYSIS OF POLARIZED LIGHT

<table>
<tr><td colspan="3">A. No intensity variation with analyzer alone</td></tr>
<tr><td>I. If with λ/4 plate in front of analyzer</td><td colspan="2">II. If with λ/4 plate in front of analyzer one finds a maximum, then</td></tr>
<tr><td>1. One has no intensity variation,
one has
natural unpolarized light</td><td>2. If one position of analyzer gives zero intensity,
one has
circularly polarized light</td><td>3. If no position of analyzer gives zero intensity,
one has
mixture of circularly polarized light and unpolarized light</td></tr>
</table>

<table>
<tr><td colspan="4">B. Intensity variation with analyzer alone</td></tr>
<tr><td>I. If one position of analyzer gives</td><td colspan="3">II. If no position of analyzer gives zero intensity</td></tr>
<tr><td rowspan="3">1. Zero intensity,
one has
plane-polarized light</td><td colspan="3">2. Insert a λ/4 plate in front of analyzer with optic axis parallel to position of maximum intensity</td></tr>
<tr><td rowspan="2">(a) If get zero intensity with analyzer,
one has
elliptically polarized light</td><td colspan="2">(b) If get no zero intensity,</td></tr>
<tr><td>(1) But the same analyzer setting as before gives the maximum intensity,
one has
mixture of plane-polarized light and unpolarized light</td><td>(2) But some other analyzer setting than before gives a maximum intensity,
one has
mixture of elliptically polarized light and plane-polarized light</td></tr>
</table>

Tabelle 1: Analyse der Polarisationszustände nach Jenknis/White [6]

Bei unpolarisiertem Licht existiert kein zeitlich konstantes Phasen- oder Amplitudenverhältnis. Es kann als Überlagerung vieler kurzer unterschiedlich polarisierter Wellenzüge aufgefasst werden und unterscheidet sich in der Projektion im zeitlichen Mittel nicht von zirkular polarisiertem Licht. Eine Überlagerung von polarisiertem und unpolarisiertem Licht erscheint wie elliptisch polarisiertes Licht.

Um unpolarisiertes Licht mathematisch zu beschreiben, gewann Stokes 1852 vier direkt messbare Größen, indem er die nicht ohne weiteres zugänglichen zeitlich veränderlichen E-Feldterme in der Polarisationsellipse durch ihre zeitlichen Mittelwerte ersetzte. Dies führt auf die Gleichung [3, S.34f.]:

$$(E_{0x}^2 + E_{0y}^2)^2 - (E_{0x}^2 - E_{0y}^2)^2 - (2E_{0x}E_{0y}\cos(\delta))^2 = (2E_{0x}E_{0y}\sin(\delta))^2$$

Die vier Stokes-Parameter lassen sich nun folgendermaßen identifizieren:

$$S_0 = E_{0x}^2 + E_{0y}^2 \tag{4a}$$
$$S_1 = E_{0x}^2 - E_{0y}^2 \tag{4b}$$
$$S_2 = 2E_{0x}E_{0y}\cos(\delta) \tag{4c}$$
$$S_3 = 2E_{0x}E_{0y}\sin(\delta) \tag{4d}$$
$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{4e}$$

Die letzte Gleichung gilt in dieser Form für vollständig polarisiertes Licht. Die Gesamtintensität ist in jedem Fall durch S_0 gegeben. Für nicht vollständig polarisiertes Licht wird sie zu einer Ungleichung, die den Grad der Polarisation (**D**egree **O**f **P**olarisation) quantifiziert:

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2 \qquad \text{DOP} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}$$

Es ist im Hinblick auf eine Beschreibung von unpolarisiertem Licht sowie für den Müller-Matrix-Formalismus praktisch, die Stokes-Parameter in einem Spaltenvektor, dem Stokes-Vektor, anzuordnen. Auch wenn es sich im mathematischen Sinn nicht um einen Vektor handelt, kann gezeigt werden [3, S. 52f.], dass sich die Stokes-Vektoren von unabhängigen Lichtstrahlen zu einem resultierenden Stokes-Vektor addieren lassen. So lässt sich unter anderem teilpolarisiertes Licht als Summe von vollständig unpolarisiertem und vollständig polarisiertem Licht beschreiben:

$$\underbrace{\mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}}_{\text{teilpolarisiert}} = \underbrace{\begin{pmatrix} S_0^u \\ S_1^u \\ S_2^u \\ S_3^u \end{pmatrix}}_{\text{unpolarisiert}} + \underbrace{\begin{pmatrix} S_0^p \\ S_1^p \\ S_2^p \\ S_3^p \end{pmatrix}}_{\text{polarisiert}} \tag{5}$$

$$\text{mit } S_0^u = (1 - \text{DOP})S_0, \quad S_1^u = S_2^u = S_3^u = 0 \quad \text{und} \quad S_0^p = \text{DOP}\,S_0, \quad S_1^p = S_1, \quad S_2^p = S_2, \quad S_3^p = S^3$$

Die einzelnen Komponenten des Stokes-Vektors **S** lassen sich jeweils mit bestimmten Polarisationszuständen identifizieren. Während S_0 die Gesamtintensität angibt, gibt S_1 den Anteil linear horizontal oder vertikal polarisierten Lichts an, S_2 den um $\pm 45°$ polarisierten Anteil und S_3 den links- bzw. rechtsdrehend zirkular polarisierten Anteil.

$$\mathbf{S}(\updownarrow) = I_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{S}(\leftrightarrow) = I_0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{S}(\nearrow) = I_0 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{S}(\nwarrow) = I_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{S}(\circlearrowleft) = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{S}(\circlearrowright) = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Des Weiteren lassen sich einige Identitäten herleiten, mit denen die elliptischen Parameter aus den Stokes-Parametern bestimmt werden können:

$$\tan(2\psi) = \frac{S_2}{S_1} \qquad \sin(2\chi) = \frac{S_3}{S_0} \qquad \mathbf{S} = \begin{pmatrix} 1 \\ \cos(2\chi)\cos(2\psi) \\ \cos(2\chi)\sin(2\psi) \\ \sin(2\chi) \end{pmatrix}$$

Der in Abhängigkeit der elliptischen Parameter dargestellte Stokes-Vektor erinnert stark an die Transformationsformel, die Kugelkoordinaten in karthesische Koordinaten umwandelt. Die Darstellung des Polarisationszustandes als Punkte auf einer Kugeloberfläche, wobei die x-,y- und z-Koordinaten S_1,S_2 und S_3 entsprechen, wird Poincaré Darstellung genannt und die entsprechende Kugel Poincaré-Sphäre (siehe Abb. 2).

1.1.3. Müller-Matrix-Formalismus

Der Müller-Matrix-Formalismus ermöglicht es, die Auswirkungen von polarisationsoptischen Komponenten auf einen einfallenden Stokes-Vektor und durch Berechnung des Matrixproduktes mathematisch zu bestimmen. Der einfallende Lichtstrahl mit dem Stokes-

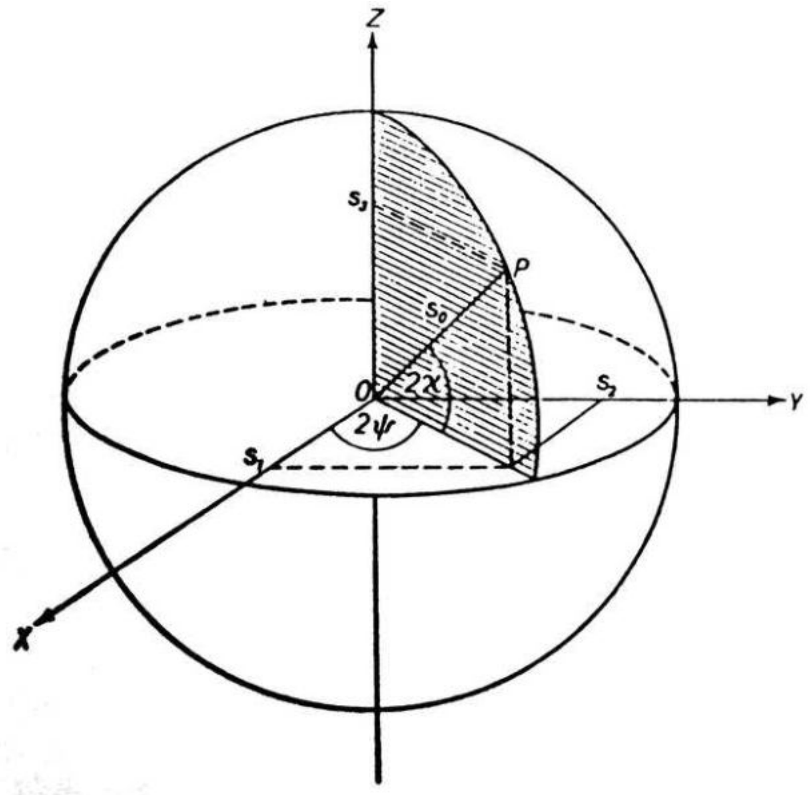

Abbildung 2: Poincaré Sphäre: Die Punkte auf der Kugeloberfläche können sowohl durch die Stokes-Parameter als auch durch die elliptischen Parameter dargestellt werden.

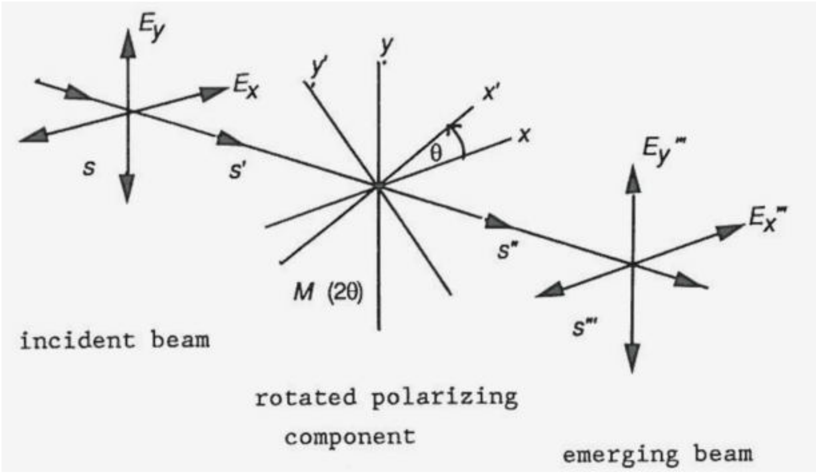

Abbildung 3: Definition der Koordinatenachsen und Drehwinkel für die Berechnung der Müller-Matrix gedrehter Elemente. Abbildung aus [3, S.80]. Der Drehwinkel α ist hier als θ bezeichnet.

Vektor $\mathbf{S}$ wird durch eine Polarisationsoptik mit der Müller-Matrix $\mathbf{M}$ in den resultierenden Stokes-Vektor $\mathbf{S}'$ umgewandelt:

$$\mathbf{S}' = \mathbf{M} \cdot \mathbf{S} \qquad \text{bzw.} \qquad \begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \cdot \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

Müller-Matrizen lassen sich für alle polarisationsoptischen Komponenten aufstellen. Für Komponenten, die den Polarisationszustand nicht ändern, entspricht die Müller-Matrix einer Einheitsmatrix. Im Folgenden werden einige für diesen Versuch wichtige Müller-Matrizen zusammengestellt.

1. Müller-Matrix für einen idealen Polarisator mit Durchlassrichtung entlang der x-Achse, der die y-Komponente des elektrischen Feldes vollständig auslöscht:

$$\mathbf{M}_{pol-x} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Müller-Matrix für einen Verzögerer, dessen schnelle Achse in y-Richtung orientiert eine Phasenverschiebung von δ zwischen der x- und der y-Komponente erzeugt:

$$\mathbf{M}_{verz}(\delta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\delta) & -\sin(\delta) \\ 0 & 0 & \sin(\delta) & \cos(\delta) \end{pmatrix}$$

3. Müller-Matrix für einen Rotator, der die orthogonalen Komponenten jeweils um einen Winkel α rotiert:

$$\mathbf{M}_{rot}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\alpha) & \sin(2\alpha) & 0 \\ 0 & -\sin(2\alpha) & \cos(2\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Aus diesen drei Matrizen lassen sich Matrizen für viele polarisationsändernde Komponenten bestimmen. So lässt sich z.B. die Müller-Matrix für eine um α gegenüber dem Laborkoordinatensystem rotierte Verzögerungsplatte (Blickrichtung entgegen der Strahlpropagationsrichtung, siehe Abb. 3) wie folgt bestimmen:

$$\mathbf{M}_{verz}(\alpha, \delta) = \mathbf{M}_{rot}(-\alpha)\mathbf{M}_{verz}(\delta)\mathbf{M}_{rot}(\alpha)$$

Die zweite Rotation in Gegenrichtung ist hierbei notwendig, da der Rotator das Koordinatensystem um seine Drehachse dreht, die Polarisation aber weiterhin im ursprünglichen Koordinatensystem betrachtet werden soll.

Wird in die für $\mathbf{M}_{verz}(\alpha, \delta)$ ermittelte Matrix für δ 180°eingesetzt, so ergibt sich eine Matrix, die der eines Rotators, der um den doppelten Winkel dreht, ähnelt:

$$\mathbf{M}_{verz}(\beta, 180^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(4\beta) & \sin(4\beta) & 0 \\ 0 & \sin(4\beta) & -\cos(4\beta) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

1.1.4. Doppelbrechung

Beim Übergang einer elektromagnetischen Welle in ein anisotropes Medium wird die Welle je nach Symmetrie des Kristalls in zwei oder mehr Komponenten zerlegt, die unterschiedlich gebrochen werden. Im Folgenden wird von einem einachsigen Kristall ausgegangen. Die Zerlegung erfolgt in einen Strahl, der orthogonal zur optischen Achse polarisiert ist, und einen Strahl, der parallel zur optischen Achse polarisiert ist. Ersterer wird analog zur Brechung in isotropen Medien nach dem Snellius'schen Brechungsgesetz gebrochen und wird daher als "ordentlicher Strahl" bezeichnet. Der andere Strahl wird als "außerordentlicher Strahl" bezeichnet. Der Brechungsindex des außerordentlichen Strahls hängt von seiner Ausbreitungsrichtung relativ zur optischen Achse ab.

In optisch anisotropen Materialien schwingen die Oszillatoren im Allgmeinen nicht parallel zum E-Feldvektor, da der Brechungsindex und damit die Ausbreitungsrichtung richtungsabhängig ist. Eine anschauliche Analogie bietet ein mechanisches Modell, in dem ein Massepunkt zwischen Federn verschiedener Federkonstanten aufgehängt ist. Die Federkonstanten sind die mechanischen Analogi zu den Brechungsindices. Durch die Richtungsabhängigkeit des Brechungsindex sind Phasenfronten mit dem Normalenvektor $\vec{k}$ nicht mehr parallel zur Ausbreitungsrichtung mit Poyntingvektor $\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$ als Normalenvektor. Während $\vec{S}$ noch orthogonal zu $\vec{E}$ ist, ist $\vec{k}$ orthogonal zu $\vec{D}$. Die Verschiebungsdichte $\vec{D}$ und Feldstärke $\vec{E}$ sind durch die elektrische Feldkonstante ε_0 und die Dielektrizitätskonstante ε', die in anisotropen Medien ein Tensor ist, verknüpft. Es gilt folgende Beziehung (hier formuliert für ein Hauptachsensystem):

$$\vec{D} = \varepsilon' \varepsilon_0 \vec{E} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \varepsilon_0 \vec{E} = \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix} \varepsilon_0 \vec{E}$$

Für einen einachsigen Kristall gilt $n_x = n_y \neq n_z$. Das sogenannte Brechungsindex-Ellipsoid ist gegeben durch

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1. \tag{7}$$

Es beschreibt die Richtungsabhängigkeit des Brechungsindex (siehe Abb. 4a): Für eine Welle, deren Ausbreitungsrichtung für den außerordentlichen Strahl durch $\vec{k}$ gegeben ist, gibt die Länge der Strecke in der Richtung von $\vec{D}$ vom Nullpunkt zum Schnittpunkt mit dem Ellipsoid den Brechungsindex n_a des außerordentlichen Strahls an [4]. Der Brechungsindex n_0 des ordentlichen Strahls ist, wie in der zweidimensionalen Darstellung in Abb. 4b zu sehen ist, nicht von der Ausbreitungsrichtung abhängig.

1.2. Lichtquellen

1.2.1. Lichtemittierende Dioden und kantenemittierende Halbleiterlaser

Lichtemittierende Dioden (LED) und Halbleiterlaser spielen heutzutage eine tragende Rolle bei vielen technischen Anwendungen wie einfachen Beleuchtungen, in der Kommunikation, wo Glasfaserkabel eine immer wichtigere Rolle in der Informationsübermittlung einnehmen, und vielen weiteren Anwendungen. Die Gründe für die vielfältige Anwendbarkeit liegen unter anderem in ihrer geringen Größe, ihren niedrigen Herstellungskosten und ihrer hohen Effizienz. Die Funktionsweise von LEDs und Halbleiterlasern besteht prinzipiell in der strahlenden Rekombination von Elektronen-Lochpaaren in der Raumladungszone eines pn-Übergangs.

Ein pn-Übergang entsteht, wenn ein p-dotierter Halbleiter, in dem Löcher die Majoritätsladungsträger bilden, und ein n-dotierter Halbleiter, in dem Elektronen die Majoritätsladungsträger bilden, in Kontakt gebracht werden. Die Majoritätsladungsträger diffundieren in Richtung der anderen Schicht und rekombinieren an der Grenzschicht, bis sich die Fermi-Energien der beiden Materialien angeglichen haben (siehe Abb. 5a). Es entsteht eine Verarmungs- oder Raumladungszone, in der keine freien Ladungsträger existieren. Wird eine Spannung in Durchlassrichtung angelegt, dass heißt die n-dotierte Schicht wird mit dem negativen Pol der Spannungsquelle und die p-dotierte Schicht wird mit dem positiven Pol der Spannungsquelle verbunden, so steigt die Ladungsträgerdichte in der n-dotierten Schicht. Die neu hinzugekommenen Ladungsträger diffundieren bis zur Verarmungszone, wo sie mit einem Loch rekombinieren (siehe Abb. 5b). Bei dieser Rekombination wird Energie frei, die betragsmäßig der Größe der Bandlücke entspricht. Wird diese Energie in Form von Photonen frei, so emittiert die Diode Licht, dessen Wellenlänge von der Größe der Bandlücke abhängt.

Weiße LEDs lassen sich entweder durch Integration mehrer unterschiedlicher Dioden in einem Bauelement konstruieren oder dadurch, dass man eine blaue LED mit einem oder mehreren Lumineszenzfarbstoffen beschichtet, die das blaue Licht absorbieren und danach Licht höherer Wellenlängen emittieren.

Kanten-Emitter-Laser (engl.: edge emitting Laser, EEL) funktionieren prinzipiell sehr ähnlich wie die LEDs [7]. Kanten-Emitter-Laser haben im Gegensatz zu LEDs einen Resonator, der im einfachsten Fall (siehe Abb. 6) durch die Kanten, die bei der Spaltung der Wafer entstehen, realisiert ist. Bis es durch stimulierte Emission zur Rekombination kommt, werden die Photonen zwischen Kanten

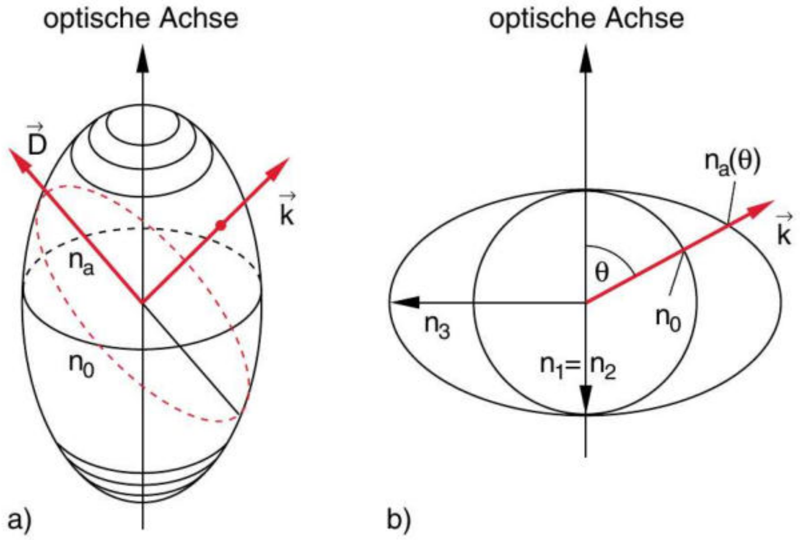

Abb. 8.31. (**a**) Rotationssymmetrisches Indexellipsoid mit der Symmetrieachse in Richtung der optischen Achse. (**b**) Zweidimensionale Darstellung von $n_a(\theta)$ und der nicht von θ abhängigen Größe n_0 für einen positiv einachsigen Kristall

Abbildung 4: Quelle: Demtröder [4]

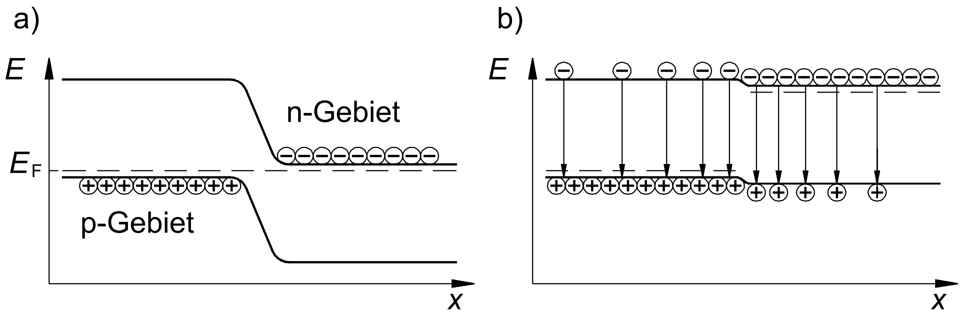

Abbildung 5: Energiebänder in einem pn-Übergang ohne (a) bzw. mit angeleger Spannung (b) [7]

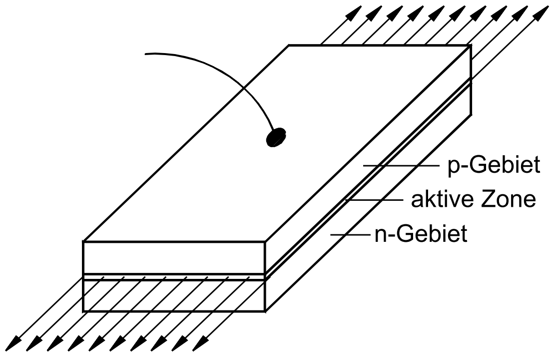

Abbildung 6: Schematische Darstellung eines einfachen Kanten-Emitter-Laser, der nur aus einem p- und einem n-dotierten Halbleiter besteht. Die Emission findet durch die teilreflektierenden Kanten statt [7].

des Lasers, die poliert werden und ca. 30 % des Strahls reflektieren und wie ein Fabry-Perot-Resonator funktionieren, hin und her reflektiert. Die hier geschilderte Grundkonstruktion kann duch Doppel-Hetero-Strukturen, für deren Entwicklung u.a. Herbert Krömer mit dem Nobelpreis ausgezeichnet wurde, effzienter gestaltet werden. Durch die unterschiedlichen Materialien kann dafür gesorgt werden, dass die aktive Zone einen besonders hohen Brechungsindex hat, woduch die Photonen durch Totalreflexion in der aktiven Zone gehalten werden. Diese Art von Laser wird auch Wellenleiter (engl.: Waveguide) genannt.

1.2.2. VCSEL

Neben den Kantenemittern nehmen oberfächenemittierende Halbleiterlaser mit vertikalem Resonator(engl.: vertical-cavity-surface-emitting-laser, VCSEL) eine immer wichtigere Rolle ein. Der Aufbau von VCSELn unterscheidet sich in einigen Aspekten deutlich von dem der Kantenemitter [7]. Im Gegensatz zu Kantenemittern erfolgt die Emission nicht parallel zum pn-Übergang, sondern senkrecht dazu und wird an der Oberfläche ausgekoppelt. Dies bietet den prinzipiellen Vorteil, dass VCSEL auf Wafern in Arraystruktur hergestellt und direkt auf dem Wafen ankontaktiert sowie getestet werden können. Es ist nicht notwendig, den Wafen erst zu zersägen und die Kanten zu spalten. Die Herstellung wird hierdurch so günstig, dass die Herstellungskosten für einen VCSEL im einstelligen Centbereich liegen. Ein weiterer Vorteil von VCSELn ist die kreisrunde astigmatismusfreie Emissionscharakteristik, die unter anderem das Einkoppeln in Glasfasern erleichtert.

Bei VCSELn ist die aktive Schicht äußerst dünn. Sie besteht aus Quantenfilmen, die sich zwischen zwei DBR-Spiegeln (Distributed-Bragg-Reflector) befinden, die den Resonator bilden (siehe exemplarische Darstellung in Abb. 7). Der Resonator ist im Gegensatz zu Kantenemittern vertikal ausgerichtet und sehr kurz. Die DBR-Spiegel bestehen aus vielen Schichten unterschiedlicher Brechungsindices, deren Dicke ein Viertel der emittierten Wellenlänge beträgt. Da bei Reflexionen ein Phasensprung stattfindet, interferieren die reflektierten Photonen im Resonator konstruktiv. Durch diese Konstruktion wird eine Reflektivität von über 99% erreicht, aufgrund der die extrem kurze Resonatorlänge in VCSELn auch notwendig ist, um eine stimulierte Emission zu erzielen. Die Oxidapertur in der Peripherie der aktiven Schicht des VCSELs zwingt zum einen den Strom, vom elektrischen Ringkontakt aus durch die Mitte der aktiven Schicht zu fließen. Zum anderen wird durch einen hohen Brechungsindexsprung auch das Licht in transversale Richtung geführt.

Durch die extrem kurze Resonatorlänge, die etwa im Bereich einer Wellenlänge liegt, existiert in der Regel nur eine einzige longitudinale Mode. Hierdurch wird unter anderem ein sehr geringer Schwellstrom, also der Strom, bei dem die Besetzungsinversion zustande kommt und die stimulierte Emission überwiegt, erreicht [8]. Die emittierte Leistung steigt nach Überschreiten der Schwelle näherungsweise linear an, bis sie ein Maximum erreicht und danach aufgrund thermischer Effekte wieder abfällt. Thermische Effekte führen dazu, dass sich die Länge des Resonators ändert, was auch eine Verschiebung der Wellenlänge zur Folge hat. Bei zu großen Wellenlängen nimmt die Effizienz des Resonators ab. Die starke Temperaturabhängigkeit des Resonators ist in erster Linie in seiner geringen Länge begründet.

Neben der Longitudinalmode entstehen auch Transversalmoden. Insbesondere entstehen bei großen Aperturdurchmessern viele Transversalmoden höherer Ordnung, deren Intensität am Rand des Oxidrings besonders hoch ist. Hierdurch wird im Nah- und Fernfeld eine ringförmige Intensitätsverteilung sichtbar.

Durch die Kreissymmetrie des VCSELs ist es nicht möglich, bestimmte Polarisationsrichtungen durch das Design des Resonators zu bevorzugen, wie es bei Kantenemittern möglich ist. Die Richtung ist in der Praxis nicht vollständig beliebig, sondern durch die Kristallstruktur, thermische Einflüsse, Doppelbrechung und weitere Effekte eingeschränkt, sodass lineare Polarisationszustände beobachtet werden können. Die Richtung dieser Polarisationszustände ist jedoch hochgradig instabil und kann sich auch im Betrieb ändern [9]. Man spricht in diesem Fall von einem Polarisationsswitch. Eine Kontrolle der Polarisationsrichtung ist beispielsweise durch das Aufbringen einer Gitterstruktur möglich.

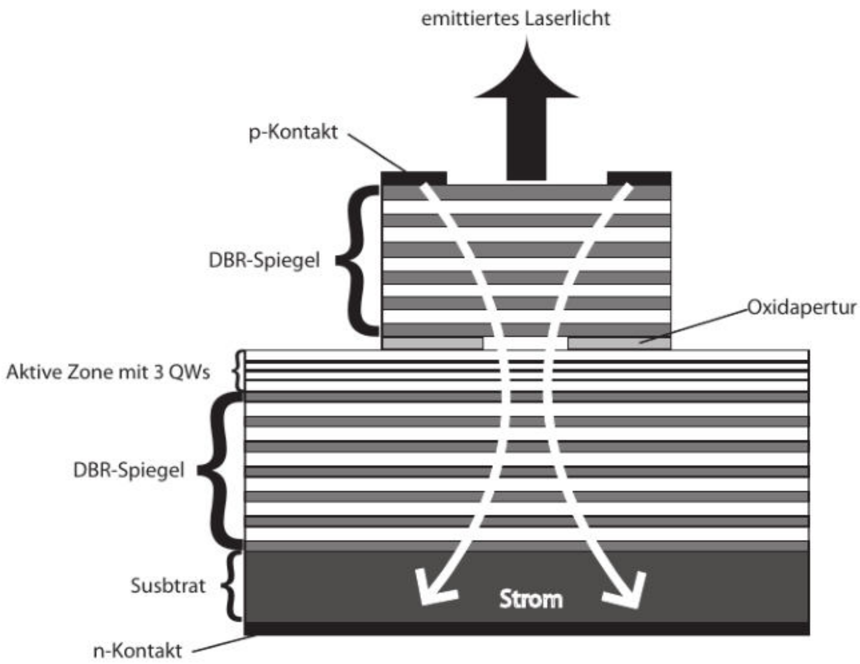

Abbildung 7: Schematischer Aufbau eines VCSEL (Quelle: Diplomarbeit von Andreas Molitor)

2. Aufbau

Für den Versuch stehen eine Reihe von Komponenten zur Verfügung, mit der verschiedene Polarisationszustände erzeugt und analysiert werden können. Die Komponenten können auf höhenverstellbaren Reitern auf einer Schiene bewegt werden. **Bei jeglicher Handhabung der Komponenten ist darauf zu achten, die Optiken nicht zu berühren.**

2.1. Komponenten

2.1.1. Polarisatoren

Glan-Thompson-Polarisationsprismen bestehen aus zwei zu einem Quader verkitteten doppelbrechenden Prismen, deren optische Achse parallel zur Eintrittsfläche liegt. Die Winkel der Prismen und der Brechungsindex des Kittes sind so gewählt, dass der ordentliche Strahl an der Grenzfläche der Prismen total reflektiert und dann an den speziell beschichteten Außenflächen absorbiert wird und der außerordentliche Strahl ohne Ablenkung wieder austritt (siehe Abb. 8).

Ein Maß für die Güte eines Polarisators ist das Auslöschungsverhältnis $I_{\mathrm{min}}/I_{\mathrm{max}}$. Dabei ist I_{min} die Intensität, die gemessen wird, wenn linear polarisiertes Licht senkrecht zur durchlassenden Richtung des Polarisators eingestrahlt wird und I_{max} entsprechend die Intensität, wenn es parrallel dazu eingestrahlt wird.

Glan-Thompson-Polarisationsprismen zeichnen sich durch ein besonders hohes Auslöschungsvermögen von 10^{-5}-10^{-7} über einen weiten Spektralbereich aus, eignen sich jedoch nicht für hohe Strahlintensitäten.

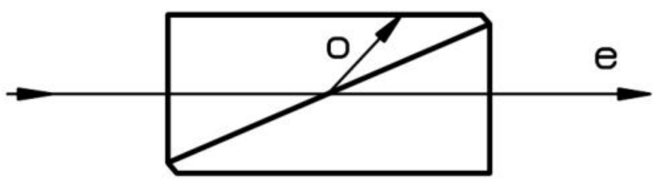

Abbildung 8: Schemaskizze eines Glan-Thompson-Polarisators (Quelle: Datenblatt des Polarisators)

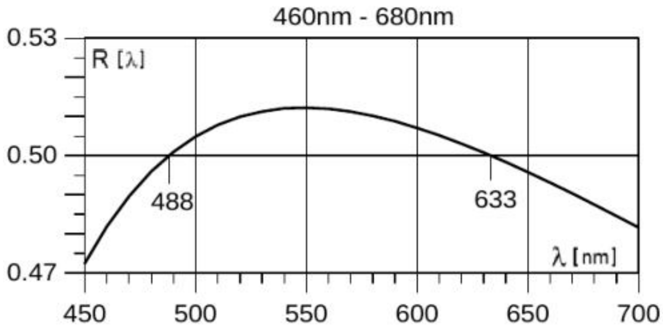

Abbildung 9: Verzögerung in Einheiten von π als Funktion der Wellenlänge für ein achromatisches Verzögerungsplättchen (Quelle: Datenblatt)

2.1.2. Verzögerungsplatten

Verzögerungsplatten bestehen aus einem doppelbrechenden Material, das so in den Strahlengang gebracht wird, dass die optische Achse senkrecht zur Ausbreitungsrichtung steht. Das Strahlenbündel wird in einen ordentlichen und einen außerordentlichen Teilstrahl aufgespalten, die mit verschiedenen Geschwindigkeiten, aber in gleicher Richtung durch das Material propagieren. Für eine Dicke d des Verzögerungsplättchens ergibt sich für die Phasenverschiebung δ zu

$$\delta = \frac{2\pi d}{\lambda}(n_a - n_o)$$

Die zu realisierende Dicke der Platten, um einen Phasenunterschied von $\frac{\pi}{2}$ im sichtbaren Wellenlängenbereich zu erzeugen, ist zu gering, um sie herstellen und handhaben zu können. Um dieses Problem zu lösen, kann entweder eine Dicke verwendet werden, die eine Phasenverschiebung von $\delta = (2k + 0.5)\pi$ erzeugt. Man spricht dann von Verzögerungsplatten k-ter Ordnung. Eine weitere Möglichkeit ist die Kombination zweier paralleler Platten, deren optische Achsen senkrecht zueinander sind. In diesem Fall ist nicht die Gesamtdicke, sondern die Differenz der Dicken der beiden Platten entscheidend:

$$\delta = \frac{2\pi(d_1 - d_2)}{\lambda}(n_a - n_e)$$

Derartige Platten werden Verzögerungsplatten nullter Ordnung (zero order) bzw. zusammengesetzte Verzögerungsplatten nullter Ordnung (compound zero order) genannt.

Die so konstruierten Verzögerungsplatten sind stark wellenlängenabhängig. Ähnlich wie bei der Verwendung von unterschiedlich stark brechenden Gläsern in achromatischen Linsensystemen lassen sich durch Verwendung von mehreren unterschiedlichen doppelbrechenden Materialien für die beiden Platten achromatische Verzögerungsplatten herstellen. Für Verzögerungsplatten nullter Ordnung, die aus zwei Platten a und b verschiedener Materialien bestehen, ergeben sich zwei Wellenlängen λ_1 und λ_2, für die der beabsichtigte Phasenunterschied erreicht wird. Die Dicken d_a und d_b lassen sich dann wie folgt bestimmen [1]:

$$\delta = \frac{2\pi}{\lambda_1}(d_a(n_{a\lambda 1} - n_{o\lambda 1}) - d_b(n_{a\lambda 1} - n_{o\lambda 1}))$$
$$\delta = \frac{2\pi}{\lambda_2}(d_a(n_{a\lambda 2} - n_{o\lambda 2}) - d_b(n_{a\lambda 2} - n_{o\lambda 2}))$$

Für Wellenlängen, die in der Nähe von λ_1 und λ_2 liegen, ergeben sich jedoch, wie in Abb. 9 beispielhaft für ein $\frac{\lambda}{2}$-Plättchen gezeigt, bereits leichte Abweichungen von der gewünschten Verzögerung. Neben der Wellenlänge beeinflussen auch Temperatur, Interferenzerscheinungen und der Einfallswinkel das Verzögerungsverhalten. Die Abhängigkeit der Phasenverschiebung vom Einfallswinkel kann hierbei auch ausgenutzt werden, um die Phasenverschiebung von Verzögerungsplatten für die jeweilige Wellenlänge anzupassen. Das kann nötig sein, wenn z.B. zirkular polarisiertes Licht mit hoher Genauigkeit erzeugt werden muss. Eine Rotation um die schnelle Achse bewirkt eine Verkleinerung der Phasenverschiebung und eine Rotation um die langsame Achse eine Vergrößerung. Eine Übersicht über die möglichen auftretenden Probleme wird z.B. von Hale [5] gegeben.

Die beiden im Versuch verwendeten achromatischen Verzögerungsplatten bestehen aus Quarz und MgF_2. Die Änderung des Gangunterschiedes beträgt laut Herstellerangabe ca. $\pm\frac{1\text{nm}}{\text{Grad}^2}\delta\varphi^2$.

Babinet-Soleil-Kompensator Ein Babinet–Soleil-Kompensator ist ein Verzögerer, der eine variable Phasenverschiebung erzeugt. Der prinzipielle Aufbau ähnelt dem eines zusammengesetzten Verzögerungsplättchens nullter Ordnung, wobei eine der beiden Platten, deren optische Achsen orthogonal zueinander sind, durch zwei Keile ersetzt wurde (siehe Abb. 10). Diese Keile können übereinandergeschoben werden, was eine Änderung der Dicke und damit eine Änderung der Phasenverschiebung zur Folge hat. Da die Phasenverschiebung wellenlängenabhängig ist, muss die Skala für jede Wellenlänge von A.U. auf Wellenlängendifferenzen bzw. Phasendifferenzen kalibriert werden.

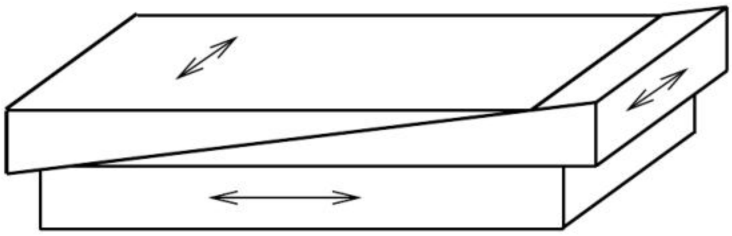

Abbildung 2: *Die Richtung der optischen Achsen im SOLEIL'schen Kompensator.*

Abbildung 10: Quelle: Anleitungsblatt Polarisation und Doppelbrechung

(a) Ansicht von vorne

(b) Ansicht von oben

Abbildung 11: Fotos des Babinet-Kompensators mit Nonius-Skala von vorne und von oben. Die Trommelskala dient zur Einstellung der Verschiebung der beiden keilförmigen Platten gegeneinander.

2.1.3. Interferenzfilter

Um spektral aufgelöste Messungen durchführen zu können, stehen im Versuch vier Bandpassfilter zur Verfügung, die als Interferenzfilter ausgeführt sind.

Mit Interferenzfiltern lässt sich ein spektral besonders schmaler Transmissionsbereich realisieren. Sie funktionieren nach dem Prinzip eines Fabry-Perot-Interferometers und bestehen aus vielen Schichten dielektrischen Materials unterschiedlicher Brechungsindices und unterschiedlicher Dicke, die die einfallenden Lichtstrahlen jeweils teilweise reflektieren und transmittieren und durch konstruktive bzw. destruktive Interferenz dafür sorgen, dass nur ein bestimmter Bereich um die Zentralwellenlänge transmittiert wird. Die Halbwertsbreite des Transmissionsbereichs der im Versuch verwendeten Filter beträgt jeweils (10 ± 2)nm. Die Zentralwellenlängen sind in folgender Tabelle zusammengestellt:

Filter	$\lambda_{zentral}$ in nm
1	488
2	543,5
3	633
4	694,3

Die Filter sind am Aufbau entsprechend nummeriert.

2.1.4. Linsen

Sowohl beim VCSEL als auch bei der LED handelt es sich um divergente Lichtquellen. Zur Kollimation und Fokussierung der Lichtquellen stehen drei unterschiedliche Linsen mit Brennweiten von $19\,\mathrm{mm}$, $60\,\mathrm{mm}$ und $80\,\mathrm{mm}$ zur Verfügung. Die Linse mit der Brennweite von $19\,\mathrm{mm}$ ist am Rand des Reiters angebracht, sodass eine Positionierung nahe an einer Lichtquelle ungeachtet des breiten Reiters möglich ist.

2.1.5. Drehwinkelaufnehmer

Je ein Polarisator und ein Verzögerungsplättchen sind in einem Drehwinkelaufnehmer montiert. Mittels einer Reflexionslichtschranke, unter der eine in 5°-Schritten in 72 stark und weniger stark reflektierende Bereiche eingeteilte Kreisscheibe (siehe Abb. 12) gedreht wird, wird bei jeder steigenden und fallenden Flanke der detektierten Spannung ein Signal generiert. Die Software liest, sobald sie ein Signal vom Drehwinkelaufnehmer erhält, einen Leistungswert vom Detektor. Somit lässt sich halbautomatisch die Intensität als Funktion des Drehwinkels aufnehmen.

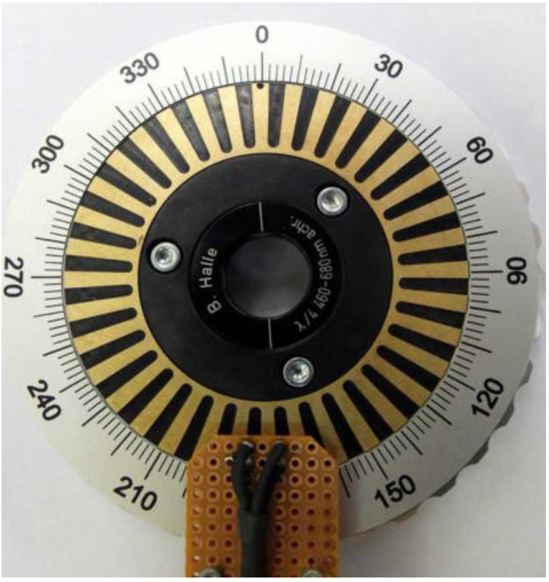

Abbildung 12: Foto des im Versuch verwendeten Drehwinkelaufnehmers mit der in verschieden stark reflektierende Zonen eingeteilten Kreisscheibe. Eingebaut ist ein $\lambda/4$-Platte, markiert ist die Richtung der schnellen Achse.

Bei der Nutzung ist zu beachten, dass ein zu schnelles Drehen aufgrund der Antwortzeit des Detektors dazu führen kann, dass der eingelesene Leistungswert nicht der dem Winkel zugehörige ist. Als Anhaltspunkt für die Geschwindigkeit, mit der gedreht werden darf, kann die Schwankung der ermittelten Stokes-Parameter in Abhängigkeit von der Drehgeschwindigkeit dienen. Bei zu schnellem Drehen kommt es hier zu auffälligen Schwankungen. Für die Anzahl der Werte, über die die Detektorelektronik mittelt, sollte ein verhältnismäßig kleiner Wert von z.B. 100 eingestellt werden, um eine hinreichende Drehgeschwindigkeit zu erlauben.

Bei der Drehrichtung ist zu beachten, dass die schnelle Achse mit dem **linken** Rand der ausgezeichneten reflektierenden Fläche zur Deckung gebracht wurde.

2.1.6. Lichtquellen

Für den Versuch stehen ein VCSEL VC670M-TO46FW-2 mit einer Wellenlänge von $670\,\mathrm{nm}$ und eine weiße LED M57L5111 (Spektrum siehe Abb.13) zur Verfügung. Für den VCSEL sind Justagebrillen vorhanden. Außerdem besteht die Möglichkeit, ihn mit einem ND-Filter (OD=0,4) abzuschwächen. Die grundlegenden Bestimmungen zum Laserschutz sind zu beachten (siehe Anhang A).

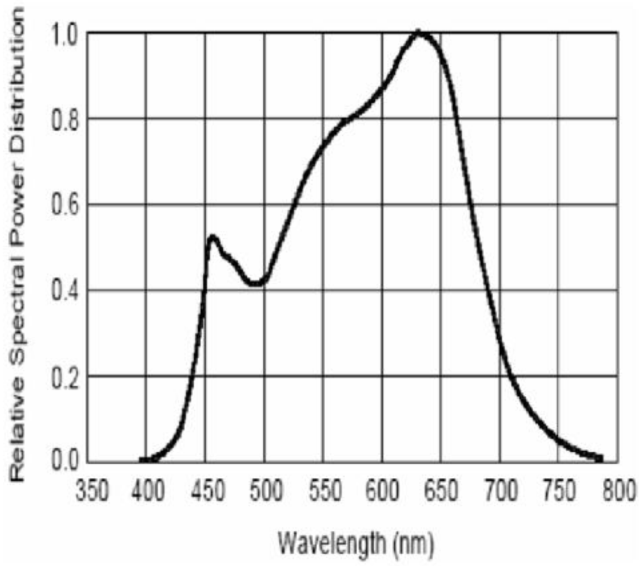

Abbildung 13: Spektrum der M57L5111 LED; Quelle: Datenblatt

2.1.7. Stromquelle

Als Stromquelle dient eine ILX Lightwave LDX-3412. Es handelt sich hierbei um eine spezielle Stromquelle für Diodenlaser und LEDs, die den Strom oder die Leistung sehr genau auf einen eingestellten Wert regelt und die Spannung entsprechend anpasst. Es lassen sich Ströme zwischen 0 und 200mA einstellen. Außerdem besteht die Möglichkeit, ein Limit einzustellen, um ein versehentliches Einstellen zu hoher Ströme zu verhindern. Ein Foto der vorhandenen Bedienelemente ist in Abb. 14 zu sehen.

Abbildung 14: Frontansicht der ILX Stromquelle mit Kennzeichnung der Knöpfe

Das Gerät wird während des Versuchs nicht ein- und ausgeschaltet. Stattdessen wird der Drehregler auf Null heruntergedreht und der Output (Knopf B) deaktiviert.

2.2. Messverfahren

2.2.1. Grundlegende Justage

Vor jedem Versuch muss der Aufbau justiert werden. Dabei müssen je nach Aufgabe die Lichtquellen auf den Detektor kollimiert bzw. fokussiert werden.

Je nach verwendeter Lichtquelle bieten sich verschiedene Linsen oder Kombinationen von Linsen an, um zum einen eine möglichst hohe Intensität auf dem Detektor zu erzielen und zum anderen möglichst ebene Wellenfronten zu erzeugen. Letzeres ist wichtig, da die Form der Wellenfronten Einfluss auf die Wirkung der polarisationsmodifizierenden Optiken hat.

Je nach Messmethode ist auch eine Ausrichtung der Achsen der Polarisationsoptiken erforderlich, die die durch die Verwendung der Skalen erzielbare Einstellgenauigkeit übersteigt. In diesem Fall kann der Umstand, dass für bestimmte Stellungen der Achsen zueinander Intensitätsminima oder -maxima zu erwarten sind, genutzt werden.

2.2.2. Bestimmung der Stokes-Vektoren mittels Fourieranalyse

Es existieren verschiedene Verfahren zur Bestimmung der Stokes-Parameter. Eine Beschreibung verschiedener Verfahren einschließlich ihrer Vor- und Nachteile ist z.B. bei Collet [3] zu finden. Die folgende Beschreibung des Verfahrens, das im Versuch mehrfach zur Anwendung kommt, stützt sich auf ein Paper von H.G. Berry und A.E. Livingston [2], in dem eine sehr allgemeine und umfassende Beschreibung des Verfahrens geliefert wird.

Verwendet wird eine Verzögerungsplatte, die eine Verzögerung von δ erzeugt und mit der x-Achse den Winkel β einschließt, und ein Polarisator, der mit der x-Achse den Winkel α einschließt und zwischen dem Detektor und der Verzögerungsplatte positioniert ist. Durch Multiplikation der entsprechenden Müllermatrizen erhält man folgenden Ausdruck für die resultierende S_0-Komponente und damit für die Intensität:

$$\begin{aligned} I_T(\alpha,\beta,\delta) = \frac{1}{2}(S_0 + (S_1\cos(2\beta) + S_2\sin(2\beta))\cos(2\alpha-2\beta) + \\ ((S_2\cos(2\beta) - S_1\sin(2\beta))\cos(\delta) - S_3\sin(\delta))\sin(2\alpha-2\beta)) \end{aligned} \tag{8}$$

Dieser Ausdruck kann durch das Anwenden von Additionstheoremen in die Form einer Fourierreihe gebracht werden:

$$\begin{aligned} I_T(\alpha,\beta,\delta) =& \frac{1}{2}\Bigg(S_0 + \frac{1}{2}(S_1\cos(2\alpha) + S_2\sin(2\alpha))(1+\cos(\delta)) \\ & - S_3\sin(\delta)\sin(2\alpha-2\beta_0)\cos(2\beta) \\ & + S_3\cos(2\alpha-2\beta_0)\sin(\delta)\sin(2\beta) \\ & + \frac{1}{2}(S_1\cos(2\alpha-4\beta_0) - S_2\sin(2\alpha-4\beta_0))(1-\cos(\delta))\cos(4\beta) \\ & + \frac{1}{2}(S_1\sin(2\alpha-4\beta_0) + S_2\cos(2\alpha-4\beta_0))(1-\cos(\delta))\sin(4\beta)\Bigg) \\ I_T(\beta) =& c^{(0)} + c^{(2)}\cos(2\beta) + s^{(2)}\sin(2\beta) + c^{(4)}\cos(4\beta) + s^{(4)}\sin(4\beta) \end{aligned} \tag{9}$$

Hier wurde außerdem die Substitution $\beta \rightarrow \beta + \beta_0$ verwendet, mit der ein Anfangswinkel β_0 in die Rechnung eingeht, relativ zu dem der Winkel β gemessen wird. Dies ist bei der Umsetzung relevant, wenn der Anfangswinkel einer experimentellen Unsicherheit unterliegt. Rotiert man die Verzögerungsplatte eine gerade Anzahl N gleich großer Winkel und nimmt für jede der Winkelpositionen die Intensität $I(\beta)$ auf, so kann man die Fourierkoeffizienten mittels diskreter Fourieranalyse bestimmen:

$$c^{(0)} = \frac{1}{N}\sum_{i=1}^{N} I_{Ti} \qquad c^{(2)} = \frac{2}{N}\sum_{i=1}^{N} I_{Ti}\cos(2\beta_i) \qquad c^{(4)} = \frac{2}{N}\sum_{i=1}^{N} I_{Ti}\cos(4\beta_i)$$

$$s^{(2)} = \frac{2}{N}\sum_{i=1}^{N} I_{Ti}\sin(2\beta_i) \qquad s^{(4)} = \frac{2}{N}\sum_{i=1}^{N} I_{Ti}\sin(4\beta_i)$$

In diesem Versuch ist $N = 72$ durch die Anzahl der Segmente des Drehwinkelaufnehmers der $\lambda/4$-Platte gegeben. Mit β_i ist hierbei die Winkelposition und mit I_{Ti} die Intensität für den i-ten Messwert bezeichnet. Die Stokes-Paramter lassen sich aus den Fourierkoeffizienten dann wie folgt berechnen (*Man beachte, dass das Paper von Berry Fehler enthält*):.

$$S_1 = \frac{4}{1-\cos(\delta)}(c^{(4)}\cos(2\alpha-4\beta_0) + s^{(4)}\sin(2\alpha-4\beta_0)) \tag{10a}$$

$$S_2 = \frac{4}{1-\cos(\delta)}(s^{(4)}\cos(2\alpha-4\beta_0) - c^{(4)}\sin(2\alpha-4\beta_0)) \tag{10b}$$

$$S_3 = \frac{2s^{(2)}}{\sin(\delta)\cos(2\alpha-2\beta_0)} = \frac{-2c^{(2)}}{\sin(\delta)\sin(2\alpha-2\beta_0)} \tag{10c}$$

$$S_0 = 2c^{(0)} - 2\frac{1+\cos(\delta)}{1-\cos(\delta)}(c^{(4)}\cos(4\alpha-4\beta_0) + s^{(4)}\sin(4\alpha-4\beta_0)) \tag{10d}$$

Dabei ist zu berücksichtigen, dass in diesem Versuchsaufbau $\alpha = 0$ und im Idealfall auch $\beta_0 = 0$ gelten, so dass der erste Teil der Gleichung für S_3 verwendet werden sollte. Beim analysieren der Messdaten bietet es sich an, die Stokesparameter auf S_0 zu normieren.

Hier fällt auf, dass die Frequenzen, die den Fourierkoeffizienten, aus denen die Stokes-Parameter S_1 und S_2 bestimmt werden, zugrunde liegen, doppelt so hoch sind wie die Frequenzen der Fourierkoeffizienten, die S_3 zugrunde liegen. Dieser Umstand ermöglicht es, die linear und zirkular polarisierten Anteile besonders gut trennen zu können und zeichnet diese Methode unter anderem aus.

Ein weiterer Vorteil dieser Methode ist, dass sie sich hervorragend automatisieren lässt. So existieren kommerziell erhältliche Systeme, bei denen schnell rotierende Verzögerungsplatten zum Einsatz kommen (z.B. Thorlabs PAX Series Polarimeters; hier rotiert das $\frac{\lambda}{4}$-Plättchen mit 30 Hz).

3. Aufgaben

3.1. Vorbereitung

- Machen Sie sich mit den Richtlinien zum Laserschutz (Anhang A) vertraut. Drucken Sie sie aus und bringen Sie sie mit zum Versuch. Machen Sie sich außerdem mit der Funktionsweise und den physikalischen Prinzipien der verwendeten Elemente und mit dem Stokes-Formalismus vertraut.
- Erklären Sie kurz die Idee des im Aufgabentext von Aufgabe 3 beschriebenen Messverfahrens und die Bedeutung der einzelnen Schritte.
- In Aufgabe 4 sollen Sie rechts- und linkszirkulares Licht herstellen. Berechnen Sie, wie Sie die Drehwinkel eines Linearpolarisators und einer idealen $\lambda/4$-Platte zur x-Achse einstellen müssen, um mit diesen beiden Elementen die gewünschte Polarisation zu erzeugen.

  Die Gleichungen (3) gelten nur für $(-\pi/4 < \chi, \psi < \pi/4)$. Wie müssen die Gleichungen für Ellipsen angepasst werden, deren langen Halbachse sehr steil steht, also einen Winkel von weniger als $45°$ mit der y-Achse einschließt?

3.2. Durchführung

1. Nehmen Sie eine P-I-Kennlinie der LED auf. Stellen Sie dazu das Limit an der Stromquelle auf $53\,\mathrm{mA}$ und messen Sie von $0\,\mathrm{mA}$ bis $50\,\mathrm{mA}$ in $5\,\mathrm{mA}$-Schritten. Berücksichtigen Sie bei dieser und allen weiteren Aufgaben Umgebungslicht und führen Sie entsprechende Untergrundmessungen durch.
2. Bestimmen die das Auslöschungsverhältnis der Linearpolarisatoren. Wählen Sie den Abstand zwischen den Polarisatoren und zum Leistungsmessgerät so, dass Sie in Aufgabe 3 noch den Babinet-Soleil-Kompensator, die $\lambda/4$-Platte und das Filterrad einfügen können. Tipp: Es bietet sich an, das Filterrad nach dem zweiten Polarisator einzufügen.

   An vorderer Stelle wird der Polarisator mit der Nonius-Skala eingefügt. Die Durchlassrichtung des hinteren Polarisators wird in x-Richtung orientiert und im Verlauf des gesamten Versuchs nicht mehr verändert. Der vordere Polarisator wird nun so rotiert, dass keine Intensität mehr transmittiert wird. Da hier eine hohe Genauigkeit wichtig ist, wird mit der Detektorelektronik über 1000 Werte gemittelt.
3. Bestimmen Sie mit den Filtern für die verschiedenen Wellenlängen die Phasenverzögerung des achromatischen Verzögerungsplättchens mit dem Babinet-Soleil-Kompensator.

   Gehen Sie folgendermaßen vor:

   a) Zuerst wird der Kompensator zwischen den beiden Polarisatoren in den Strahlengang gebracht. Im Allgemeinen fällt nach dem Einbringen des Kompensators wieder Licht auf den Detektor.

   b) Der Kompensator wird nun rotiert, bis am Detektor ein Intensitätsminimum gemessen wird.

   c) Der Kompensator wird nun unter Verwendung der Nonius-Skala um $45°$ in eine Position rotiert, so dass Sie die Trommel gut erreichen und ablesen können.

   d) Für die zu untersuchenden Wellenlängen werden alle drei Einstellungen der Trommelskala bestimmt, für die ein Intensitätsminimum erzeugt wird. Es bietet sich an, zunächst mit dem Auge ein Intensitätsminimum zu finden und erst danach den Detektor zu verwenden. Achten Sie darauf, den Winkel des Kompensators dabei nicht unbemerkt zu verstellen. Mit der Trommelskala dürfen Sie unter keinen Umständen den **Wertebereich von 0 bis 55** verlassen, da sonst der Kompensator zerstört wird! Beim Vorliegen eines Intensitätsminiums ist die Phasenverschiebung zwischen der x- und der y- Komponente ein ganzzahliges Vielfaches von 2π. Mit den aufgezeichneten Werten kann so in der Auswertung die Trommelskala kalibriert werden.

   e) Der Filter 4 ist noch ausgewählt und der Babinet-Soleil-Kompensator auf das dritte Minimum eingestellt. Das Vorgehen zur Vermessung der $\lambda/4$-Platte ist nun ähnlich wie bei der Kalibrierung des Babinet-Soleil-Kompensators: Die $\lambda/4$-Platte mit der Nonius-Skala wird zwischen den ersten Polarisator und den Babinet-Soleil-Kompensator in den Strahlengang gestellt. Im Allgemeinen fällt nach dem Einbringen der $\lambda/4$-Platte wieder Licht auf den Detektor. Sie wird nun rotiert, bis am Detektor ein Intensitätsminimum gemessen wird, und anschließend unter Verwendung der Nonius-Skala um $45°$

gedreht. Für die zu untersuchenden Wellenlängen werden wieder die Einstellungen der Trommelskala bestimmt, für die ein Intensitätsminimum erzeugt wird. Aus der Verschiebung der Minima relativ zu den Minima ohne die $\lambda/4$-Platte und dem Kalibrierungsfaktor können sie die Verzögerung der Platte für die verschiedenen Wellenlängen bestimmen.

4. Stellen Sie Licht der folgenden Polarisationszustände her:
   - Lineare Polarisation: vertikal und unter $45°$ zur Horizontalen
   - Rechts- und linkszirkulare Polarisation

   Charakterisieren Sie die Polarisationszustände mit Hilfe des Stokes-Formalismus. Nutzen Sie hierfür die LED und den $694{,}3\,\mathrm{nm}$-Filter. Verwenden Sie bei der Auswertung außerdem Ihre Erkenntnisse aus der Voraufgabe über die tatsächliche Verzögerung der $\lambda/4$-Platte. Beachten Sie bei der Bestimmung der Drehwinkel der Elemente die Winkeldefinition in Abb. 3. Bei den Lambda-Platten zeigt die Markierung auf der Fassung die Richtung der schnellen Achse an, bei den Polarisatioren die Durchlassrichtung.

5. Vermessen Sie zuerst die Polarisation des Lichts der LED. Vermessen Sie nun den Polarisationszustand, der entsteht, wenn das Licht der LED durch die Brille aus dem 3D-Kino tritt. Untersuchen Sie die Propagationsrichtung, wie sie im 3D-Kino vorliegt (d.h. das Licht propagiert von der Leinwand zum Auge) und auch die umgekehrte Propagationsrichtung (d.h. vom Auge zur Leinwand).

6. Nehmen Sie eine P-I-Kennlinie des VCSELs bis $5\,\mathrm{mA}$ auf.

   Zum Wechsel der Lichtquelle wird
   1. zuerst der Strom **vollständig** herunter gedreht (Knopf A),
   2. der Output deaktiviert,
   3. 30 Sekunden gewartet,
   4. die alte Lichtquelle an der Steckverbindung vom Gerät getrennt,
   5. das Limit (Knopf C) neu eingestellt,
   6. die neue Lichtquelle angeschlossen,
   7. der Output aktiviert und
   8. der gewünschte Strom eingestellt (Knopf A).

   Mit angeschlossener Lichtquelle darf auf keinen Fall das Limit verändert oder der Ausschalter betätigt werden!

   Andernfalls führt die entstehende Stromspitze und mit sehr hoher Wahrscheinlichkeit zur sofortigen Zerstörung der Lichtquelle!

   Stellen Sie an der Stromquelle eine Limit von $5{,}3\,\mathrm{mA}$ ein. Messen Sie von $0\,\mathrm{mA}$ bis $1{,}2\,\mathrm{mA}$ in Schritten von $0{,}2\,\mathrm{mA}$. Messen Sie dann bis $2{,}2\,\mathrm{mA}$ weiter in Schritten von $0{,}1\,\mathrm{mA}$. Oberhalb von $2{,}2\,\mathrm{mA}$ bis $5\,\mathrm{mA}$ können Sie die Schritte wieder auf $0{,}2\,\mathrm{mA}$ vergrößern.

   Beachten Sie die Vorschriften zum Laserschutz!

7. Bestimmen Sie den Polarisationszustand des VCSELs unter Nutzung des Stokes-Formalismus in Abhängigkeit des Stroms. Messen Sie die Polarisation bei $0{,}4\,\mathrm{mA}$, $0{,}8\,\mathrm{mA}$ und $1{,}2\,\mathrm{mA}$. Messen Sie im Bereich der Laserschwelle (von $1{,}2\,\mathrm{mA}$ bis $2{,}2\,\mathrm{mA}$) alle $0{,}1\,\mathrm{mA}$ und anschließend noch bei $3\,\mathrm{mA}$, $4\,\mathrm{mA}$ und $5\,\mathrm{mA}$.

3.3. Auswertung

Zur Auswertung aller Aufgaben gehört eine ausführliche Dokumentation des Versuchsaufbaus mit Begründung und dessen Justage und Kalibration. Zu den Ergebnissen ist eine Betrachtung der Messunsicherheiten und eine ausführliche Diskussion der Ergebnisse durchzuführen.

Auswertung der einzelnen Aufgaben:

1: Plotten Sie die Kennlinie der LED und interpretieren sie deren Verlauf.

2: Berechnen Sie das Auslöschungsverhältnis des Linearpolarisators und die Unsicherheit des Ergebnisses.

3: Bestimmen Sie für die untersuchten Wellenlängen den Kalibrierungsfaktor γ, mit dem sich die Skaleneinheiten in die Phasenverschiebung in Radiant umrechnen lassen. Bestimmen Sie für die untersuchten Wellenlängen die Verzögerung der $\lambda/4$-Platte. Mitteln Sie jeweils über die drei vorliegenden Werte und schätzen Sie die Unsicherheit ab.

4: Stellen Sie für die betrachteten Polarisationszustände jeweils die ideale Intensitätsverteilung und die gemessene Verteilung in Polardiagrammen gegenüber und erläutern sie den Intensitätsverlauf. Vergleichen Sie auch die erwarteten Polarisationsellipsen mit den gemessenen Ellipsen, die Sie anhand der ermittelten Stokes-Parameter plotten. Führen Sie eine Fehlerbetrachtung für die Stokes-Parameter durch. Falls Sie für die Stokes-Parameter unrealistische Werte erhalten (z.B. einige Parameter oder der DOP sind größer als Eins), untersuchen Sie, ob ein systematischer Fehler von β_0 die Ursache sein kann.

5: Stellen Sie die gemessenen Intensitäten mit der 3D-Brille in Polardiagrammen dar und berechnen Sie die Stokes-Parameter. Plotten Sie auch die Polarisationsellipsen und interpretieren Sie das Ergebnis. Wie funktioniert die Brille?

6: Plotten und erläutern Sie die Kennlinie der Laserdiode.

7: Berechnen Sie die Stokes-Parameter und die Polarisationsellipse und verwenden Sie dazu einen sinnvollen Wert für die tatsächliche Verzögerung δ der $\lambda/4$-Platte bei der Laserwellenlänge. Stellen Stokes-Parameter, Polarisationsellipsen und Polardiagramme in Abhängigkeit des Pumpstroms dar. Erläutern Sie das Ergebnis.

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A. Richtlinien zum Laserschutz der Abteilung A des F-Praktikums

Wichtige Punkte zum Laserschutz

Der Laserschutz spielt in unseren Labors eine **sehr zentrale** Rolle. Ganz allgemein gilt: **Im Umgang mit Lasern ist der gesunde Menschenverstand nicht zu ersetzen!** Einige spezielle Hinweise werden im folgenden angeführt.

1. Die Laserschutzvorschriften sind **immer** zu beachten.
2. Kopf **niemals** auf Strahlhöhe. Daher **nie** im Sitzen am Lasertisch arbeiten.
3. Richtige Schutzbrille aufsetzen; Wellenlänge und Leistung müssen bei der Wahl berücksichtigt werden. Bitte beim Betreuer oder Laserschutzbeauftragten, oder an den Aushängen an den Labortüren informieren!
4. Achtung: praktisch alle Laser für Laboranwendungen sind mindestens Klasse 3, also von vornherein für die Augen gefährlich, ggf. auch für die Haut – evtl. auch hiefür Schutzmaßnahmen ergreifen
5. Zur Justage kann der Laserstrahl mittels Wandlerkarten sichtbar gemacht werden. Zu beachten ist: Diese halten keine sehr hohen Leistungen aus und besitzen im allgemeinen eine reflektierende Oberfläche. Achtung deshalb vor **Reflektionen**! Auch Kameras besitzen eine **Zerstörschwelle**!
6. Spiegel und sonstige Komponenten nie in den **ungeblockten** Laserstrahl einbauen! Vor Einbau immer überlegen, in welche Richtung der Reflex geht! Diese Richtung zunächst blocken, bevor der Strahl wieder frei gegeben wird.
7. **Nie mit reflektierenden Werkzeugen im Strahlengang hantieren**! Unkontrollierbare Reflexe! Vorsicht ist z.B. auch mit BNC-Kabeln geboten, die in den Strahlengang gelangen könnten!

   Gleiches gilt auch für Uhren und Ringe. Diese vorsichtshalber ausziehen, wenn Sie mit den Händen im Strahlengang arbeiten.
8. Auch Leistungsmessgeräte können Reflexe verursachen! Unbeschichtete Silizium-Fotodioden reflektieren über 30% des Lichtes!
9. Achtung im Umgang mit **Strahlteilerwürfeln**! Diese haben immer einen zweiten Ausgang! Ggf. abblocken!
10. **Warnlampen** bei Betrieb des Lasers anschalten und nach Beendigung der Arbeit wieder ausschalten
11. Dafür sorgen, dass auch Dritte im Labor die richtigen Schutzbrillen tragen, oder sich außerhalb des Laserschutzbereiches befinden
12. Filtergläser in Laserschutzbrillen dürfen ***grundsätzlich nicht*** aus- oder umgebaut werden!!!
13. In besonderem Maße auf Beistehende achten.

Hiermit erkläre ich, dass ich die vorstehenden Punkte gelesen und verstanden habe. Ich bestätige, dass ich eine Einführung in den Umgang mit Lasern sowie eine arbeitsplatzbezogene Unterweisung erhalten habe.

Name: Arbeitsgruppe:

Unterschrift: Datum:

B. Literatur: Edward Collet, Polarized Light

A Historical Note

At the midpoint of the nineteenth century the wave theory of light developed by Augustin Jean Fresnel (1788–1827) and his successors was a complete triumph. The wave theory completely explained the major optical phenomena of interference, diffraction, and polarization. Furthermore, Fresnel had successfully applied the wave theory to the problem of the propagation and polarization of light in anisotropic media, that is, crystals. A further experiment was carried out in 1851 by Armand Hypolite Louis Fizeau (1819–1896), who showed that the speed of light was less in an optically dense medium than in a vacuum, a result predicted by the wave theory. The corpuscular theory, on the other hand, had predicted that in an optically dense medium the speed of light would be greater than in a vacuum. Thus, in practically all respects Fresnel's wave theory of light appeared to be triumphant.

By the year 1852, however, a crisis of quite proportions was slowly simmering in optics. The crisis, ironically, had been brought on by Fresnel himself 35 years earlier. In the year 1817 Fresnel, with the able assistance of his colleague Dominique François Arago (1786–1853), undertook a series of experiments to determine the influence of polarized light on the interference experiments of Thomas Young, (1773–1829). At the beginning of these experiments Fresnel and Arago held the view that light vibrations were longitudinal. At the end of their experiments they were unable to understand their results on the basis of longitudinal vibrations. Arago communicated the puzzling results to Young, who then suggested that the experiments could be understood if the light vibrations were transverse, consisted of only two orthogonal components, and there was no longitudinal component. Indeed, this did make some, but not all, of the results comprehensible. At the conclusion of their experiments Fresnel and Arago summarized their results in a series of statements that have come down to us as the four interference laws of Fresnel and Arago.

All physical experiments are described in terms of verbal statements from which mathematical statements can then be written (e.g., Kepler's laws of planetary motion and Newton's laws of motion). Fresnel understood this very well. Upon completing his experiments, he turned to the problem of developing the mathematical statements for the four interference laws. Fresnel's wave theory was an amplitude description of light and was completely successful in describing completely polarized light, that is, elliptically polarized light and its degenerate states, linearly and circularly polarized light. However, the Fresnel-Arago experiments were carried out not with completely polarized light but with another state of polarized light called unpolarized light. In order to describe the Fresnel–Arago experiments it would be necessary for Fresnel to provide the mathematical statements for unpolarized light, but much to his surprise, on the basis of his amplitude formulation of light, he was unable to write the mathematical statements for unpolarized light! And he never succeeded. With his untimely death in 1827 the task of describing unpolarized light (or for that matter any state of polarized light within the framework of classical optics) along with providing the mathematical statements of the Fresnel–Arago interference laws passed to others. For many years his successors were no more successful than he had been.

By 1852 35 years had elapsed since the enunciation of the Fresnel–Arago laws and there was still no satisfactory description of unpolarized light or the interference laws. It appeared that unpolarized light, as well as so-called partially polarized light, could not be described within the framework of the wave theory of light, which would be a crisis indeed.

The year 1852 is a watershed in optics because in that year Sir George Gabriel Stokes (1819–1903) published two remarkable papers in optics. The first appeared with the very bland title, "On the Composition and Resolution of Streams of Polarized Light from Different Sources," a title that appears to be far removed from the Fresnel–Arago interference laws; the paper itself does not appear to have attracted much attention. It is now, however, considered to be one of the great papers of classical optics. After careful reading of his paper, one discovers that it provides the mathematical formulation for describing any state of polarized light and, most importantly, the mathematical statements for unpolarized light: the mathematical statements for the Fresnel–Arago interference laws could now be written. Stokes had been able to show, finally, that unpolarized light and partially polarized light could be described within the framework of the wave theory of light.

Stokes was successful where all others had failed because he developed a highly novel approach for describing unpolarized and partially polarized light. He abandoned the fruitless attempts of his predecessors to describe unpolarized light in terms of amplitudes and, instead, resorted to an experimental definition of unpolarized light. In other words, he was led to a formulation of polarized light in terms of measured quantities, that is, intensities (observables). This was a completely unique point of view for the nineteenth century. The idea of observables was not to reappear again in physics until the advent of quantum mechanics in 1925 by Werner Heisenberg (1901–1976) and later in optics with the observable formulation of the optical field in 1954 by Emil Wolf (1922–).

Stokes showed that his intensity formulation of the polarized light could be used to describe not only unpolarized and partially polarized light but completely polarized light as well. Thus, his formulation was applicable to any state of polarized light. His entire paper is devoted to describing in all the detail of mid-nineteenth-century algebra the properties of various combinations of polarized and unpolarized light. Near the end of his paper Stokes introduced his discovery that four parameters, now known as the Stokes polarization parameters, could characterize any state of polarized light. Unlike the amplitude formulation of the optical field, his parameters were directly accessible to measurement. Furthermore, he then used these parameters to obtain a correct mathematical statement for unpolarized light. The stage had now been set to write the mathematical statements for the Fresnel–Arago interference laws.

At the end of Stoke's paper he turns, at long last, to his first application, the long awaited mathematical statements for the Fresnel–Arago interference laws. In his paper he states, "Let us now apply the principles and formulae which have just been established to a few examples. And first let us take one of the fundamental experiments by which MM. Arago and Fresnel established the laws of interference of polarized light, *or rather an analogous experiment mentioned by Sir John Herschel*." Thus, with these few words Stokes abandoned his attempts to provide the mathematical statements for the Fresnel–Arago laws. At this point Stokes knew that to apply his formulation to the formulation of the Fresnel–Arago interference laws was a considerable undertaking. It was sufficient for Stokes to know that his mathematical formulation of polarized light would explain them. Within several more pages, primarily devoted to correcting several experiments misunderstood by his colleagues, he concluded his paper.

This sudden termination is remarkable in view of its author's extraordinary effort to develop the mathematical machinery to describe polarized light, culminating in the Stokes polarization parameters. One must ask why he brought his paper to such a rapid conclusion. In my opinion, and this shall require further historical research, the answer lies in the paper that immediately follows Stoke's polarization paper, published only two months later. Its title was, "On the Change of the Refrangibility of Light."

In the beginning of this Historical Note it was pointed out that by 1852 there was a crisis in optics over the inability to find a suitable mathematical description for unpolarized light and the Fresnel–Arago interference laws. This crisis was finally overcome with the publication of Stokes' paper on polarized light in 1852. But this next paper by Stokes dealt with a new problem of very disconcerting proportions. It was the first in a series of papers that would lead, 75 years later, to quantum mechanics. The subject of this second paper is a topic that has become known as the fluorescence of solutions. It is a monumental paper and was published in two parts. The first is a 20-page abstract! The second is the paper itself, which consists of nearlv 150 pages. After reading this paper it is easy to understand why Stokes had concluded his paper on the Fresnel–Arago interference laws. He was deeply immersed in numerous experiments exploring the peculiar phenomenon of fluorescence. After an enormous amount of experimental effort Stokes was able to enunciate his now famous law of fluorescence, namely, that the wavelength of the emitted fluorescent radiation was greater than the excitation wavelength; he also found that the fluorescence radiation appeared to be unpolarized. Stokes was never able to find the reason for this peculiar behavior of fluorescence or the basis of his law. He would spend the next 50 years searching for the reason for his empirical law until his death in 1903. Ironically, in 1905, two years after Stoke's death, a young physicist by the name of Albert Einstein (1879–1955) published a paper entitled "On a Heuristic Point of View Concerning the Generation and Conversion of Light" and showed that Stokes' law of fluorescence could be easily explained and understood on the basis of the quantum hypothesis of Max Planck (1858–1947). It is now clear that Stokes never had the slightest chance of explaining the phenomenon of fluorescence within the framework of classical optics. Thus, having helped to remove one of the last barriers to the acceptance of the wave theory of light, Stoke's investigations on the nature of light had led him to the discovery of the first law ever associated with the quantum phenomenon. Unknowingly, Stokes had stumbled onto the quantum nature of light. Thirty-five years later, in 1888, a similar chain of events was repeated when Heinrich Hertz (1857–1894), while verifying the electromagnetic field theory of James Clerk Maxwell (1831–1879), the ultimate proof of the truth of the classical wave theory of light, also discovered a new and unexplainable phenomenon, the photoelectric effect. We now know that this too can be understood only in terms of the quantum theory. Science is filled with ironies.

Within two months of the publication in March 1852 of his paper on polarized light, in which the formulation of classical optics appeared to be complete, with the May 1852 publication of his paper on fluorescence, Stokes went from complete triumph to complete dismay. He would constantly return to the subject of fluorescence for the remainder of his life, always trying but never succeeding in understanding the origin of his law of fluorescence.

Stoke's great paper on polarization was practically forgotten because by the mid-nineteenth century classical optics was believed to be complete and physicists had turned their attention to the investigation of the electromagnetic field and the statistical mechanics of molecules. His paper was buried in the scientific literature for nearly a century. Its importance was finally recognized with its "discovery" in the 1940s by the Nobel laureate Subrahmanya Chandrasekhar (1910–) who used the Stokes parameters to include the effects of polarized light in the equations of radiative transfer.

In this book we shall see that the Stokes polarization parameters provide a rich and powerful tool for investigating and understanding polarized light and its interaction with matter. The use of these parameters provides a mathematical formulation of polarized light whose power is far greater than was ever imagined by their originator and serves as a tribute to his genius.

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where

$$A = a\cos\delta \qquad B = a\sin\delta \tag{40}$$

Another form for (37) is to express $\cos\omega_0 t$ and $\sin\omega_0 t$ in terms of exponents; that is,

$$\cos\omega_0 t = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \tag{41a}$$

$$\sin\omega_0 t = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \tag{41b}$$

Substituting (41a) and (41b) into (39) and grouping terms leads to

$$x(t) = Ce^{i\omega_0 t} + De^{-i\omega_0 t} \tag{42a}$$

where

$$C = \frac{A - iB}{2} \qquad D = \frac{A + iB}{2} \tag{42b}$$

and C and D are complex constants. Thus, we see that the solution of the harmonic oscillator can be written in terms of purely real quantities or complex quantities.

The form of (35a) is of particular interest. The differential equation (33a) clearly describes the amplitude motion of the harmonic oscillator. Let us retain the original form of (33a) and multiply through by $dx/dt = v$, so we can write

$$mv\frac{dv}{dt} = -kx\frac{dx}{dt} \tag{43}$$

We now integrate both sides of (43), and we are led to

$$\frac{mv^2}{2} = \frac{-kx^2}{2} + C \tag{44}$$

where C is a constant of integration. Thus, by merely carrying out a formal integration we are led to a new form for describing the motion of the harmonic oscillator. At the beginning of the eighteenth century the meaning of (44) was not clear. Only slowly did physicists come to realize that (44) describes the motion of the harmonic oscillator in a completely new way, namely the description of motion in terms of energy. The terms $mv^2/2$ and $-kx^2/2$ correspond to the kinetic energy and the potential energy for the harmonic oscillator, respectively. Thus, early on in the development of physics a connection was made between the amplitude and energy for oscillatory motion. The energy of the wave could be obtained by merely squaring the amplitude. This point is introduced because of its bearing on Young's interference experiment, specifically, and on optics, generally. The fact that a relation exists between the amplitude of the harmonic oscillator and its energy was taken directly over from mechanics into optics and was critical for Young's interference experiment. In optics, however, the energy would become known as the intensity.

2.2.5. A Note on the Equation of a Plane

The equation of a plane was stated in (11) to be

$$\mathbf{s}\cdot\mathbf{r} = \text{constant} \tag{11}$$

We can show that (11) does indeed describe a plane by referring to Figure 2. Inspecting the figure, we see that $\mathbf{r}$ is a vector with its origin at the origin of the coordinates, so,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{45}$$

and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors. Similarly, from Figure 2 we see that

$$\mathbf{s} = s_x\mathbf{i} + s_y\mathbf{j} + s_z\mathbf{k} \tag{46}$$

Suppose we now have a vector $\mathbf{r}_0$ along $\mathbf{s}$ and the plane is perpendicular to $\mathbf{s}$. Then $\overline{OP}$ is the vector $\mathbf{r} - \mathbf{r}_0$ and is perpendicular to $\mathbf{s}$. Hence, the equation of the plane is

$$\mathbf{s}\cdot(\mathbf{r} - \mathbf{r}_0) = 0 \tag{47}$$

or

$$\mathbf{s}\cdot\mathbf{r} = \zeta \tag{48}$$

where $\zeta = \mathbf{s}\cdot\mathbf{r}_0$ is a constant. Thus, the name *plane-wave solutions* arises from the fact that the wave front is characterized by a plane of infinite extent.

2.3. YOUNG'S INTERFERENCE EXPERIMENT

In the previous section we saw that the developments in mechanics in the eighteenth century led to the mathematical formulation of the wave equation and the concept of energy.

Around the year 1800, Thomas Young performed a simple, but remarkable, optical experiment known as the two-pinhole interference experiment. He showed that this experiment could be understood in terms of waves; the experiment gave the first clear-cut support for the wave theory of light. In order to understand the pattern which he observed, he adopted the ideas developed in mechanics and applied them to optics, an extremely novel and radical approach. Until the advent of Young's work, very little progress had been made in optics since the researches of Newton (the corpuscular theory of light) and Huygens (the wave theory of light). The simple fact was that by the year 1800, aside from Snell's law of refraction and the few things learned about polarization, there was no theoretical basis on which to proceed. Young's work provided the first critical step in the development and acceptance of the wave theory of light.

The experiment carried out by Young is shown in Figure 3. A source of light, σ, is placed behind two pinholes s_1 and s_2, which are equidistant from σ. The pinholes then act as secondary monochromatic sources which are in phase, and the beams from them are superposed on the screen Σ at an arbitrary point P. Remarkably, when the screen is then observed, one does not see a

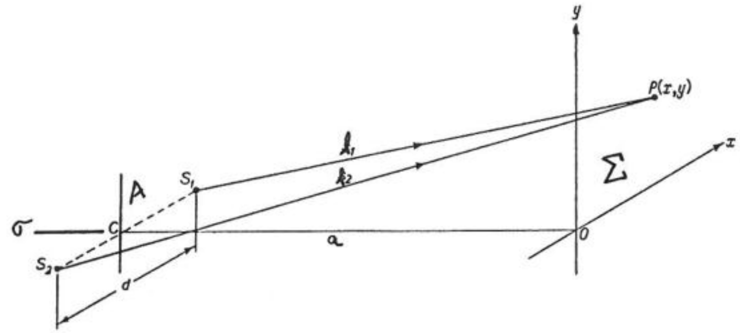

Figure 3 Young's interference experiment.

uniform distribution of light. Instead, a distinct pattern consisting of bright bands alternating with dark bands is observed. In order to explain this behavior, Young assumed that each of the pinholes, s_1 and s_2, emitted waves of the form

$$u_1 = u_{01}\sin(\omega t - kl_1) \tag{49a}$$

$$u_2 = u_{02}\sin(\omega t - kl_2) \tag{49b}$$

At the source plane A, l_1 and l_2 are zero. The pattern is observed on the plane Oxy normal to the perpendicular bisector of $\overline{s_1s_2}$ and with the x axis parallel to $\overline{s_1s_2}$. The separation of the pinholes is d, and a is the distance between the line joining the pinholes and the plane of observation Σ. For the point $P(x,y)$ on the screen, Figure 3 shows that

$$l_1^2 = a^2 + y^2 + \left(x - \frac{d}{2}\right)^2 \tag{50a}$$

$$l_2^2 = a^2 + y^2 + \left(x + \frac{d}{s}\right)^2 \tag{50b}$$

Thus,

$$l_2^2 - l_1^2 = 2xd \tag{51}$$

Equation (51) can be written as

$$(l_2 - l_1)(l_1 + l_2) = 2xd \tag{52}$$

Now if x and y are small compared to a, then $l_1 + l_2 \simeq 2a$. Thus,

$$l_2 - l_1 = \Delta l = \frac{xd}{a} \tag{53}$$

At this point we now return to the wave theory. The secondary sources s_1 and s_2 are assumed to be equal, so $u_{01} = u_{02} = u_0$. In addition, the assumption

is made that the optical disturbances u_1 and u_2 can be superposed at $P(x,y)$ (the principle of coherent superposition), so

$$\begin{aligned} u(t) &= u_1 + u_2 \\ &= u_0\,[\sin(\omega t - kl_2) + \sin(\omega t - kl_2)] \end{aligned} \tag{54}$$

A serious problem now arises. While (54) certainly describes an interference behavior, the parameter of time enters in the term ωt. In the experiment the observed pattern does not vary over time, so the time factor cannot enter the final result. This suggests that we average the amplitude $u(t)$ over the time of observation T. The time average of $u(t)$ written as $\langle u(t)\rangle$, is then defined to be

$$\langle u(t)\rangle = \lim_{T\to\infty} \frac{\int_0^T u(t)\,dt}{\int_0^T dt} \tag{55a}$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)\,dt \tag{55b}$$

Substituting (54) into (55) yields

$$\langle u(t)\rangle = \lim_{T\to\infty} \int_0^T [\sin(\omega t - kl_1) + \sin(\omega t - kl_2)]\,dt \tag{56}$$

Using the trigonometric identity

$$\sin(\omega t - kl) = \sin(\omega t)\cos(kl) - \cos(\omega t)\sin(kl) \tag{57}$$

and averaging over one cycle (56) yields

$$\langle u(t)\rangle = 0 \tag{58}$$

This is not observed. That is, the time average of the amplitude is calculated to be zero, but observation shows that the pattern exhibits nonzero intensities. At this point we must abandon the idea that the interference phenomenon can be explained only in terms of amplitudes $u(t)$. Another idea must now be borrowed from mechanics. Namely, the optical disturbance must be described in terms of squared quantities, analogous to energy, $u^2(t)$. But this, too, contains a time factor. Again, a time average is introduced, and a new quantity, I, called the intensity in optics, is defined:

$$I = \langle u^2(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T u^2(t)\,dt \tag{59}$$

Substituting $u^2(t) = (u_0\sin(\omega t - kl))^2$ into (59) and averaging over one cycle yields

$$\begin{aligned} I = \langle u^2(t)\rangle &= \lim_{T\to\infty} \frac{1}{T}\int_0^T u_0^2 \sin^2(\omega t - kl)\,dt \\ &= \frac{u_0^2}{2} = I_0 \end{aligned} \tag{60}$$

Thus, the intensity is constant over time; this behavior is observed.

The time average of $u^2(t)$ is now applied to the superposed amplitudes (54). Squaring $u^2(t)$ yields

$$u^2(t) = u_0^2[\sin^2(\omega t - kl_1) + \sin^2(\omega t - kl_2) + 2\sin(\omega t - kl_1)\sin(\omega t - kl_2)] \tag{61}$$

The last term is called the interference. Equation (13) can be rewritten with help of the well-known trigonometric identity

$$2\sin(\omega t - kl_1)\sin(\omega t - kl_2) = \cos(k[l_2 - l_1]) - \cos(2\omega t - k[l_1 + l_2]) \tag{62}$$

Thus, (61) can be written as

$$u^2(t) = u_0^2[\sin^2(\omega t - kl_1) + \sin^2(\omega t - kl_2) + \cos(k[l_2 - l_1]) - \cos(2\omega t - k[l_1 + l_2])] \tag{63}$$

Substituting (63) into (59), we obtain the intensity on the screen to be

$$I = \langle u^2(t)\rangle = 2I_0[1 + \cos k(l_2 - l_1)] \tag{64a}$$

or

$$I = 4I_0 \cos^2\left[\frac{k(l_2 - l_1)}{2}\right] \tag{64b}$$

where, from (53),

$$l_2 - l_1 = \Delta l = \frac{xd}{a} \tag{53}$$

Equation (64b) is Young's famous interference formula. We note that from (60) we would expect the intensity from a single source to be $u_0^2/2 = I_0$, so the intensity from two independent optical sources would be $2I$. Equation (64a) (or (64b)) shows a remarkable result. Namely, when the intensity is observed from a single source in which the beam is divided, the observed intensity varies between 0 and $4I_0$; the intensity can be double or even zero from that expected from two independent optical sources! We see from (64b) that there will be maximum intensities $(4I_0)$ at

$$x = \frac{a\lambda n}{d} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{65a}$$

and minimum intensities (null) at

$$x = \frac{a\lambda}{d}\left(\frac{2n+1}{2}\right) \qquad n = 0, \pm 1, \pm 2, \ldots \tag{65b}$$

Thus, in the vicinity of O on the plane Σ an interference pattern consisting of bright and dark bands are aligned parallel to the OY axis (at right angles to the line $\overline{s_1 s_2}$ joining the two sources).

Young's experiment is of great importance because it was the first step in establishing the wave theory of light and was the first theory to provide an explanation of the observed interference pattern. It also provides a method, albeit one of low precision, of measuring the wave length of light by measuring d, a and the fringe spacing according to (65a) or (65b). The separation Δx between the central bright line and the first bright line is, from (65a),

$$\Delta x = x_1 - x_0 = \frac{a\lambda}{d} \tag{66}$$

The expected separation on the observing screen can be found by assuming the following values:

$$a = 100 \text{ cm} \qquad d = 0.1 \text{ cm}$$
$$\lambda = 5 \times 10^{-5} \text{ cm} \qquad \Delta x = 0.05 \text{ cm} = 0.5 \text{ mm} \tag{67}$$

The resolution of the human eye at a distance of 25 cm is, approximately, of the same order of magnitude, so the fringes can be observed with the naked eye.

Young's interference gave the first real support for the wave theory. However, aside from the important optical concepts introduced here to explain the interference pattern, there is another reason for discussing Young's interference experiment. Around 1818, Fresnel and Arago repeated his experiments with polarized light to determine the effects, if any, of polarized light on the interference phenomenon. The results were surprising to understand in their entirety. To explain these experiments it was necessary to understand the nature and properties of polarized light. Before we turn to the subject of polarized light, however, we discuss another topic of importance, namely, the reflection and transmission of a wave at an interface separating two different media.

2.4. REFLECTION AND TRANSMISSION OF A WAVE AT AN INTERFACE

The wave theory and the wave equation allow us to treat an important problem, namely, the reflection and transmission of wave at an interface between two different media. Specifically, in optics, light is found to be partially reflected and partially transmitted at the boundary of two media characterized by different refractive indices. The treatment of this problem was first carried out in mechanics, however, and shows how the science of mechanics paved the way for the introduction of the wave equation into optics.

Two media can be characterized by their ability to support two different velocities v_1 and v_2. In Figure 4 we show an incident wave coming from the left which is partially transmitted and reflected at the interface (boundary).

We saw earlier that the solution of the wave equation in complex form is

$$u(x) = Ae^{-ikx} + Be^{+ikx} \tag{68}$$

where $k = \omega/v$. The time factor $\exp(i\omega t)$ has been suppressed. The term Ae^{-ikx} describes propagation to the right, and the term Be^{+ikx} describes propagation to the left. The fields to the left and right of the interface (boundary) can be

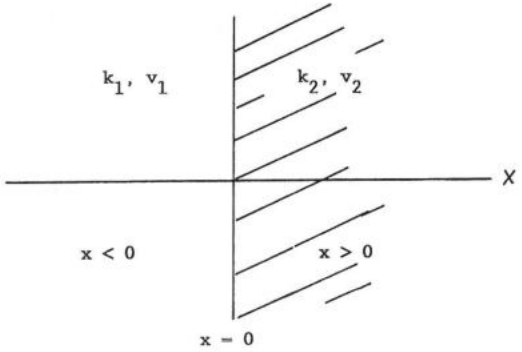

Figure 4 Reflection and transmission of a wave at the interface between two media.

described by a superposition of waves propagating to the right and left, that is,

$$u_1(x) = Ae^{-ik_1x} + Be^{+ik_1x} \qquad x < 0 \tag{69a}$$

$$u_2(x) = Ce^{-ik_2x} + De^{+ik_2x} \qquad x > 0 \tag{69b}$$

where $k_1 = \omega/v_1$ and $k_2 = \omega/v_2$.

We must now evaluate A, B, C, and D. To do this, we assume that at the interface the fields are continuous—that is,

$$u_1(x)\,|_{x=0} = u_2(x)\,|_{x=0} \tag{70}$$

and that the slopes of $u_1(x)$ and $u_2(x)$ are continuous at the interface—that is, the derivatives of $u_1(x)$ and $u_2(x)$, so

$$\left.\frac{\partial u_1(x)}{\partial x}\right|_{x=0} = \left.\frac{\partial u_2(x)}{\partial x}\right|_{x=0} \tag{71}$$

We also assume that there is no source of waves in the medium to the right of the interface. This means that the wave which appears to the left of the interface is due only to reflection of the incident wave. This requires that we set $D = 0$ in (69b).

Applying these conditions to (69a) and (69b) we easily find

$$A + B = C \tag{72a}$$

$$k_1A - k_1B = k_2C \tag{72b}$$

We solve for B and C in terms of the amplitude of the incident wave, A, and find

$$B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)A \tag{73a}$$

$$C = \left(\frac{2k_1}{k_1 + k_2}\right)A \tag{73b}$$

The B term is associated with the reflected wave in (69a). If $k_1 = k_2$, then (73a) and (73b) show that $B = 0$ and $C = A$; that is, there is no reflected wave, and we have complete transmission as expected.

We can write (69a) as the sum of an incident wave $u_i(x)$ and a reflected wave $u_r(x)$:

$$u_1(x) = u_i(x) + u_r(x) \tag{74a}$$

and we can write (69b) as a transmitted wave:

$$u_2(x) = u_t(x) \tag{74b}$$

The energies corresponding to $u_i(x)$, $u_r(x)$, and $u_t(x)$ are then the squares of these quantities. We can use complex quantities to bypass the formal time-averaging procedure and define the energies of these waves to be

$$\mathcal{E}_i = u_i(x)u_i^*(x) \tag{75a}$$

$$\mathcal{E}_r = u_r(x)u_r^*(x) \tag{75b}$$

$$\mathcal{E}_t = u_t(x)u_t^*(x) \tag{75c}$$

The principle of conservation of energy requires that

$$\mathcal{E}_i = \mathcal{E}_r + \mathcal{E}_t \tag{76}$$

The fields $u_i(x)$, $u_r(x)$, and $u_t(x)$ from (69a) and (69b) are

$$u_i(x) = Ae^{-ik_1x} \tag{77a}$$

$$u_r(x) = Be^{+k_1x} \tag{77b}$$

$$u_t(x) = Ce^{-ik_2x} \tag{77c}$$

The energies corresponding to (77) are then substituted in (76), and we find

$$A^2 = B^2 + C^2 \tag{78a}$$

or

$$\left(\frac{B}{A}\right)^2 + \left(\frac{C}{A}\right)^2 = 1 \tag{78b}$$

The quantities $(B/A)^2$ and $(C/A)^2$ are the normalized reflection and transmission coefficients, which we write as R and T, respectively. Thus (78b) becomes

$$R + T = 1 \tag{79a}$$

where

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \tag{79b}$$

$$T = \left(\frac{2k_1}{k_1 + k_2}\right)^2 \tag{79c}$$

from (73a) and (73b). Equation (79b) and (79c) can be seen to satisfy the conservation condition (79a).

The coefficients B and C show an interesting behavior, which is as follows. From (73a) and (73b) we write

$$\frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1} \tag{80a}$$

$$\frac{C}{A} = \frac{2}{1 + k_2/k_1} \tag{80b}$$

where

$$\frac{k_2}{k_1} = \frac{\omega/v_2}{\omega/v_1} = \frac{v_1}{v_2} \tag{80c}$$

Now if $v_2 = 0$, that is, there is no propagation in the second medium, (80c) becomes

$$\lim_{v_2 \to 0} \frac{k_2}{k_1} = \frac{v_1}{v_2} = \infty \tag{81}$$

With this limiting value, (81), we see that (80a) and (80b) become

$$\frac{B}{A} = -1 = e^{i\pi} \tag{82a}$$

$$\frac{C}{A} = 0 \tag{82b}$$

Equation (82a) shows that there is a 180° (π rad) phase reversal upon reflection. Thus, the reflected wave is completely out of phase with the incident wave, and we have total cancellation. This behavior is described by the term *standing waves*. We now derive the equation which specifically shows that the resultant wave does not propagate.

The field to the left of the interface is given by (69a) and is

$$u_1(x,t) = e^{i\omega t}(Ae^{-ik_1x} + Be^{ik_1x}) \qquad x < 0 \tag{83}$$

where we have reintroduced the (suppressed) time factor $\exp(i\omega t)$. From (82a) we can then write

$$u_1(x,t) = Ae^{i\omega t}(e^{-ik_1x} - e^{ik_1x}) \tag{84a}$$

$$= Ae^{i(\omega t - k_1x)} - Ae^{i(\omega t + k_1x)} \tag{84b}$$

$$= u_-(x,t) - u_+(x,t) \tag{84c}$$

where

$$u_-(x,t) = Ae^{i(\omega t - k_1x)} \tag{84d}$$

$$u_+(x,t) = Ae^{i(\omega t + k_1x)} \tag{84e}$$

The phase velocity v_p of a wave can be defined in terms of amplitude as

$$v_p = -\frac{(\partial u/\partial t)}{(\partial u/\partial x)} \tag{85}$$

Applying (85) to (84d) and (84e), respectively, we find that

$$v_p(-) = \frac{\omega}{k} \tag{86a}$$

$$v_p(+) = \frac{-\omega}{k} \tag{86b}$$

so the total velocity of the wave is

$$v = v_p(-) + v_p(+) = 0 \tag{87}$$

Thus, the resultant velocity of the wave is zero according to (87); that is, the wave does not propagate and it appears to be standing in place. The equation for the standing wave is given by (84a), which is written as

$$u_1(x,t) = 2Ae^{i\omega t}\sin(k_1x) \tag{88}$$

It is customary to take the real part of (88)

$$u(x,t) = 2A\cos(\omega t)\sin(kx) \tag{89}$$

where we have dropped the subscript 1. We see that there is no propagator $\omega t - kx$, so (89) does not describe propagation. We emphasize that the argument or term $\omega t - kx$ describes the propagation of a wave. This is easily seen by setting

$$\omega t - kx = \text{constant} \tag{90}$$

Equation (90) is set to a constant value because the velocity of the wave is determined by a point on the wave which moves with a constant velocity, so the amplitude at this point is also constant. We now differentiate (90),

$$\omega\, dt - k\, dx = 0 \tag{91}$$

so

$$v_p = \frac{dx}{dt} = \frac{\omega}{k} \tag{92}$$

and the phase velocity v_p is constant as required.

Thus, we see that the wave equation and wave theory lead to a correct description of the transmission and reflection of a wave at a boundary. While this behavior was first studied in mechanics in the eighteenth century, it was applied with equal success to optics in the following century. It appears that this was first done by Fresnel, who derived the equations for reflection and transmission at an interface between two media characterized by refractive indices n_1 and n_2. We shall not derive Fresnel's equations but merely refer the reader to the references at the end of this chapter for their derivation. We shall use Fresnel's equations later, however, when we discuss the change in polarization of light at an optical interface.

With this material on the wave equation behind us, we can now turn to the study of one of the most interesting properties of light, its polarization.

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3
The Polarization Ellipse

3.1. INTRODUCTION

Christian Huygens was the first to suggest that light was not a scalar quantity based on his work on the propagation of light through crystals; it appeared that light had "sides" in the words of Newton. This vectorial nature of light is called *polarization*. If we follow mechanics and equate an optical medium to an isotropic elastic medium, it should be capable of supporting three independent oscillations (optical disturbances): $u_x(r,t)$, $u_y,(r,t)$, and $u_z(r,t)$. Correspondingly, three independent wave equations are then required to describe the propagation of the optical disturbance, namely,

$$\nabla^2 u_i(\mathbf{r},t) = \frac{1}{v^2}\frac{\partial^2 u_i(\mathbf{r},t)}{\partial t^2} \qquad i = x,y,z \tag{1}$$

where v is the velocity of propagation of the oscillation and $\mathbf{r} = \mathbf{r}(x,y,z)$. In a Cartesian system the components $u_x(\mathbf{r},t)$ and $u_y(\mathbf{r},t)$ are said to be the transverse components, and the component $u_z(\mathbf{r},t)$ is said to be the longitudinal component. Thus, according to (1) the optical field components should be

$$u_x(\mathbf{r},t) = u_{0x}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_x) \tag{2a}$$

$$u_y(\mathbf{r},t) = u_{0y}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_y) \tag{2b}$$

$$u_z(\mathbf{r},t) = u_{0z}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_z) \tag{2c}$$

In 1818 Fresnel and Arago carried out a series of fundamental investigations on Young's interference experiment using polarized light. After a considerable amount of experimentation they were forced to conclude that the longitudinal component (2c) did not exist. That is, light consisted only of the transverse components (2a) and (2b). If we take the direction of propagation to be in the z direction, then the optical field in free space must be described only by

$$u_x(z,t) = u_{0x} \cos(\omega t - \mathbf{k}z + \delta_x) \tag{3a}$$

$$u_y(z,t) = u_{0y} \cos(\omega t - \mathbf{k}z + \delta_y) \tag{3b}$$

where u_{0x} and u_{0y} are the maximum amplitudes and δ_x and δ_y are arbitrary phases. There is no reason, a priori, for the existence of only transverse components on the basis of an elastic medium (the "ether" in optics). It was considered to be a defect in Fresnel's theory. Nevertheless, in spite of this Eqs. (3a) and (3b) were found to satisfactorily describe the phenomenon of interference using polarized light.

The "defect" in Fresnel's theory was overcome by the development of a new theory, which we now call Maxwell's electrodynamic theory and his equations. One of the immediate results of solving his equations was that in free space only transverse components arose; there was no longitudinal component. This was one of the first triumphs of Maxwell's theory. Nevertheless, Maxwell's theory took nearly 40 years to be accepted in optics due, in large part, to the fact that up to the end of the nineteenth century it led to practically nothing that could not be explained or understood by Fresnel's theory.

Equations (3a) and (3b) are spoken of as the polarized or polarization components of the optical field. In this chapter we consider the consequences of these equations. The results are very interesting and lead to a surprising number of revelations about the nature of light.

3.2. THE INSTANTANEOUS OPTICAL FIELD AND THE POLARIZATION ELLIPSE

In previous sections we pointed out that the experiments of Fresnel and Arago led to the discovery that light consisted only of two transverse components. The components were perpendicular to each other and could be chosen for convenience to be propagating in the z direction. The waves are said to be "instantaneous" in the sense that the time duration for the wave to go through one complete cycle is only 10^{-15} sec at optical frequencies. In this chapter we find the equation which arises when the propagator is eliminated between the transverse components. In order to do this we show in Figure 1 the transverse optical field propagating in the z direction.

The transverse components are represented by

$$E_x(z,t) = E_{0x} \cos(\tau + \delta_x) \tag{4a}$$

$$E_y(z,t) = E_{0y} \cos(\tau + \delta_y) \tag{4b}$$

where $\tau = \omega t - \kappa z$ is the propagator. The subscripts x and y refer to the components in the x and y directions, E_{0x} and E_{0y} are the maximum amplitudes, and δ_x and δ_y are the phases, respectively. As the field propagates, $E_x(z,t)$ and $E_y(z,t)$ give rise to a resultant vector. This vector describes a locus of points in space,

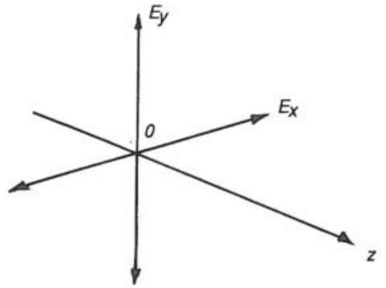

Figure 1 Propagation of the transverse optical field.

and the curve generated by those points will now be derived. In order to do this (4a) and (4b) are written as

$$\frac{E_x}{E_{0x}} = \cos\tau \cos\delta_x - \sin\tau \sin\delta_x \tag{5a}$$

$$\frac{E_y}{E_{0y}} = \cos\tau \cos\delta_y - \sin\tau \sin\delta_y \tag{5b}$$

Hence,

$$\frac{E_x}{E_{0x}} \sin\delta_y - \frac{E_y}{E_{0y}} \sin\delta_x = \cos\tau \sin(\delta_y - \delta_x) \tag{6a}$$

$$\frac{E_x}{E_{0x}} \cos\delta_y - \frac{E_y}{E_{0y}} \cos\delta_x = \sin\tau \sin(\delta_y - \delta_x) \tag{6b}$$

Squaring (6a) and (6b) and adding gives

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}} \cos\delta = \sin^2\delta \tag{7a}$$

where

$$\delta = \delta_y - \delta_x \tag{7b}$$

Equation (7a) is recognized as the equation of an ellipse and shows that at any instant of time the locus of points described by the optical field as it propagates is an ellipse. This behavior is spoken of as *optical polarization*, and (7a) is called the *polarization ellipse*. In Figure 2 the ellipse is shown inscribed within a rectangle whose sides are parallel to the coordinate axes and whose lengths are $2E_{0x}$ and $2E_{0y}$.

We now determine the points where the ellipse is tangent to the sides of the rectangle. For convenience we write E_x and E_y simply as x and y and E_{0x} and E_{0y} as a and b, respectively. We then write (7a) as

$$a^2y^2 - (2abx\cos\delta)y + b^2(x^2 - a^2\sin^2\delta) = 0 \tag{8}$$

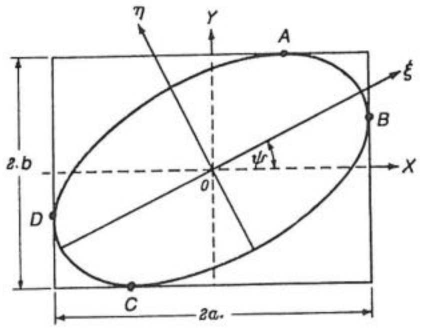

Figure 2 An elliptically polarized wave and the polarization ellipse.

The solution of this quadratic equation (8) is

$$y = \frac{bx\cos\delta}{a} \pm \frac{b\sin\delta}{a}(a^2 - x^2)^{1/2} \tag{9}$$

At the top and bottom of the ellipse where it is tangent to the rectangle the slope is $dy/dx = y' = 0$. We now differentiate (9), set $y' = dy/dx = 0$, and find that

$$x = \pm a\cos\delta \tag{10a}$$

Substituting (10a) into (9), the corresponding values of y are found to be

$$y = \pm b \tag{10b}$$

Similarly, by considering (9) where the slope is $y' = \infty$ on the sides of the rectangle, the tangent points are

$$x = \pm a \tag{11a}$$

$$y = \pm b\cos\delta \tag{11b}$$

Equations (10) and (11) show that the maximum length of the sides of the ellipse are $x = \pm a$ and $y = \pm b$. The ellipse is tangent to the sides of the rectangle at $(\pm a, \pm b\cos\delta)$ and $(\pm a\cos\delta, \pm b)$, or in terms of E_{0x} and E_{0y}, $(\pm E_{0x}, \pm E_{0y}\cos\delta)$ and $(\pm E_{0x}\cos\delta, \pm E_{0y})$, respectively. We also see that (10) and (11) show that the maximum value of x and y are $\pm a$ and $\pm b$ or E_{0x} and E_{0y}, respectively.

In Figure 2 the ellipse is shown touching the rectangle at points A, B, C, and D, the coordinates of which are

$$A: \quad +E_{0x}\cos\delta,\ +E_{0y} \tag{12a}$$

$$B: \quad +E_{0x},\ +E_{0y}\cos\delta \tag{12b}$$

$$C: \quad -E_{0x}\cos\delta,\ -E_{0y} \tag{12c}$$

$$D: \quad -E_{0x},\ -E_{0y}\cos\delta \tag{12d}$$

The presence of the "cross term" in (7a) shows that the polarization ellipse is, in general, rotated, and this behavior is shown in Figure 2 where the ellipse is shown rotated through an angle ψ. More will be said about this later.

It is also of interest to determine the maximum and minimum areas of the polarization ellipse which can be inscribed within the rectangle. We see that along the x axis the ellipse is tangent at the extrema $x = -a$ and $x = +a$. The area of the ellipse above the x axis is given by

$$A = \int_{-a}^{+a} y\,dx \tag{13}$$

Substituting (9) into (13) and evaluating the integrals, we find that the area of the polarization ellipse is

$$A = \pi ab\sin\delta \tag{14}$$

or, in terms of the original parameters,

$$A = \pi E_{0x}E_{0y}\sin\delta \tag{15}$$

Thus, the area of the polarization ellipse depends on the lengths of the major and minor axes, E_{0x} and E_{0y}, and the phase shift δ between the orthogonal transverse components. We see that for $\delta = \pi/2$ the area is $\pi E_{0x}E_{0y}$, whereas for $\delta = 0$ the area is zero. The significance of these results will soon become apparent.

In general, completely polarized light is elliptically polarized. However, there are certain degenerate forms of the polarization ellipse which are continually encountered in the study of polarized light. Because of the importance of these special degenerate forms we now discuss them as special cases in the following chapter. These are the cases where either E_{0x} or E_{0y} is zero or equal and/or where $\delta = 0$, $\pi/2$, or π radians.

3.3. SPECIALIZED (DEGENERATE) FORMS OF THE POLARIZATION ELLIPSE

The polarization ellipse (7a) degenerates to special forms for different values of E_{0x}, E_{0y}, and δ. We now consider these special forms.

1. $E_{0y} = 0$. In this case $E_y(z,t)$ is zero so we must refer to (4). We then have

$$E_x(z,t) = E_{0x}\cos(\tau + \delta_x) \tag{16a}$$

$$E_y(z,t) = 0 \tag{16b}$$

In this case there is an oscillation only in the x direction. The light is then said to be linearly polarized in the x direction, and we call this *linear horizontally polarized light*. Similarly, if $E_{0x} = 0$ and $E_y(z,t) \neq 0$, then we have a linear oscillation along the y axis, and we speak of *linear vertically polarized light*.

2. $\delta = 0$ or π. Equation (7a) reduces to

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} \pm 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}} = 0 \tag{17}$$

Equation (17) can be written as

$$\left(\frac{E_x}{E_{0x}} \pm \frac{E_y}{E_{0y}}\right)^2 = 0 \tag{18}$$

whence

$$E_y = \pm\left(\frac{E_{0y}}{E_{0x}}\right)E_x \tag{19}$$

Equation 19 is recognized as the equation of a straight line with slope $\pm(E_{0y}/E_{0x})$ and zero intercept. Thus, we say that we have linearly polarized light with slope $\pm(E_{0y}/E_{0x})$. The value $\delta = 0$ yields a negative slope, and the value $\delta = \pi$ a positive slope. If $E_{0x} = E_{0y}$, then we see that

$$E_y = \pm E_x \tag{20}$$

The positive value is said to represent *linear +45° polarized light*, and the negative values *is* said to represent *linear −45° polarized light*.

3. $\delta = \pi/2$ or $3\pi/2$. The polarization ellipse reduces to

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} = 1 \tag{21}$$

This is the standard equation of an ellipse. Note that $\delta = \pi/2$ or $\delta = 3\pi/2$ yields the identical polarization ellipse.

4. $E_{0x} = E_{0y} = E_0$ and $\delta = \pi/2$ or $\delta = 3\pi/2$. The polarization ellipse now reduces to

$$\frac{E_x^2}{E_0^2} + \frac{E_y^2}{E_0^2} = 1 \tag{22}$$

Equation (22) describes the equation of a circle. Thus, for this condition the light is said to be right or left circularly polarized ($\delta = \pi/2$ and $3\pi/2$, respectively). Again, we note that (22) shows that it alone cannot determine if the value of δ is $\pi/2$ or $3\pi/2$.

Finally, in the previous chapter we showed that the area of the polarization ellipse was

$$A = \pi E_{0x}E_{0y}\sin\delta \tag{23}$$

We see that for $\delta = 0$ or π the area of the polarization ellipse is zero, which is to be expected for linearly polarized light. For $\delta = \pi/2$ or $3\pi/2$ the area of the ellipse is a maximum; that is, $\pi E_{0x}E_{0y}$. It is important to note that even if the phase shift between the orthogonal components is $\pi/2$ or $3\pi/2$, the light is, in general, elliptically polarized. Furthermore, the polarization ellipse shows that it is in the standard form as given by (21).

For the more restrictive condition where the orthogonal amplitudes are equal so that $E_{0x} = E_{0y} = E_0$ and therefore we have a circle (23) becomes

$$A = \pi E_0^2 \tag{24}$$

which is, of course, the area of a circle.

The previous special forms of the polarization ellipse are spoken of as being degenerate forms of the polarization ellipse. We can summarize these results by saying that the degenerate states of the polarization ellipse are (1) linearly horizontal or vertically polarized light, (2) linear +45° or −45° polarized light, and (3) right or left circularly polarized light.

Aside from the fact that these degenerate states appear quite naturally as special cases of the polarization ellipse, there is a fundamental reason for their importance: they are relatively easy to create in an optical laboratory and can be used to create "null-intensity" conditions. Polarization measurements, which are based on null-intensity conditions enable very accurate measurements to be made.

3.4. THE ELLIPTICAL PARAMETERS OF THE POLARIZATION ELLIPSE

The polarization ellipse has the form

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - \frac{2E_xE_y\cos\delta}{E_{0x}E_{0y}} = \sin^2\delta \tag{25}$$

where $\delta = \delta_x - \delta_y$. In general, the axes of the ellipse are not in the Ox and Oy directions. In (25) the presence of the "product" term E_xE_y shows that it is actually a rotated ellipse; in the standard form of an ellipse the product term is not present. In this section we find the mathematical relations between the parameters of the polarization ellipse, E_{0x}, E_{0y}, and δ and the angle of rotation ψ, and another important parameter, χ, the ellipticity angle.

In Figure 3 we show the rotated ellipse. Let Ox and Oy be the initial, unrotated, axes, and let Ox' and Oy' be a new set of axes along the rotated ellipse. Furthermore, let $\psi\,(0 \le \psi \le \pi)$ be the angle between Ox and the direction Ox' of the major axis.

The components E_x' and E_y' are related by

$$E_x' = E_x\cos\psi + E_y\sin\psi \tag{26a}$$

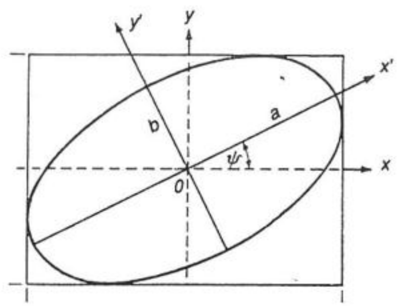

Figure 3 The rotated polarization ellipse.

$$E'_y = -E_x \sin\psi + E_y \cos\psi \tag{26b}$$

If $2a$ and $2b$ $(a \geq b)$ are the lengths of the major and minor axes, respectively, then the equation of the ellipse in terms of Ox' and Oy' can be written as

$$E'_x = a\cos(\tau + \delta') \tag{27a}$$

$$E'_y = \pm b\sin(\tau + \delta') \tag{27b}$$

where τ is the propagator and δ' is an arbitrary phase. The $\pm$ sign describes the two possible senses in which the end point of the field vector can describe the ellipse. The form of (27) is chosen because it is easy to see that it leads to the standard form of the ellipse, namely,

$$\frac{E'^2_x}{a^2} + \frac{E'^2_y}{b^2} = 1 \tag{28}$$

We can relate a and b in (27) to the parameters E_{0x} and E_{0y} in (25) by recalling that the original equations for the optical field are

$$\frac{E_x}{E_{0x}} = \cos(\tau + \delta_x) \tag{29a}$$

$$\frac{E_y}{E_{0y}} = \cos(\tau + \delta_y) \tag{29b}$$

We then substitute (27) and (29) into (26), expand the terms, and write

$$\begin{aligned} a(\cos\tau\cos\delta' - \sin\tau\sin\delta') &= E_{0x}(\cos\tau\cos\delta_x - \sin\tau\sin\delta_x)\cos\psi \\ &\quad + E_{0y}(\cos\tau\cos\delta_y - \sin\tau\sin\delta_y)\sin\psi \end{aligned} \tag{30a}$$

$$\begin{aligned} \pm b(\sin\tau\cos\delta' + \cos\tau\sin\delta') &= -E_{0x}(\cos\tau\cos\delta_x - \sin\tau\sin\delta_x)\sin\psi \\ &\quad + E_{0y}(\cos\tau\cos\delta_y - \sin\tau\sin\delta_y)\cos\psi \end{aligned} \tag{30b}$$

Equating the coefficients of $\cos\tau$ and $\sin\tau$ leads to the following equations:

$$a\cos\delta' = E_{0x}\cos\delta_x\cos\psi + E_{0y}\cos\delta_y\sin\psi \tag{31a}$$

$$a\sin\delta' = E_{0x}\sin\delta_x\cos\psi + E_{0y}\sin\delta_y\sin\psi \tag{31b}$$

$$\pm b\cos\delta' = E_{0x}\sin\delta_x\sin\psi - E_{0y}\sin\delta_y\cos\psi \tag{31c}$$

$$\pm b\sin\delta' = E_{0x}\cos\delta_x\sin\psi - E_{0y}\cos\delta_y\cos\psi \tag{31d}$$

Squaring and adding (31a) and (31b) and using $\delta = \delta_y - \delta_x$, we find

$$a^2 = E^2_{0x}\cos^2\psi + E^2_{0y}\sin^2\psi + 2E_{0x}E_{0y}\cos\psi\sin\psi\cos\delta \tag{32a}$$

Similarly, from (31c) and (31b) we find that

$$b^2 = E^2_{0x}\sin^2\psi + E^2_{0y}\cos^2\psi - 2E_{0x}E^2_{0y}\cos\psi\sin\psi\cos\delta \tag{32b}$$

Hence,

$$a^2 + b^2 = E^2_{0x} + E^2_{0y} \tag{33}$$

Next, we multiply (31a) by (31c), (31b) by (31d), and add. This gives

$$\pm ab = E_{0x}E_{0y}\sin\delta \tag{34}$$

Further, dividing (31d) by (31a) and (31c) by (31b) leads to

$$(E^2_{0x} - E^2_{0y})\sin 2\psi = 2E_{0x}E_{0y}\cos\delta\cos 2\psi \tag{35a}$$

or

$$\tan 2\psi = \frac{2E_{0x}E_{0y}\cos\delta}{E^2_{0x} - E^2_{0y}} \tag{35b}$$

which relates the angle of rotation ψ to E_{0x}, E_{0y}, and δ.

We note that in terms of the phase δ, ψ is equal to zero only for $\delta = 90°$ or $270°$. Similarly, in terms of amplitude, only if E_{0x} or E_{0y} is equal to zero is ψ equal to zero.

An alternative method for determining ψ is to transform (25) directly to (28). To show this we write (26a) and (26b) as

$$E_x = E'_x\cos\psi - E'_y\sin\psi \tag{36a}$$

$$E_y = E'_x\sin\psi + E'_y\cos\psi \tag{36b}$$

Equation (36) can be obtained from (26) by solving for E_x and E_y or, equivalently, replacing ψ by $-\psi$, E_x by E'_x, and E_y by E'_y. Upon substituting (36a) and (36b) into (25), the cross term is seen to vanish only for the condition given by (35).

It is useful to introduce an auxiliary angle $\alpha\,(0 \leq \alpha \leq \pi/2)$ for the polarization ellipse defined by

$$\tan\alpha = \frac{E_{0y}}{E_{0x}} \tag{37}$$

Then (35) is easily shown by using (36) to reduce to

$$\tan 2\psi = \frac{2E_{0x}E_{0y}}{E^2_{0x}E^2_{0y}}\cos\delta = \frac{2\tan\alpha}{1 - \tan^2\alpha}\cos\delta \tag{38}$$

which then yields

$$\tan 2\psi = (\tan 2\alpha)\cos\delta \tag{39}$$

We see that for $\delta = 0$ or π the angle of rotation is

$$\psi = \pm\alpha \tag{40}$$

For $\delta = \pi/2$ or $3\pi/2$ we have $\psi = 0$, so the angle of rotation is also zero.

Another important parameter of interest is the angle of ellipticity, χ. This is defined by

$$\tan\chi = \frac{\pm b}{a} \qquad -\frac{\pi}{4} \le \chi \le \frac{\pi}{4} \tag{41}$$

We see that for linearly polarized light $b = 0$, so $\chi = 0$. Similarly, for circularly polarized light $b = a$, so $\chi = \pm\pi/4$. Thus, (41) describes the extremes of the ellipticity of the polarization ellipse.

Using (33), (34), and (37), we easily find that

$$\frac{\pm 2ab}{a^2 + b^2} = \frac{2E_{0x}E_{0y}}{E_{0x}^2 + E_{0y}^2}\sin\delta = (\sin 2\alpha)\sin\delta \tag{42}$$

Next, using (41) we easily see that the left side of (42) reduces to $\sin 2\chi$, so we can write (42) as

$$\sin 2\chi = (\sin 2\alpha)\sin\delta \tag{43}$$

which is the relation between the ellipticity of the polarization ellipse and the parameters E_{0x}, E_{0y}, and δ of the polarization ellipse.

We note that only for $\delta = \pi/2$ or $3\pi/2$ does (43) reduce to

$$\chi = \pm\alpha \tag{44}$$

which is to be expected.

The results that we have obtained here will be used again, so it is useful to summarize them. The elliptical parameters E_{0x}, E_{0y}, and δ of the polarization ellipse are related to the orientation angle ψ and ellipticity angle χ by the following equations:

$$\tan 2\psi = (\tan 2\alpha)\cos\delta \qquad 0 \le \psi < \pi \tag{45a}$$

$$\sin 2\chi = (\sin 2\alpha)\sin\delta \qquad -\frac{\pi}{4} < \chi \le \frac{\pi}{4} \tag{45b}$$

where $0 \le \alpha \le \pi/2$ and

$$a^2 + b^2 = E_{0x}^2 + E_{0y}^2 \tag{45c}$$

$$\tan\alpha = \frac{E_{0y}}{E_{0x}} \tag{45d}$$

$$\tan\chi = \frac{\pm b}{a} \tag{45e}$$

We emphasize that the polarization ellipse can be described either in terms of the orientation and ellipticity angles ψ and χ on the left sides of (45a) and (45b) or the major and minor axes E_{0x} and E_{0y} and the phase shift δ on the right sides of (45a) and (45b).

Finally, a few words must be said on the terminology of polarization. Two cases of polarization are distinguished according to the sense in which the end point of the field vector describes the ellipse. It seems natural to call the polarization right-handed or left-handed according to whether the rotation of **E** and the direction of propagation form a right-handed or left-handed screw. The traditional terminology, however, is just the opposite and is based on the apparent behavior of **E** when viewed face on by the observer. In this book we shall conform to the traditional, that is, customary usage. Thus, the polarization is *right-handed* when to an observer looking in the direction from which the light is coming, the end point of the electric vector would appear to describe the ellipse in the *clockwise* sense. If we consider the value of (4) for two time instants separated by a quarter of a period, we see that in this case $\sin\delta > 0$, or by (45), $0 < \chi \le \pi/4$. For *left-handed* polarization the opposite is the case; i.e., to an observer looking in the direction from which the light is propagated, the electric vector would appear to describe the ellipse *counterclockwise*; in this case $\sin\delta < 0$, so that $-\pi/4 \le \chi < 0$.

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4
The Stokes Polarization Parameters

4.1. INTRODUCTION

In Chapter 3 we saw that the elimination of the propagator between the transverse components of the optical field led to the polarization ellipse. Analysis of the ellipse showed that for special cases it led to forms which can be interpreted as linearly polarized light and circularly polarized light. This description of light in terms of the polarization ellipse is very useful because it enables us to describe by means of a single equation various states of polarized light. However, this representation is inadequate for several reasons. As the beam of light propagates through space, we find that in a plane, transverse to the direction of propagation, the light vector traces out an ellipse or some special form of an ellipse, such as a circle or a straight line, in a time interval of the order of 10^{-15} sec. This period of time is clearly too short to allow us to follow the tracing of the ellipse. This fact, therefore, immediately prevents us from ever observing the polarization ellipse. Another limitation is that the polarization ellipse is only applicable to describing light which is completely polarized. It cannot be used to describe either unpolarized light or partially polarized light. This is a particularly serious limitation because in nature light is very often unpolarized or partially polarized. Thus, the polarization ellipse is an idealization of the true behavior of light; it is only correct at any given instant of time. These limitations force us to consider an alternative description of polarized light in which only observed or measured quantities enter. We are, therefore, in the same situation as when we dealt with the wave equation and its solutions, neither of which can be observed. We must again turn to using average values of the optical field which in the present case requires that we represent polarized light in terms of observables.

In 1852, Sir George Gabriel Stokes (1819–1903) discovered that the polarization behavior could be represented in terms of observables. He found that any state of polarized light could be completely described by four measurable quantities now known as the Stokes polarization parameters. The first parameter expresses the total intensity of the optical field. The remaining three parameters describe the polarization state. Stokes was led to his formulation in order to provide a suitable mathematical description of the Fresnel–Arago interference laws (1818). These laws were based on experiments carried out with an unpolarized light source, a quantity which Fresnel and his successors were never able to characterize mathematically. Stokes succeeded where others had failed because he abandoned the attempts to describe unpolarized light in terms of amplitude. He resorted to an experimental definition, namely, unpolarized light is light whose polarization state is unaffected by its propagation through a wave plate or polarizer or both. Stokes also showed that his parameters could be applied not only to unpolarized light but to partially polarized and completely polarized light as well. Unfortunately, Stokes' paper was forgotten for nearly a century. Its importance was finally brought to the attention of the scientific community by the Nobel laureate S. Chandrasekhar in 1947, who used them to formulate the radiative transfer equations for the scattering of partially polarized light. The Stokes parameters have been a prominent part of the optical literature on polarized light ever since.

We saw earlier that the amplitude of the optical field cannot be observed. However, the quantity which can be observed is the intensity, which is derived by taking a time average of the square of the amplitude. This suggests that if we take a time average of the unobserved polarization ellipse we will be led to the observables of the polarization ellipse. When this is done, as we shall show shortly, we obtain four parameters which are exactly the Stokes parameters. Thus, the Stokes parameters are a logical consequence of the wave theory. Furthermore, the Stokes parameters give a complete description of any polarization state of light. Most important, the Stokes parameters are exactly those quantities which are measured. Aside form this important formulation, however, when the Stokes parameters are used to describe physical phenomena, e.g., the Zeeman effect, one is lead to a very interesting representation. Originally, the Stokes parameters were used only to describe the measured intensity and polarization state of the optical field. But by forming the Stokes parameters in terms of a column matrix, the so-called Stokes vector, we are led to a formulation in which we obtain not only measurables but observables which can be seen in a spectroscope. As a result, we shall see that the formalism of the Stokes parameters is far more versatile than originally envisioned and possesses a greater usefulness than is commonly known.

4.2. THE DERIVATION OF THE STOKES POLARIZATION PARAMETERS

We consider a pair of plane waves which are orthogonal to each other at a point in space, conveniently taken to be $z = 0$, and not necessarily monochromatic to be represented by the equations

$$E_x(t) = E_{0x}(t)\cos\left[\omega t + \delta_x(t)\right] \tag{1a}$$

$$E_y(t) = E_{0y}(t)\cos\left[\omega t + \delta_y(t)\right] \tag{1b}$$

where $E_{0x}(t)$ and $E_{0y}(t)$ are the instantaneous amplitudes, ω is the instantaneous angular frequency, and $\delta_x(t)$ and $\delta_y(t)$ are the instantaneous phase factors. At all times the amplitudes and phase factors fluctuate slowly compared to the rapid vibrations of the cosinusoids. The explicit removal of the term ωt between (1a) and (1b) yields the familiar polarization ellipse, which is valid, in general, only at a given instant of time:

$$\frac{E_x^2(t)}{E_{0x}^2(t)} + \frac{E_y^2(t)}{E_{0y}^2(t)} - \frac{2E_x(t)E_y(t)}{E_{0x}(t)E_{0y}(t)}\cos\delta(t) = \sin^2\delta(t) \tag{2}$$

where $\delta(t) = \delta_y(t) - \delta_x(t)$.

For monochromatic radiation, the amplitudes and phases are constant for all time, so (2) reduces to

$$\frac{E_x^2(t)}{E_{0x}^2} + \frac{E_y^2(t)}{E_{0y}^2} - \frac{2E_x(t)E_y(t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta \tag{3}$$

While E_{0x}, E_{0y}, and δ are constants, E_x and E_y continue to be implicitly dependent on time, as we see from (1a) and (1b). Hence, we have written $E_x(t)$ and $E_y(t)$ in (3). In order to represent (3) in terms of the observables of the optical field field, we must take an average over the time of observation. Because this is a long period of time relative to the time for a single oscillation, this can be taken to be infinite. However, in view of the periodicity of $E_x(t)$ and $E_y(t)$, we need average (3) only over a single period of oscillation. The time average is represented by the symbol $\langle\cdots\rangle$, and so we write (3) as

$$\frac{\langle E_x^2(t)\rangle}{E_{0x}^2} + \frac{\langle E_y^2(t)\rangle}{E_{0y}^2} - \frac{2\langle E_x(t)E_y(t)\rangle}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta \tag{4a}$$

where

$$\langle E_i(t)E_j(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T E_i(t)E_j(t)\,dt \qquad i,j = x,y \tag{4b}$$

Multiplying (4) by $4E_{0x}^2E_{0y}^2$, we see that

$$\begin{aligned}4E_{0y}^2\langle E_x^2(t)\rangle + 4E_{0x}^2\langle E_y^2(t)\rangle - 8E_{0x}E_{0y}\langle E_x(t)E_y(t)\rangle\cos\delta \\ = (2E_{0x}E_{0y}\sin\delta)^2\end{aligned} \tag{5}$$

From (1a) and (1b), we then find the average values of (5) using (4b) are

$$\langle E_x^2(t)\rangle = \frac{1}{2}E_{0x}^2 \tag{6a}$$

$$\langle E_y^2(t)\rangle = \frac{1}{2}E_{0y}^2 \tag{6b}$$

$$\langle E_x(t)E_y(t)\rangle = \frac{1}{2}E_{0x}E_{0y}\cos\delta \tag{6c}$$

Substituting (6a), (6b), and (6c) into (5) yields

$$2E_{0x}^2E_{0y}^2 + 2E_{0x}^2E_{0y}^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2 \tag{7}$$

Since we wish to express the final result in terms of intensity this suggests that we add and subtract the quantity $E_{0x}^4 + E_{0y}^4$ to the left-hand side of (7); doing this leads to perfect squares. Upon doing this and grouping terms, we are led to the following equation:

$$(E_{0x}^2 + E_{0y}^2)^2 - (E_{0x}^2 - E_{0y}^2)^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2 \tag{8}$$

We now write the quantities inside the parentheses as

$$S_0 = E_{0x}^2 + E_{0y}^2 \tag{9a}$$

$$S_1 = E_{0x}^2 - E_{0y}^2 \tag{9b}$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta \tag{9c}$$

$$S_3 = 2E_{0x}E_{0y}\sin\delta \tag{9d}$$

and then express (8) as

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{10}$$

The four equations given by (9) are the Stokes polarization parameters for a plane wave. They were introduced into optics by Sir George Gabriel Stokes in 1852. We see that the Stokes parameters are real quantities, and they are simply the observables of the polarization ellipse and, hence, the optical field. The first Stokes parameter S_0 is the total intensity of the light. The parameter S_1 describes the amount of linear horizontal or vertical polarization, the parameter S_2 describes the amount of linear +45° or −45° polarization, and the parameter S_3 describes the amount of right or left circular polarization contained within the beam; this correspondence will be shown shortly. We note that the four Stokes parameters are expressed in terms of intensities, and we again emphasize that the Stokes parameters are *real* quantities.

If we now have partially polarized light, then we see that the relations given by (9) continue to be valid for very short time intervals, since the amplitudes and phases fluctuate slowly. Using Schwarz's inequality, one can show that for any state of polarized light the Stokes parameters always satisfy the relation

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2 \tag{11}$$

The equality sign applies when we have completely polarized light, and the inequality sign when we have partially polarized light or unpolarized light.

In Chapter 3, Eq. (35) we saw that orientation angle ψ of the polarization ellipse was given by

$$\tan 2\psi = \frac{2E_{0x}E_{0y}\cos\delta}{E_{0x}^2 - E_{0y}^2} \tag{12}$$

Inspecting (9) we see that if we divide (9c) by (9b), ψ can be expressed in terms of the Stokes parameters:

$$\tan 2\psi = \frac{S_2}{S_1} \tag{13}$$

Similarly, from (42) and (43) in Chapter 3 the ellipticity angle χ was given by

$$\sin 2\chi = \frac{2E_{0x}E_{0y}\sin\delta}{E_{0x}^2 + E_{0y}^2} \tag{14}$$

Again, inspecting (9) and dividing (9d) by (9a), we can see that χ can be expressed in terms of the Stokes parameters:

$$\sin 2\chi = \frac{S_3}{S_0} \tag{15}$$

The Stokes parameters enable us to describe the degree of polarization P for any state of polarization. By definition,

$$P = \frac{I_{\text{pol}}}{I_{\text{tot}}} = \frac{(S_1^2 + S_2^2 + S_3^2)^{1/2}}{S_0} \qquad 0 \leq P \leq 1 \tag{16}$$

where I_{pol} is the intensity of the sum of the polarization components and I_{tot} is the total intensity of the beam. The value of $P = 1$ corresponds to completely polarized light, $P = 0$ corresponds to unpolarized light, and $0 < P < 1$ corresponds to partially polarized light.

To obtain the Stokes parameters of an optical beam, one must always take a time average of the polarization ellipse. However, the time-averaging process can be formally bypassed by representing the (real) optical amplitudes, (1a) and (1b), in terms of complex amplitudes,

$$E_x(t) = E_{0x}\exp[i(\omega t + \delta_x)] = \mathcal{E}_{0x}\exp(i\omega t) \tag{17a}$$

$$E_y(t) = E_{0y}\exp[i(\omega t + \delta_y)] = \mathcal{E}_{0y}\exp(i\omega t) \tag{17b}$$

where

$$\mathcal{E}_{0x} = E_{0x}\exp(i\delta_x) \tag{17c}$$

and

$$\mathcal{E}_{0y} = E_{0y}\exp(i\delta_y) \tag{17d}$$

are complex amplitudes. The Stokes parameters for a plane wave are now obtained from the formulas

$$S_0 = E_xE_x^* + E_yE_y^* \tag{18a}$$

$$S_1 = E_xE_x^* - E_yE_y^* \tag{18b}$$

$$S_2 = E_xE_y^* + E_yE_x^* \tag{18c}$$

$$S_3 = i(E_xE_y^* - E_yE_x^*) \tag{18d}$$

We shall use (18), the complex representation, henceforth, as the defining equations for the Stokes parameters. Substituting (17c) and (17d) into (18) gives

$$S_0 = E_{0x}^2 + E_{0y}^2 \tag{19a}$$

$$S_1 = E_{0x}^2 - E_{0y}^2 \tag{19b}$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta \tag{19c}$$

$$S_3 = 2E_{0x}E_{0y}\sin\delta \tag{19d}$$

which are the Stokes parameters obtained formally from the polarization ellipse (9).

As examples of the representation of polarized light in terms of the Stokes parameters, we consider (1) linear horizontal and linear vertical polarized light, (2) linear +45° and linear −45° polarized light, and (3) right and left circularly polarized light.

Linear Horizontal Polarized Light (LHP)

For this case $E_{0y} = 0$. Then, from (19) we have

$$S_0 = E_{0x}^2 \tag{20a}$$

$$S_1 = E_{0x}^2 \tag{20b}$$

$$S_2 = 0 \tag{20c}$$

$$S_3 = 0 \tag{20d}$$

Linear Vertical Polarized Light (LVP)

For this case $E_{0x} = 0$. From (19) we have

$$S_0 = E_{0y}^2 \tag{21a}$$

$$S_1 = -E_{0y}^2 \tag{21b}$$

$$S_2 = 0 \tag{21c}$$

$$S_3 = 0 \tag{21d}$$

Linear +45° Light (L + 45)

The conditions to obtain $L + 45$ polarized light are $E_{0x} = E_{0y} = E_0$ and $\delta = 0^0$. Using these conditions and the definition of the Stokes parameters (19), we find that

$$S_0 = 2E_0^2 \tag{22a}$$

$$S_1 = 0 \tag{22b}$$

$$S_2 = 2E_0^2 \tag{22c}$$

$$S_3 = 0 \tag{22d}$$

Linear −45° Light (L − 45)

The conditions on the amplitude are the same as for $L + 45$ light, but the phase difference is $\delta = 180°$. Then from (19) we see that the Stokes parameters are

$$S_0 = 2E_0^2 \tag{23a}$$

$$S_1 = 0 \tag{23b}$$

$$S_2 = -2E_0^2 \tag{23c}$$

$$S_3 = 0 \tag{23d}$$

Right Circularly Polarized Light (RCP)

The conditions to obtain RCP light are $E_{0x} = E_{0y} = E_0$ and $\delta = 90°$. From (19) the Stokes parameters are then

$$S_0 = 2E_0^2 \tag{24a}$$

$$S_1 = 0 \tag{24b}$$

$$S_2 = 0 \tag{24c}$$

$$S_3 = 2E_0^2 \tag{24d}$$

Left Circularly Polarized Light (LCP)

For LCP light the amplitudes are again equal, but the phase shift between the orthogonal, transverse components is $\delta = 270°$. The Stokes parameters from (19) are then

$$S_0 = 2E_0^2 \tag{25a}$$

$$S_1 = 0 \tag{25b}$$

$$S_2 = 0 \tag{25c}$$

$$S_3 = -2E_0^2 \tag{25d}$$

Finally, the Stokes parameters for elliptically polarized light are, of course, given by (19).

Inspection of the four Stokes parameters suggests that they can be arranged in the form of a column matrix. This column matrix is called the Stokes vector. This step, while simple, provides a formal method for treating numerous complicated problems involving polarized light. We now discuss the Stokes vector.

4.3. THE STOKES VECTOR

The four Stokes parameters can be arranged in a column matrix and written as

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{26}$$

The column matrix (26) is called the Stokes vector. Mathematically, it is not a vector, but through custom it is called a vector. Equation (26) should correctly be called the Stokes column matrix. The Stokes vector for elliptically polarized light is then written from (19) as

$$S = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix} \tag{27}$$

Equation (27) is also called the Stokes vector for a plane wave.

The Stokes vector for linearly polarized light and circularly polarized light are readily found from (27). We now derive these Stokes vectors.

Linearly Horizontally Polarized Light (LHP)

For this case $E_{0y} = 0$, and we find from (27) that

$$S = I_0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{28}$$

where $I_0 = E_{0x}^2$ is the total intensity.

Linearly Vertically Polarized Light (LVP)

For this case $E_{0x} = 0$, and we find that (27) reduces to

$$S = I_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{29}$$

where, again, I_0 is the total intensity.

Linear +45° Polarized Light (L + 45)

In this case $E_{0x} = E_{0y} = E_0$ and $\delta = 0$, so (27) becomes

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{30}$$

where $I_0 = 2E_0^2$.

Linear −45° Polarized Light (L − 45)

Again, $E_{0x} = E_{0y} = E_0$, but now $\delta = 180°$. Then (27) becomes

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{31}$$

and $I_0 = 2E_0^2$.

Right Circularly Polarized Light (RCP)

In this case $E_{0x} = E_{0y} = E_0$ and $\delta = 90°$. Then (27) becomes

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{32}$$

and $I_0 = 2E_0^2$.

Left Circularly Polarized Light (LCP)

Again, we have $E_{0x} = E_{0y}$, but now the phase shift δ between the orthogonal amplitudes is $\delta = 270°$ (or $-90°$. Equation (27) then reduces to

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{33}$$

and $I_0 = 2E_0^2$.

We also see from (27) that if $\delta = 0°$ or $180°$, then (27) reduces to

$$S = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ \pm E_{0x}E_{0y} \\ 0 \end{pmatrix} \tag{34}$$

We recall that the ellipticity angle χ and the orientation angle ψ for the polarization ellipse are given, respectively, by

$$\sin 2\chi = \frac{S_3}{S_0} \qquad \frac{-\pi}{4} \le \chi \le \frac{\pi}{4} \tag{35a}$$

$$\tan 2\psi = \frac{S_2}{S_1} \qquad 0 \le \psi < \pi \tag{35b}$$

We see that S_3 is zero, so the ellipticity angle χ is zero and, hence, (34) is the Stokes vector for linearly polarized light. The orientation angle according to (35b) is

$$\tan 2\psi = \frac{\pm E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2} \tag{36}$$

The form of (34) is a useful representation for linearly polarized light. Another useful representation can be made by expressing the amplitudes E_{0x} and E_{0y} in terms of an angle. To show this, we first rewrite the total intensity S_0 as

$$S = E_{0x}^2 + E_{0y}^2 = E_0^2 \tag{37}$$

Equation (37) suggests Figure 1. From Figure 1 we see that

$$E_{0x} = E_0 \cos\alpha \tag{38a}$$

$$E_{0y} = E_0 \sin\alpha \qquad 0 \le \alpha \le \frac{\pi}{2} \tag{38b}$$

The angle α is called the auxiliary angle; it is identical to the auxiliary angle used to represent the orientation angle and ellipticity equations summarized earlier. Substituting (38) into (34) leads to the following Stokes vector for linearly polarized light:

$$S = I_0 \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix} \tag{39}$$

where $I_0 = E_0^2$ is the total intensity. Equation (38) can also be used to represent the Stokes vector for elliptically polarized light, (27). Substituting (38) into (27)

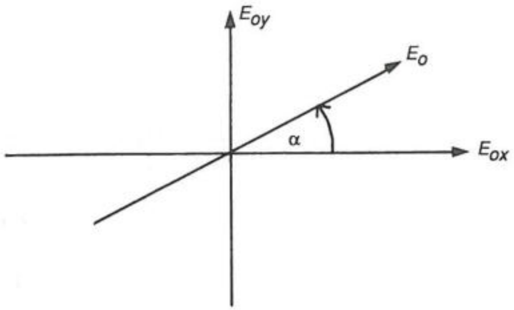

Figure 1 Resolution of the optical field components.

gives

$$S = I_0 \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos\delta \\ \sin 2\alpha \sin\delta \end{pmatrix} \tag{40}$$

It is customary to write the Stokes vector in normalized form by setting $I_0 = 1$. Thus, (40) is written merely as

$$S = \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos\delta \\ \sin 2\alpha \sin\delta \end{pmatrix} \tag{41}$$

The orientation angle ψ and the ellipticity angle χ of the polarization ellipse are given by (35a) and (35b). Substituting S_1, S_2, and S_3 into (41) into (35a) and (35b) gives

$$\tan 2\psi = \tan 2\alpha \cos\delta \tag{42a}$$

$$\sin 2\chi = \sin 2\alpha \sin\delta \tag{42b}$$

which are identical to the relations we found earlier.

The use of the auxiliary angle α enables us to express the orientation and ellipticity in terms of α and δ. Expressing (42) in this manner shows that there are two unique polarization states. For $\alpha = 45°$, (42) reduces to

$$S = \begin{pmatrix} 1 \\ 0 \\ \cos\delta \\ \sin\delta \end{pmatrix} \tag{43}$$

Thus, the polarization ellipse is expressed only in terms of the phase shift δ between the orthogonal amplitudes. The orientation angle ψ is seen to be always 45°. The ellipticity angle, however, (43b) is

$$\sin 2\chi = \sin\delta \tag{44}$$

so $\chi = \delta/2$. The Stokes vector (43) expresses that the polarization ellipse is rotated 45° from the horizontal axis and that the polarization state of the light can vary from linearly polarized ($\delta = 0, \pi$) to circularly polarized ($\delta = 90°, 270°$).

Another unique polarization state occurs when $\delta = 90°$ or $270°$. For this condition (41) reduces to

$$S = \begin{pmatrix} 1 \\ \cos 2\alpha \\ 0 \\ \pm\sin 2\alpha \end{pmatrix} \tag{45}$$

We see that we now have a Stokes vector and a polarization ellipse which depends only on the auxiliary angle α. From (42a) the orientation angle ψ is always zero. However, (42b) and (45) show that the ellipticity angle χ is now given by

$$\sin 2\chi = \pm \sin\alpha \tag{46}$$

so $\chi = \pm\alpha/2$. In general, (46) shows that we will have elliptically polarized light. For $\alpha = +90°$ and $-90°$ we obtain right and left circularly polarized light. Similarly, for $\alpha = 0°$ and $180°$ we obtain linearly horizontal and vertical polarized light.

The Stokes vector can also be expressed only in terms of S_0, ψ, and χ. To show this we write (35a) and (35b) as

$$S_2 = S_1 \tan 2\psi \tag{47a}$$

$$S_3 = S_0 \sin 2\chi \tag{47b}$$

In Section 4.2 we found that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{48}$$

Substituting (47a) and (47b) into (48), we find that

$$S_1 = S_0 \cos 2\chi \cos 2\psi \tag{49a}$$

$$S_2 = S_0 \cos 2\chi \sin 2\psi \tag{49b}$$

$$S_3 = S_0 \sin 2\chi \tag{49c}$$

Arranging (49) in the form of a Stokes vector, we have

$$S = S_0 \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{pmatrix} \tag{50}$$

The Stokes parameters (49) are almost identical in form to the well-known equations relating Cartesian coordinates to spherical coordinates. We recall that the spherical coordinates r, θ, and ϕ are related to Cartesian coordinates x, y, and z by

$$x = r \sin\theta \cos\phi \tag{51a}$$

$$y = r \sin\theta \sin\phi \tag{51b}$$

$$z = r \cos\theta \tag{51c}$$

Comparing (51) with (49), we see that the equations are identical if the angles are related by

$$\theta = 90° - 2\chi \tag{52a}$$

$$\phi = 2\psi \tag{52b}$$

In Figure 2 we have drawn a sphere whose center is also at the center of the Cartesian coordinate system. We see that expressing the polarization state of an optical beam in terms of χ and ψ allows us to describe its ellipticity and orientation on a sphere; the radius of the sphere is taken to be unity. The representation of the polarization state on a sphere was first introduced by H. Poincaré in 1892 and is, appropriately, called the Poincaré sphere. However, at that time, Poincaré introduced the sphere in an entirely different way, namely, by representing the polarization equations in a complex plane and then projecting the plane onto a sphere, a so-called stereographic projection. In this way he was led to (49). He does not appear to have known that (49) were directly related to the Stokes parameters. Because the Poincaré sphere is of historical interest and is still used to describe the polarization state of light, we shall discuss it in detail later. It is especially useful for describing the change in polarized light when it interacts with polarizing elements.

The discussion in this chapter shows that the Stokes parameters and the Stokes vector can be used to describe an optical beam which is completely polarized. We have, at first sight, only provided an alternative description of completely polarized light. All of the equations derived here are based on the polarization ellipse given in Chapter 3, that is, the amplitude formulation. However, we have pointed out that the Stokes parameters can also be used to describe unpolarized

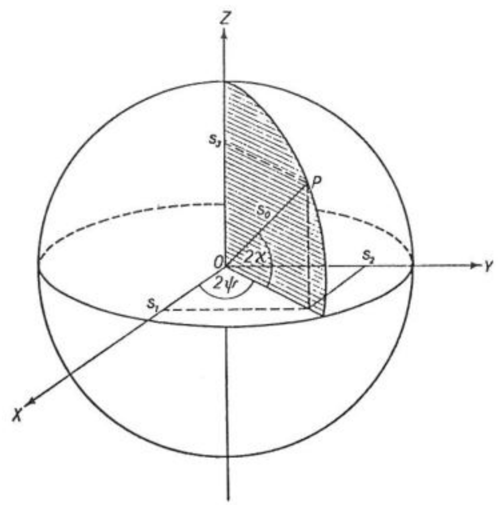

Figure 2 The Poincaré representation of polarized light on a sphere.

and partially polarized light, quantities which cannot be described within an amplitude formulation of the optical field. In order to extend the Stokes parameters to unpolarized and partially polarized light, we must now consider the classical measurement of the Stokes polarization parameters.

4.4. THE CLASSICAL MEASUREMENT OF THE STOKES POLARIZATION PARAMETERS

The Stokes polarization parameters are immediately useful because, as we shall now see, they are directly accessible to measurement. This is due to the fact that they are an intensity formulation of the polarization state of an optical beam. In this section we shall describe the measurement of the Stokes polarization parameters. This is done by allowing an optical beam to pass through two optical elements known as retarder and polarizer. Specifically, the incident field is described in terms of its components, and the field emerging from the polarizing components is then used to determine the intensity of the emerging beam. Later, we shall carry out this same problem by using a more formal but powerful approach known as the Mueller matrix calculus. In the following chapter we shall also see how this measurement method enables us to determine the Stokes parameters for unpolarized and partially polarized light.

We begin by referring to the Figure 3, which shows an incident optical beam incident on a polarizing element called a retarder. This polarizing element is then followed by another polarizing element called a polarizer. The components of the incident beam are

$$E_x(t) = E_{0x}e^{i\delta_x}e^{i\omega t} \quad (53a)$$

$$E_y(t) = E_{0y}e^{i\delta_y}e^{i\omega t} \quad (53b)$$

In Section 4.2 we saw that the Stokes parameters for a plane wave written in complex notation could be obtained from

$$S_0 = E_xE_x^* + E_yE_y^* \quad (54a)$$

$$S_1 = E_xE_x^* - E_yE_y^* \quad (54b)$$

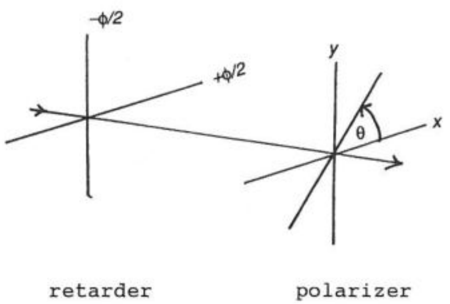

Figure 3 Measurement of the Stokes polarization parameters.

$$S_2 = E_xE_y^* + E_yE_x^* \quad (54c)$$

$$S_3 = i(E_xE_y^* - E_yE_x^*) \quad (54d)$$

where $i = \sqrt{-1}$ and the asterisk represents the complex conjugate.

In order to measure the Stokes parameters the incident field propagates through a phase-shifting element which has the property that the phase of the x component (E_x) is advanced by $\phi/2$ and the phase of the y component E_y is retarded by $\phi/2$, written as $-\phi/2$. The components E_x' and E_y' emerging from the phase-shifting element component are then

$$E_x' = E_xe^{i\phi/2} \quad (55a)$$

$$E_y' = E_ye^{-i\phi/2} \quad (55b)$$

The optical component which behaves in this manner is called a retarder in optics; it will be discussed in more detail later.

Next, the field described by (55) is incident on a component which is called a polarizer. It has the property that the optical field can only pass along an axis known as the transmission axis. Ideally, if the transmission axis of the polarizer is at an angle θ only the components of E_x' and E_y' can be transmitted perfectly along the transmission axis; they are attenuated completely at any other angle. A polarizing element which behaves in this manner is called a polarizer. This behavior is described in Figure 4. The component of E_x' along the transmission axis is $E_x'\cos\theta$. Similarly, the component of E_y' is $E_y'\sin\theta$. The field transmitted along the transmission axis is the sum of these components so the total field E emerging from the polarizer is

$$E = E_x'\cos\theta + E_y'\sin\theta \quad (56)$$

Substituting (55) into (56), the field emerging from the polarizer is

$$E = E_xe^{i\phi/2}\cos\theta + E_ye^{-i\phi/2}\sin\theta \quad (57)$$

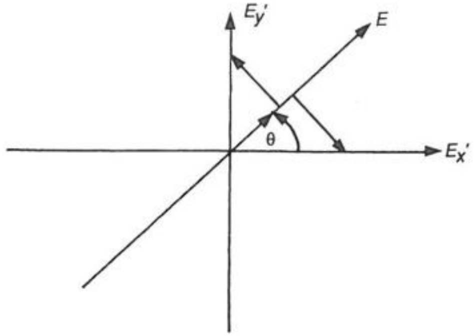

Figure 4 Resolution of the optical field components by a polarizer.

The intensity of the beam is defined by

$$I = E \cdot E^* \tag{58}$$

Taking the complex conjugate of (57) and forming the product in accordance with (58), the intensity of the emerging beam is

$$\begin{aligned} I(\theta, \phi) = {} & E_x E_x^* \cos^2\theta + E_y E_y^* \sin^2\theta \\ & + E_x^* E_y e^{-i\phi} \sin\theta \cos\theta + E_x E_y^* e^{i\phi} \sin\theta \cos\theta \end{aligned} \tag{59}$$

Equation (59) can be rewritten by using the well-known trigonometric half-angle formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{60a}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{60b}$$

$$\sin\theta \cos\theta = \frac{\sin 2\theta}{2} \tag{60c}$$

Using (60) in (59) and grouping terms, we find that the intensity $I(\theta, \phi)$ becomes

$$\begin{aligned} I(\theta, \phi) = \frac{1}{2} \big[& (E_x E_x^* + E_y E_y^*) + (E_x E_x^* - E_y E_y^*) \cos 2\theta \\ & + (E_x E_y^* + E_y E_x^*) \cos\phi \sin 2\theta + i(E_x E_y^* - E_y E_x^*) \sin\theta \sin 2\theta \big] \end{aligned} \tag{61}$$

The terms within brackets are exactly the Stokes parameters given in (54). It was first derived by Stokes and is the manner in which the Stokes parameters were first introduced in the optical literature. Replacing the terms in (61) by those in (54), we arrive at

$$I(\theta, \phi) = \frac{1}{2} [S_0 + S_1 \cos 2\theta + S_2 \cos\phi \sin 2\theta + S_3 \sin\phi \sin 2\theta] \tag{62}$$

Equation (62) is Stokes' famous intensity formula for measuring the four Stokes parameters. Thus, we see that the Stokes parameters are directly accessible to measurement; that is, they are observable quantities.

The first three Stokes parameters are measured by removing the retarder ($\phi = 0°$) and rotating the transmission axis of the polarizer to the angles $\theta = 0°$, $+45°$, and $+90°$, respectively. The final parameter, S_3, is measured by reinserting a so-called quarter-wave-plate retarder ($\phi = 90°$) into the optical path and setting the transmission axis of the polarizer to $\theta = 45°$. The respective intensities are then found from (62) to be, respectively,

$$I(0°, 0°) = \frac{1}{2} [S_0 + S_1] \tag{63a}$$

$$I(45°, 0°) = \frac{1}{2} [S_0 + S_2] \tag{63b}$$

$$I(90°, 0°) = \frac{1}{2} [S_0 - S_1] \tag{63c}$$

$$I(45°, 90°) = \frac{1}{2} [S_0 + S_3] \tag{63d}$$

Solving (63) for the Stokes parameters, we have

$$S_0 = I(0°, 0°) + I(90°, 0°) \tag{64a}$$

$$S_1 = I(0°, 0°) - I(90°, 0°) \tag{64b}$$

$$S_2 = 2I(45°, 0°) - I(0°, 0°) - I(90°, 0°) \tag{64c}$$

$$S_3 = 2I(45°, 90°) - I(0°, 0°) - I(90°, 0°) \tag{64d}$$

Equation (64) is really quite remarkable. In order to measure the Stokes parameters it is necessary to measure the intensity at four angles. We must remember, however, that in 1852 there were no devices to measure the intensity *quantitatively*. The intensities can be measured quantitatively only with an optical detector. But when Stokes introduced the Stokes parameters, such detectors did not exist. The only optical detector was the human eye (retina), a detector capable of measuring only the null or greater-than-null state of light, and so the above method for measuring the Stokes parameters could not be used! Stokes did not introduce the Stokes parameters to describe the optical field in terms of observables as is sometimes stated. The reason for his derivation of (62) was not to measure the Stokes polarization parameters but to provide the solution to an entirely different problem, namely, a mathematical statement for unpolarized light. We shall soon see that (62) is perfect for doing this. It is possible to measure all four Stokes parameters using the human eye, however, by using a null–intensity technique. This method is described in Section 6.4.

Unfortunately, after Stokes solved this problem and published his great paper on the Stokes parameters and the nature of polarized light, he never returned to this subject again. By the end of his researches on this subject he had turned his attention to the problem of the fluorescence of solutions. This problem would become the major focus of his attention for the rest of his life. Aside from Lord Rayleigh in England and Emil Verdet in France, the importance of Stokes' paper and the Stokes parameters was not fully recognized, and the paper was, practically, forgotten for nearly a century by the optical community. Fortunately, however, Emil Verdet did understand the significance of Stokes' paper and wrote a number of subsequent papers on the Stokes polarization parameters. He thus began a tradition in France of studying the Stokes parameters. The Stokes polarization parameters did not really appear in the English-speaking world again until they were "rediscovered" by S. Chandrasekhar in the late 1940s when he was writing his monumental papers on radiative transfer. Previous to Chandrasekhar no one had included optical polarization in the equations of radiative transfer. In order to introduce polarization into his equations, he eventually found Stokes' original paper. He immediately recognized that because the Stokes parameters were an intensity formulation of optical polarization they could be introduced into radiative equations. It was only after the publication of Chandrasekhar's papers that the Stokes parameters reemerged. They have remained in the optical literature ever since.

We now describe Stokes' formulation for unpolarized light.

4.5. THE STOKES PARAMETERS FOR UNPOLARIZED AND PARTIALLY POLARIZED LIGHT

The intensity $I(\theta, \phi)$ of a beam of light emerging from the retarder/polarizer combination was seen in the previous section to be

$$I(\theta, \phi) = \frac{1}{2}[S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta \cos \phi + S_3 \sin 2\theta \sin \phi] \tag{65}$$

where S_0, S_1, S_2, and S_3 are the Stokes parameters of the incident beam, and θ is the rotation angle of the transmission axis of the polarizer, and ϕ is the phase shift of the wave plate, respectively. By setting θ to 0°, 45°, and 90° and ϕ to 0° or 90°, all four Stokes parameters can then be measured. However, it was not Stokes' intention to merely cast the polarization of the optical field in terms of the intensity rather than the amplitude. Rather, he was interested in finding a suitable mathematical description for unpolarized light. Stokes, unlike his predecessors and his contemporaries, recognized that it was impossible to describe unpolarized light in terms of amplitudes. Consequently, he abandoned the amplitude approach and sought a description based on the observed intensity.

To describe unpolarized light using (65), Stokes observed that unpolarized light had a very unique property, namely, its intensity was unaffected by (1) rotation of a linear polarizer (when a polarizer is used to analyze the state of polarization, it is called an analyzer) or (2) the presence of a retarder. Thus, for unpolarized light the only way the observed intensity $I(\theta, \phi)$ could be independent of θ, ϕ was for (65) to satisfy

$$I(\theta, \phi) = \frac{1}{2} S_0 \tag{66a}$$

and

$$S_1 = S_2 = S_3 = 0 \tag{66b}$$

Equations (66a) and (66b) are the mathematical statements for unpolarized light. Thus, Stokes had finally provided a correct mathematical statement. From a conceptual point of view S_1, S_2, and S_3 describe the polarizing behavior of the optical field. Since there is no polarization, (66a) and (66b) must be the correct mathematical statements for unpolarized light. Later, we shall show how (66) is used to formulate the interference laws of Fresnel and Arago.

In this way Stokes discovered an entirely different way to describe the polarization state of light. His formulation could be used to describe completely polarized light and completely unpolarized light as well. Furthermore, Stokes had been led to a formulation of the optical field in terms of measurable quantities, the Stokes parameters, so-called observables. This was a very unique point of view for nineteenth-century optical physics. The representation of radiation phenomena in terms of observables would not reappear again in physics until 1925 with the discovery of the laws of quantum mechanics by W. Heisenberg.

The Stokes parameters described in (65) arise from an experimental configuration. Consequently, they were associated for a long time with the experimental measurement of the polarization of the optical field. Thus, a study of classical optics shows that polarization was conceptually understood with the nonobservable polarization ellipse, whereas the measurement was made in terms of intensities, the Stokes parameters. In other words, there were two distinct ways to describe the polarization of the optical field.

We have seen, however, that the Stokes parameters are actually a consequence of the wave theory and arise naturally from the polarization ellipse. It is only necessary to transform the nonobservable polarization ellipse to the observed intensity domain, whereupon we are led directly to the Stokes parameters. Thus, the Stokes polarization parameters must be considered as part of the conceptual foundations of the wave theory.

For a completely polarized beam of light we saw that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{67}$$

and we have just seen that for unpolarized light

$$S_0^2 > 0 \tag{68}$$

Equations (67) and (68) represent extreme states of polarization. Clearly, there must be an intermediate polarization state. This intermediate state is called partially polarized light. Thus, (67) can be used to describe all three polarization conditions by writing it as

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2 \tag{69}$$

For perfectly polarized light "≥" is replaced by "=;" for unpolarized light "≥" is replaced by ">" with $S_1 = S_2 = S_3 = 0$; and for partially polarized light "≥" is replaced by ">."

An important quantity which describes these various polarization conditions is the degree of polarization P. This quantity can be expressed in terms of the Stokes parameters. To derive P we decompose the optical field into unpolarized and polarized portions which are mutually independent. Then, and this will be proved later, the Stokes parameters of a combination of independent waves are the sums of the respective Stokes parameters of the separate waves. The four Stokes parameters, S_0, S_1, S_2, and S_3 of the beam are represented by S. The total intensity of the beam is then S_0. We subtract the polarized intensity $(S_1^2 + S_2^2 + S_3^2)^{1/2}$ from the total intensity S_0 and we obtain the unpolarized intensity. Thus, we have

$$S^{(u)} = S_0 - \sqrt{S_1^2 + S_2^2 + S_3^2}, 0, 0, 0 \tag{70a}$$

and

$$S^{(p)} = \sqrt{S_1^2 + S_2^2 + S_3^2}, S_1, S_2, S_3 \tag{70b}$$

where $S^{(u)}$ represents the unpolarized part and $S^{(p)}$ represents the polarized part. The degree of polarization P is then defined to be

$$P = \frac{I_{pol}}{I_{tot}} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \qquad 0 \leq P \leq 1 \tag{71}$$

Thus, $P = 0$ indicates that the light is unpolarized, $P = 1$ that the light is (completely) polarized, and $0 < P < 1$ that the light is partially polarized.

The use of the Stokes parameters to describe polarized light rather than the amplitude formulation enables us to deal directly with the quantities measured in an optical experiment. Thus, we carry out the analysis in the amplitude domain and then transform the amplitude results to the Stokes parameters, using the defining equations. When this is done, we can easily relate the experimental results to the theoretical results. Furthermore, when we obtain the Stokes parameters, or rather the Stokes vector, we shall see that we are led to a description of radiation in which the Stokes parameters not only describe the measured quantities but can be used to truly describe the observed spectral lines in a spectroscope. In other words, we shall arrive at observables in the strictest sense of the word.

4.6. ADDITIONAL PROPERTIES OF THE STOKES POLARIZATION PARAMETERS

Before we proceed to apply the Stokes parameters to a number of problems of interest, we wish to discuss a few of their additional properties. We saw earlier that the Stokes parameters could be used to describe any state of polarized light. In particular, we saw how unpolarized light and completely polarized light could be written in terms of a single Stokes vector. The question remains as to how we can represent partially polarized light in terms of the Stokes parameters and the Stokes vector. To answer this question, we must establish a fundamental property of the Stokes parameters, the property of additivity whereby the Stokes parameters of two completely independent beams can be added. This property is another way of describing the principle of incoherent superposition. We now prove this property of additivity.

We recall that the Stokes parameters for an optical beam can be represented in terms of complex amplitudes by

$$S_0 = E_x E_x^* + E_y E_y^* \tag{72a}$$

$$S_1 = E_x E_x^* + E_y E_y^* \tag{72b}$$

$$S_2 = E_x E_y^* + E_y E_x^* \tag{72c}$$

$$S_3 = i(E_x E_y^* - E_y E_x^*) \tag{72d}$$

Consider now that we have two optical beams each of which is characterized by its own set of Stokes parameters represented as $S^{(1)}$ and $S^{(2)}$,

$$S_0^{(1)} = E_{1x} E_{1x}^* + E_{1y} E_{1y}^* \tag{73a}$$

$$S_1^{(1)} = E_{1x} E_{1x}^* - E_{1y} E_{1y}^* \tag{73b}$$

$$S_2^{(1)} = E_{1x} E_{1y}^* + E_{1y} E_{1x}^* \tag{73c}$$

$$S_3^{(1)} = i(E_{1x} E_{1y}^* - E_{1y} E_{1x}^*) \tag{73d}$$

and

$$S_0^{(2)} = E_{2x} E_{2x}^* + E_{2y} E_{2y}^* \tag{74a}$$

$$S_1^{(2)} = E_{2x} E_{2x}^* - E_{2y} E_{2y}^* \tag{74b}$$

$$S_2^{(2)} = E_{2x} E_{2y}^* + E_{2y} E_{2x}^* \tag{74c}$$

$$S_3^{(2)} = i(E_{2x} E_{2y}^* - E_{2y} E_{2x}^*) \tag{74d}$$

The superscripts and subscripts 1 and 2 refer to the first and second beams, respectively. These two beams are now superposed. Then by the principle of superposition for amplitudes the total field in the x and y direction is

$$E_x = E_{1x} + E_{2x} \tag{75a}$$

$$E_y = E_{1y} + E_{1y} \tag{75b}$$

We now form products of (75a) and (75b) according to (72):

$$\begin{aligned} E_x E_x^* &= (E_{1x} + E_{2x})(E_{1x} + E_{2x})^* \\ &= E_{1x} E_{1x}^* + E_{1x} E_{2x}^* + E_{2x} E_{1x}^* + E_{2x} E_{2x}^* \end{aligned} \tag{76a}$$

$$\begin{aligned} E_y E_y^* &= (E_{1y} + E_{2y})(E_{1y} + E_{2y})^* \\ &= E_{1y} E_{1y}^* + E_{1y} E_{2y}^* + E_{2y} E_{1y}^* + E_{2y} E_{2y}^* \end{aligned} \tag{76b}$$

$$\begin{aligned} E_x E_y^* &= (E_{1x} + E_{2x})(E_{1y} + E_{2y})^* \\ &= E_{1x} E_{1y}^* + E_{1x} E_{2y}^* + E_{2x} E_{1y}^* + E_{2x} E_{2y}^* \end{aligned} \tag{76c}$$

$$E_x E_x^* = E_{1y} E_{1x}^* + E_{2y} E_{1x}^* + E_{1y} E_{2x}^* + E_{2y} E_{2x}^* \tag{76d}$$

Let us now assume that the two beams are completely independent of each other with respect to their amplitudes and phase. We describe the degree of independence by writing an overbar on the products of E_x and E_y, that is, $\overline{E_x E_x^*}$, $\overline{E_y E_y^*}$, etc., so

$$\overline{E_i E_j^*} \qquad i,j = x,y \tag{77}$$

Since the two beams are completely independent, we express this behavior by

$$\overline{E_{1j} E_{2j}^*} = \overline{E_{2i} E_{1i}^*} = 0 \qquad i \neq j \tag{78a}$$

$$\overline{E_{1i} E_{2i}^*} \neq 0 \tag{78b}$$

$$\overline{E_{1j} E_{2j}^*} \neq 0 \qquad i,j = x,y \tag{78c}$$

The value of zero in (78a) indicates complete independence. On the other hand, the nonzero value in (78b) and (78c) means that there is some degree of dependence. Operating on (76a) through (76b) with an overbar and using the conditions expressed by (78), we find that

$$\overline{E_x E_x^*} = \overline{E_{1x} E_{1x}^*} + \overline{E_{2x} E_{2x}^*} \tag{79a}$$

$$\overline{E_y E_y^*} = \overline{E_{1y} E_{1y}^*} + \overline{E_{2y} E_{2y}^*} \tag{79b}$$

$$\overline{E_x E_y^*} = \overline{E_{1x} E_{1y}^*} + \overline{E_{2x} E_{2y}^*} \tag{79c}$$

$$\overline{E_y E_x^*} = \overline{E_{1y} E_{1x}^*} + \overline{E_{2y} E_{2x}^*} \tag{79d}$$

We now form the Stokes parameters according to (72), drop the overbar because the noncorrelated terms have been eliminated, and group terms. The result is

$$S_0 = E_xE_x^* + E_yE_y^* = (E_{1x}E_{1x}^* + E_{1y}E_{1y}^*) + (E_{2x}E_{2x}^* + E_{2y}E_{2y}^*) \tag{80a}$$

$$S_1 = E_xE_x^* - E_yE_y^* = (E_{1x}E_{1x}^* - E_{1y}E_{1y}^*) + (E_{2x}E_{2x}^* - E_{2y}E_{2y}^*) \tag{80b}$$

$$S_2 = E_xE_y^* + E_yE_x^* = (E_{1x}E_{1y}^* + E_{1y}E_{1x}^*) + (E_{2x}E_{2y}^* + E_{2y}E_{2x}^*) \tag{80c}$$

$$S_3 = i(E_xE_y^* + E_yE_x^*) = i(E_{1x}E_{1y}^* + E_{1y}E_{1x}^*) + i(E_{2x}E_{2y}^* + E_{2y}E_{2x}^*) \tag{80d}$$

From (73) and (74) we see that we can then write (80) as

$$S_0 = S_0^{(1)} + S_0^{(2)} \tag{81a}$$

$$S_1 = S_1^{(1)} + S_1^{(2)} \tag{81b}$$

$$S_2 = S_2^{(1)} + S_2^{(2)} \tag{81c}$$

$$S_3 = S_3^{(1)} + S_3^{(2)} \tag{81d}$$

Thus, the Stokes parameters of two completely independent optical beams can be added and represented by the Stokes parameters of the combined beams. In terms of the Stokes vector, that is, column matrices, we can write (81) as

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} S_0^{(1)} \\ S_1^{(1)} \\ S_2^{(1)} \\ S_3^{(1)} \end{pmatrix} + \begin{pmatrix} S_0^{(2)} \\ S_1^{(2)} \\ S_2^{(2)} \\ S_3^{(2)} \end{pmatrix} \tag{82}$$

or simply

$$S = S^{(1)} + S^{(2)} \tag{83}$$

so the Stokes vectors, $S^{(i)}$, $i = 1, 2$, are also additive.

As a first application of this result, (82), we recall that the Stokes vector for unpolarized light is

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{84}$$

We also saw that the Stokes vector could be written in terms of the orientation angle ψ and the ellipticity χ as

$$S = I_0 \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{pmatrix} \tag{85}$$

Thus, we see from (82) that we can write (84), using (85), as

$$I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{I_0}{2} \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{pmatrix} + \frac{I_0}{2} \begin{pmatrix} 1 \\ -\cos 2\chi \cos 2\psi \\ -\cos 2\chi \sin 2\psi \\ -\sin 2\chi \end{pmatrix} \tag{86}$$

We can also express (82) in terms of beams (72) and (73) using (85) as

$$I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{I_0}{2} \begin{pmatrix} 1 \\ \cos 2\chi_1 \cos 2\psi_1 \\ \cos 2\chi_1 \sin 2\psi_2 \\ \sin 2\chi_1 \end{pmatrix} + \frac{I_0}{2} \begin{pmatrix} 1 \\ \cos 2\chi_2 \cos 2\psi_2 \\ \cos 2\chi_2 \sin 2\psi_2 \\ \sin 2\chi_2 \end{pmatrix} \tag{89}$$

Comparing the Stokes parameters in the second column in (87) with (86), we see that

$$\cos 2\chi_2 \cos 2\chi_2 = -\cos 2\chi_1 \cos 2\psi_1 \tag{88a}$$

$$\cos 2\chi_2 \sin \psi_2 = -\cos 2\chi_1 \sin 2\psi_1 \tag{88b}$$

$$\sin 2\chi_2 = -\sin 2\chi_1 \tag{88c}$$

From (88c) we can write

$$\sin 2\chi_2 = \sin(-2\chi_1) \tag{89a}$$

or

$$\chi_2 = -\chi_1 \tag{89b}$$

Thus, the ellipticity of beam 2 is opposite to beam 1. We now substitute (89b) into (88a) and (88b) and we have

$$\cos 2\psi_2 = -\cos 2\psi_1 \tag{90a}$$

$$\sin 2\psi_2 = -\sin 2\psi_1 \tag{90b}$$

Equation (90a) and (90b) can only be satisfied if

$$2\psi_1 = 2\psi_2 - \pi \tag{91a}$$

or

$$\psi_2 - \psi_1 + \frac{\pi}{2} \tag{91b}$$

Thus, the polarization ellipse for the second beam is oriented 90° ($\pi/2$) from the first beam. The conditions (89b) and (92b)

$$\chi_2 = -\chi_1 \tag{89b}$$

$$\psi_2 = \psi_1 + \frac{\pi}{2} \tag{91b}$$

are said to describe two polarization ellipses of oppose polarization. Thus, unpolarized light is a superposition or mixture of two beams of equal intensity and opposite polarization. As special cases of (86) we see that unpolarized light can be decomposed into (independent) beams of linear and circular polarized light; that is,

$$I_0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{I_0}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\frac{I_0}{2}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} \tag{92a}$$

$$I_0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{I_0}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\frac{I_0}{2}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \tag{92b}$$

$$I_0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{I_0}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}+\frac{I_0}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \tag{92c}$$

Of course, the intensity of each beam is half the intensity of the unpolarized beam.

We now return to our original problem of representing partially polarized light in terms of the Stokes vector. We call that the degree of polarization P is defined by

$$P=\frac{\sqrt{S_1^2+S_2^2+S_3^2}}{S_0}\qquad 0\le P\le 1 \tag{93}$$

This equation suggests that partially polarized light can be represented by a superposition of unpolarized light and completely polarized light by using (82). A little thought shows that if we have a beam of partially polarized light, which we can write as

$$S=\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix} \tag{94}$$

Equation (94) can be written as

$$S=\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix}=(1-P)\begin{pmatrix}S_0\\0\\0\\0\end{pmatrix}+P\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix}\qquad 0\le P\le 1 \tag{95}$$

The first Stokes vector on the right side of (95) represents unpolarized light, and the second Stokes vector represents completely polarized light. For $P=0$, unpolarized light, (95) reduces to

$$S=\begin{pmatrix}S_0\\0\\0\\0\end{pmatrix} \tag{96a}$$

and for $P=1$, completely polarized light, (95) reduces to

$$S=\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix} \tag{96b}$$

We note that S_0 on the left side of (95) always satisfies

$$S_0\ge\sqrt{S_1^2+S_2^2+S_3^2} \tag{97a}$$

whereas S_0 in the Stokes vector associated with P on the right side of (95) always satisfies

$$S_0=\sqrt{S_1^2+S_2^2+S_3^2} \tag{97b}$$

Another representation of partially polarized light in terms of P is the decomposition of a beam into two completely polarized beams of opposite polarizations, namely,

$$\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix}=\frac{1+P}{2P}\begin{pmatrix}PS_0\\S_1\\S_2\\S_3\end{pmatrix}+\frac{1-P}{2P}\begin{pmatrix}PS_0\\-S_1\\-S_2\\-S_3\end{pmatrix}\qquad 0<P\le 1 \tag{98a}$$

where

$$PS_0=\sqrt{S_1^2+S_2^2+S_3^2} \tag{98b}$$

Thus, partially polarized light can also be decomposed into two oppositely polarized beams.

While we have restricted this discussion to two beams, it is easy to see that we could have described the optical field in terms of n beams, that is, extended (83) to

$$S=S^{(1)}+S^{(2)}+S^{(3)}+\cdots+S^{(n)}$$

$$= \sum_{i=1}^{n} S^{(i)} \qquad i = 1, \ldots, n \tag{99}$$

We have not done this for the simple reason that, in practice, dealing with two beams is sufficient. Nevertheless, the reader should be aware that the additivity law can be extended to n beams. Lastly, we note that for partially polarized light the intensities of the two beams is given by

$$S_0^{(1)} = \frac{1}{2}S_0 - \frac{1}{2}\sqrt{S_1^2 + S_2^2 + S_3^2} \tag{100a}$$

$$S_0^{(2)} = \frac{1}{2}S_0 + \frac{1}{2}\sqrt{S_1^2 + S_2^2 + S_3^2} \tag{100b}$$

Only for unpolarized light are the intensities of the two beams equal. This is also shown by (98a).

It is of interest to express the parameters of the polarization ellipse in terms of the Stokes parameters. To do this, we recall that

$$S_0 = E_{0x}^2 + E_{0y}^2 = I_0 \tag{101a}$$

$$S_1 = E_{0x}^2 - E_{0y}^2 = I_0 \cos 2\alpha \tag{101b}$$

$$S_2 = 2E_{0x}E_{0y} \cos\delta = I_0 \sin 2\alpha \cos\delta \tag{101c}$$

$$S_3 = 2E_{0x}E_{0y} \sin\delta = I_0 \sin 2\alpha \sin\delta \tag{101d}$$

We can then write (101) as

$$E_{0x}^2 = \frac{S_0 + S_1}{2} \tag{102a}$$

$$E_{0y}^2 = \frac{S_0 - S_1}{2} \tag{102b}$$

$$\cos\delta = \frac{S_2}{2E_{0x}E_{0y}} \tag{102c}$$

$$\sin\delta = \frac{S_3}{2E_{0x}E_{0y}} \tag{102d}$$

We recall that the instantaneous polarization ellipse is

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - \frac{2E_xE_y}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta \tag{103}$$

Substituting (102) into the appropriate terms in (103) gives

$$\frac{2E_x^2}{S_0 + S_1} + \frac{2E_y^2}{S_0 - S_1} - \frac{4S_2E_xE_y}{S_0^2 - S_1^2} = \frac{S_3^2}{S_0^2 - S_1^2} \tag{104}$$

where we have used $E_{0x}^2E_{0y}^2 = (S_0^2 - S_1^2)/4$ from (102a) and (102b). Multiplying through (104) by $(S_0^2 - S_1^2)/S_3^2$ then yields

$$\frac{2(S_0 - S_1)E_x^2}{S_3^2} + \frac{2(S_0 + S_1)E_y^2}{S_3^2} - \frac{4S_2E_xE_y}{S_3^2} = 1 \tag{105}$$

We now write (105) as

$$Ax^2 - 2Cxy + 2By^2 = 1 \tag{106a}$$

where

$$A = \frac{2(S_0 - S_1)}{S_3^2} \tag{106b}$$

$$B = \frac{2(S_0 + S_1)}{S_3^2} \tag{106c}$$

$$C = \frac{2S_2}{S_3^2} \tag{106d}$$

and for convenience we have set $x = E_x$ and $y = E_y$.

We can now find the orientation and ratio of the axes in terms of the Stokes parameters (106). To do this we first express x and y in polar coordinates:

$$x = \rho\cos\phi \tag{107a}$$

$$y = \rho\sin\phi \tag{107b}$$

Substituting (107a) and (107b) into (106) we find

$$A\rho^2\cos^2\phi - 2C\rho^2\sin\phi\cos\phi + B\rho^2\sin^2\phi = 1 \tag{108}$$

Using the half-angle formulas for $\cos^2\phi$ and $\sin^2\phi$, (108) then becomes

$$\frac{A\rho^2(1 + \cos 2\phi)}{2} - C\rho^2\sin 2\phi + \frac{B\rho^2(1 - \cos 2\phi)}{2} = 1 \tag{109}$$

We now introduce the parameter L defined in terms of ρ as

$$L = \frac{2}{\rho^2} \tag{110}$$

substitute (110) into (109), and write

$$L = (A + B) - 2C\sin 2\phi + (A - B)\cos 2\phi \tag{111}$$

The major and minor axes of the ellipse correspond to maximum and minimum values of ρ, respectively, whereas L is a minimum and maximum, (110). The angle ϕ where this maximum and minimum occur can be found in the usual way by setting $dL/d\phi = 0$ and solving for ϕ. We, therefore, have from (111)

$$\frac{dL}{d\phi} = 4C\cos 2\phi - 2(A - B)\sin 2\phi = 0 \tag{112}$$

and

$$\frac{\sin 2\phi}{\cos 2\phi} = \tan 2\phi = \frac{-2C}{A-B} \tag{113}$$

Solving for ϕ, we find that

$$\phi = \frac{-1}{2}\tan^{-1}\frac{2C}{A-B} \tag{114}$$

To find the corresponding maximum and minimum values of L in (111), we must express $\sin 2\phi$ and $\cos 2\phi$ in terms of A, B, and C. We can find unique expressions for $\sin 2\phi$ and $\cos 2\phi$ from (113) by constructing the right triangle in Figure 5. We see from the right triangle that (113) is satisfied by

$$\sin 2\phi = \frac{-2C}{\sqrt{(2C)^2 + (A-B)^2}} \tag{115a}$$

$$\cos 2\phi = \frac{A-B}{\sqrt{(2C)^2 + (A-B)^2}} \tag{115b}$$

or

$$\sin 2\phi = \frac{2C}{\sqrt{(2C)^2 + (A-B)^2}} \tag{115c}$$

$$\cos 2\phi = \frac{-(A-B)}{\sqrt{(2C)^2 + (A-B)^2}} \tag{115d}$$

Substituting (115a) and (115b) into (111) yields

$$I_{\max} = (A+B) + \sqrt{(2C)^2 + (A-B)^2} \tag{116a}$$

and, similarly, substituting (115c) and (115d) into (111) yields

$$L_{\min} = (A+B) - \sqrt{(2C)^2 + (A-B)^2} \tag{116b}$$

We have written "max" and "min" on L in (116a) and (116b) to indicate that

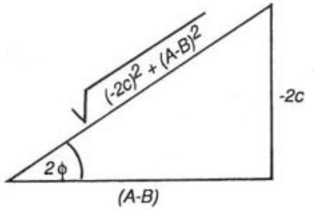

Figure 5 Right triangle corresponding to Equation (113).

these are the maximum and minimum values of L. We also note that (115a) and (115b) are related by

$$\sin 2\phi_1 = -\sin 2\phi_2 \tag{117a}$$

and (115b) and (115d) by

$$\cos 2\phi_1 = -\cos 2\phi_2 \tag{117b}$$

We see that (115a) and (115b) are satisfied by setting

$$\phi_2 = \phi_1 + \frac{\pi}{2} \tag{118}$$

Thus, the maximum and minimum lengths, that is, the major and minor axes, are at ϕ_1 and $\phi_1 + 90°$, respectively, which is exactly what we would expect. We thus see from (110) that

$$\rho_{\min}^2 = \frac{2}{L_{\max}} \tag{119a}$$

$$\rho_{\max}^2 = \frac{2}{L_{\min}} \tag{119b}$$

The ratio of the square of the lengths of the major axis to the minor axis is defined to be

$$R = \frac{\rho_{\max}^2}{\rho_{\min}^2} \tag{120}$$

so from (116a) and (116b) we have

$$R = \frac{(A+B) - \sqrt{(2C)^2 + (A-B)^2}}{(A+B) + \sqrt{(2C)^2 + (A-B)^2}} \tag{121}$$

We can now express (121) in terms of the Stokes parameters, (106b), (106c) and (106d) and we find that (121) becomes

$$R = \frac{S_0 - \sqrt{S_1^2 + S_2^2}}{S_0 + \sqrt{S_1^2 + S_2^2}} \tag{122}$$

Thus, we have found the relation between the length of the major and minor axes of the polarization ellipse and the Stokes parameters. This can be expressed directly by using (119) and (106) or as a ratio R given by (122).

Not surprisingly there are other interesting relations between the Stokes parameters and the parameters of the polarization ellipse. These relations are fundamental to the development of the Poincaré sphere, so we shall discuss them in Chapter 11.

and

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or

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$$\cos 2\phi = \frac{-(A - B)}{\sqrt{(2C)^2 + (A - B)^2}} \tag{115d}$$

Substituting (115a) and (115b) into (111) yields

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and, similarly, substituting (115c) and (115d) into (111) yields

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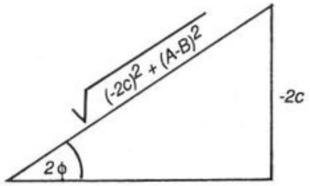

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$$\rho^2_{max} = \frac{2}{L_{min}} \tag{119b}$$

The ratio of the square of the lengths of the major axis to the minor axis is defined to be

$$R = \frac{\rho^2_{max}}{\rho^2_{min}} \tag{120}$$

so from (116a) and (116b) we have

$$R = \frac{(A + B) - \sqrt{(2C)^2 + (A - B)^2}}{(A + B) + \sqrt{(2C)^2 + (A - B)^2}} \tag{121}$$

We can now express (121) in terms of the Stokes parameters, (106b), (106c) and (106d) and we find that (121) becomes

$$R = \frac{S_0 - \sqrt{S_1^2 + S_2^2}}{S_0 + \sqrt{S_1^2 + S_2^2}} \tag{122}$$

Thus, we have found the relation between the length of the major and minor axes of the polarization ellipse and the Stokes parameters. This can be expressed directly by using (119) and (106) or as a ratio R given by (122).

Not surprisingly there are other interesting relations between the Stokes parameters and the parameters of the polarization ellipse. These relations are fundamental to the development of the Poincaré sphere, so we shall discuss them in Chapter 11.

Books

1. Born, M. and Wolf, E., *Principles of Optics*, 3rd ed., Pergamon Press, New York, 1965.
2. Shurcliff, W., *Polarized Light*, Harvard University Press, Cambridge, 1962.

5
The Mueller Matrices for Polarizing Components

5.1. INTRODUCTION

In the previous chapters we have concerned ourselves with the fundamental properties of polarized light. In this chapter we now turn our attention to the study of the interaction of polarized light with elements which can change its state of polarization and see that the matrix representation of the Stokes parameters leads to a very powerful mathematical tool for treating this interaction. In Figure 1 we

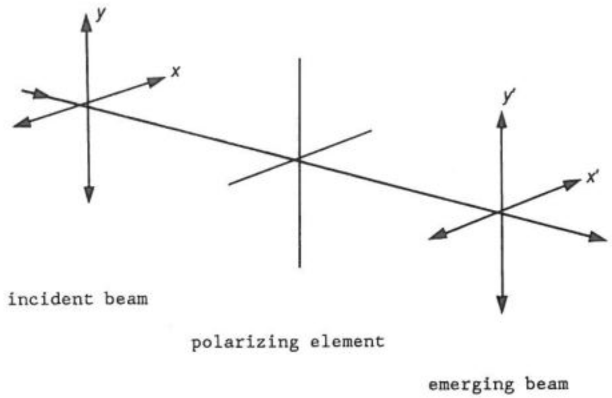

Figure 1 Interaction of a polarized beam with a polarizing element.

show an incident beam interacting with a polarizing element and the emerging beam. In Figure 1 the incident beam is characterized by its Stokes parameters S_i, where $i = 0, 1, 2, 3$. The incident polarized beam interacts with the polarizing medium, and the emerging beam is characterized by a new set of Stokes parameters S'_1, where, again, $i = 0, 1, 2, 3$. We now assume that S'_1 can be expressed as a linear combination of the four Stokes parameters of the incident beam by the relations

$$S'_0 = m_{00}S_0 + m_{01}S_1 + m_{02}S_2 + m_{03}S_3 \tag{1a}$$

$$S'_1 = m_{10}S_1 + m_{11}S_1 + m_{12}S_2 + m_{13}S_3 \tag{1b}$$

$$S'_2 = m_{20}S_2 + m_{21}S_1 + m_{22}S_2 + m_{23}S_3 \tag{1c}$$

$$S'_3 = m_{30}S_3 + m_{31}S_1 + m_{32}S_2 + m_{33}S_3 \tag{1d}$$

Equation (1) can be written in terms of the Stokes vector (Stokes column matrix). In matrix form (1) is written as

$$\begin{pmatrix} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{2}$$

Equation (2) can be simply represented as a matrix equation, namely,

$$\mathbf{S}' = \mathbf{M} \cdot \mathbf{S} \tag{3}$$

The 4×4 matrix in (2) is known as the Mueller matrix. It was introduced by H. Mueller during the early 1940s. While Mueller appears to have based his 4×4 matrix on a paper by F. Perrin and a still earlier paper by P. Soleillet, his name is firmly attached to it in the optical literature. Mueller's important contribution was that he, apparently, was the first to describe polarizing components in terms of his Mueller matrices. Remarkably, Mueller never published his work on his matrices. Their appearance in the optical literature was due to others, such as N. G. Park III, who published Mueller's ideas along with his own contributions and others shortly after the end of the Second World War.

In nature, when an optical beam interacts with matter its polarization state is almost always changed. In fact, this appears to be the rule rather than the exception. The polarization state can be changed by (1) changing the amplitudes, (2) the phase, or (3) the direction of the orthogonal field components. An optical element which changes the orthogonal amplitudes unequally is called a *polarizer* or *diattenuator*. Similarly, an optical device which introduces a phase shift between the orthogonal components is called a *retarder*; other names used for the same device are wave plate, compensator, or phase shifter. Finally, if the optical device rotates the orthogonal components of the beam through an angle θ as it propagates through the element, it is called a *rotator*. These effects are easily

understood by writing the transverse field components for a plane wave:

$$E_x(z,t) = E_{0x}\cos(\omega t - \kappa z + \delta_x) \tag{4a}$$

$$E_y(z,t) = E_{0y}\cos(\omega t - \kappa z + \delta_y) \tag{4b}$$

Equation (4) can be changed by varying the amplitudes, E_{0x} or E_{0y}, or the phase, δ_x or δ_y and, finally, the direction of $E_x(z,t)$ and $E_y(z,t)$. The corresponding devices for causing these changes are the polarizer, retarder, and rotator. The use of the names *polarizer* and *retarder* arose, historically, before the behavior of these polarizing elements was fully understood. The preferable names would be diattenuator for a polarizer and phase shifter for the retarder. All three polarizing elements change the polarization state of an optical beam.

In the following sections we derive the Mueller matrices for these polarizing elements. We then apply the Mueller matrix calculus to a number of problems of great interest and see its great utility.

5.2. THE MUELLER MATRIX OF A POLARIZER

A polarizer is an optical element which attenuates the orthogonal components of an optical beam unequally; that is, a polarizer is an anisotropic attenuator; the two orthogonal transmission axes are designated p_x and p_y. Recently, it has also been called a diattenuator. If the orthogonal components of the incident beam are attenuated equally, then the polarizer becomes a neutral density filter. We now derive the Mueller matrix for a polarizer.

In Figure 2 a polarized beam is shown incident on a polarizer along with the emerging beam. The components of the incident beam are represented by E_x and E_y, respectively. After the beam emerges from the polarizer the components are E'_x and E'_y, and they are parallel to the original axes. The fields are related by

$$E'_x = p_x E_x \qquad 0 \le p_x \le 1 \tag{5a}$$

$$E'_y = p_y E_y \qquad 0 \le p_y \le 1 \tag{5b}$$

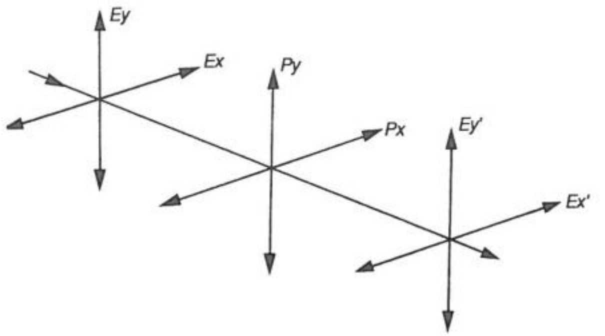

Figure 2 The Mueller matrix of a polarizer with attenuation coefficients p_x and p_y.

The factors p_x and p_y are the amplitude attenuation coefficients along its orthogonal transmission axes. For no attenuation or perfect transmission along an orthogonal axis $p_x(p_y) = 1$, whereas for complete attenuation $p_x(p_y) = 0$. If one of the axes has an absorption coefficient which is zero so that there is no transmission along this axis, the polarizer is said to have only a single transmission axis.

The Stokes polarization parameters of the incident and emerging beams are, respectively,

$$S_0 = E_x E_x^* + E_y E_y^* \tag{6a}$$

$$S_1 = E_x E_x^* - E_y E_y^* \tag{6b}$$

$$S_2 = E_x E_y^* + E_y E_x^* \tag{6c}$$

$$S_3 = i(E_x E_y^* - E_y E_x^*) \tag{6d}$$

and

$$S_0' = E_x' E_x'^* + E_y' E_y'^* \tag{7a}$$

$$S_1' = E_x' E_x'^* - E_y' E_y'^* \tag{7b}$$

$$S_2' = E_x' E_y'^* + E_y' E_x'^* \tag{7c}$$

$$S_3' = i(E_x' E_y'^* - E_y' E_x'^*) \tag{7d}$$

Substituting (5) into (7) and using (6), we then find

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & 0 & 2p_x p_y & 0 \\ 0 & 0 & 0 & 2p_x p_y \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{8}$$

The 4×4 matrix in (8) is written by itself as

$$M = \frac{1}{2} \begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & 0 & 2p_x p_y & 0 \\ 0 & 0 & 0 & 2p_x p_y \end{pmatrix} \qquad 0 \le p_{x,y} \le 1 \tag{9}$$

Equation 9 is the Mueller matrix for a polarizer. In general, the existence of the m_{33} term shows that the polarization of emerging beam of light will be elliptically polarized.

For a neutral density filter $p_x = p_y = p$ and (9) becomes

$$M = p^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

which is a unit diagonal matrix. Equation (10) shows that the polarization state is not changed by a neutral density filter, but the intensity of the incident beam is reduced by a factor of p^2. This is the expected behavior of a neutral density filter, since it only affects the magnitude the intensity and not the polarization state. According to (10), the emerging intensity I' is then

$$I' = p^2 I \tag{11}$$

where I is the intensity of the incident beam.

Equation (9) is the Mueller matrix for a polarizer which is described by unequal attenuation along the p_x and p_y axes. An *ideal* linear polarizer is one which has transmission along only one axis and no transmission along the other axis. This behavior can be described by first setting, say, $p_y = 0$. Then (9) reduces to

$$M = \frac{p_x^2}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

Equation (12) is the Mueller matrix for an ideal linear polarizer which polarizes only along the x axis. It is most often called a linear horizontal polarizer. It would be a perfect linear polarizer if the transmission factor p_x was unity ($p_x = 1$). Thus, the Mueller matrix for an ideal perfect linear polarizer with its transmission axis in the x direction is

$$M = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

We note that the maximum intensity of the emerging beam which can be obtained is only 50% of the original intensity. Thus, the use of an ideal polarizer reduces the intensity by a factor of 1/2; it is the price we pay for obtaining perfectly polarized light. It is called a linear polarizer because it affects a linearly polarized beam in a unique manner as we shall soon see.

In general, all polarizers are described by (9). However, there is only one known natural material which comes close to approaching the perfect ideal polarizer described by (13), and this is calcite. A synthetic material known as Polaroid is also used as a polarizer. It performance is not as good as calcite, but its cost is very low in comparison to natural calcite polarizers, e.g., a Glan–Thompson prism. Nevertheless, there are a few types of Polaroid which perform extremely well as "ideal" polarizers. We shall discuss the topic of calcite and Polaroid polarizers in a separate chapter.

If an ideal polarizer is used in which the role of the transmission axes is reversed, that is, $p_x = 0$ and $p_y = 1$, then (9) reduces to

$$M = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

which is the Mueller matrix for a linear vertical polarizer.

Finally, it is convenient to rewrite the Mueller matrix, (9), of a polarizer in terms of trigonometric functions. This can be done by setting

$$p_x^2 + p_y^2 = p^2 \tag{15a}$$

and

$$p_x = p\cos\alpha \qquad p_y = p\sin\alpha \tag{15b}$$

Substituting (15) into (9) yields

$$M = \frac{p^2}{2}\begin{pmatrix} 1 & \cos 2\alpha & 0 & 0 \\ \cos 2\alpha & 1 & 0 & 0 \\ 0 & 0 & \sin 2\alpha & 0 \\ 0 & 0 & 0 & \sin 2\alpha \end{pmatrix} \tag{16}$$

where $0 \le \alpha \le 90°$. For an ideal perfect linear polarizer $p = 1$. For a linear horizontal polarizer $\alpha = 0$, and for a linear vertical polarizer $\alpha = 90°$. The usefulness of the trigonometric form of the Mueller matrix, (16), will appear later.

The reason for calling (13) a linear polarizer is due to the following fact. Suppose we have an incident beam of arbitrary intensity and polarization so its Stokes vector is

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{17}$$

We now matrix multiply (17) by (13) or (14), and we can write

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ \pm 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{18}$$

Carrying out the matrix multiplication in (18), we find

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \frac{1}{2}(S_0 \pm S_1)\begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

Inspecting (19), we see that the Stokes vector of the emerging beam is always *linearly* horizontally (+) or vertically (−) polarized. Thus, an ideal linear polarizer always creates linearly polarized light regardless of polarization state of the incident beam. The simple fact is, however, that because the factor $2p_xp_y$ in (9) is never zero there is no known perfect linear polarizer and all polarizers create elliptically polarized light. While the ellipticity may be small and, in fact, negligible, there is always some present.

The above behavior of linear polarizers allows us to develop a test to determine if a polarizing element is actually a linear polarizer. If a linear polarizer is used to create linearly polarized light, we call it a *generator*. If it is used to analyze polarized light, it is called an *analyzer*. The test to determine if we have a linear polarizer is shown in Figure 3. In the test we assume that we have a linear polarizer and set its axis in the horizontal (H) direction. We then take another polarizer and set its axis in the vertical (V) direction as shown in the figure. The Stokes vector of the incident beam is S, and the Stokes vector of the beam emerging from the first polarizer (horizontal) is

$$S' = M_H S \tag{20}$$

Next, the S' beam propagates to the second polarizer (vertical), and the Stokes vector S'' of the emerging beam is now

$$S'' = M_V S' = M_V M_H S = MS \tag{21}$$

where we have used (20). We see that M is the Mueller matrix of the combined vertical and linear polarizer,

$$M = M_V M_H \tag{22}$$

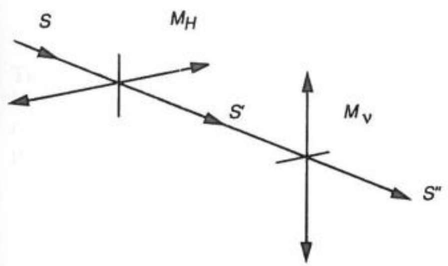

Figure 3 Testing for a linear polarizer.

where M_H and M_V are given by (13) and (14), respectively. This result, (21) and (22), shows that we can relate the Stokes vector of the emerging beam to the incident beam by merely matrix-multiplying the Mueller matrix of each component and finding the resulting Mueller matrix. In general, the matrices do not commute. We now carry out (22) and write, using (13) and (14),

$$M = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

Thus, we obtain a null Mueller matrix and, hence, a null output intensity regardless of the polarization state of the incident beam. The appearance of a null Mueller matrix (or intensity) occurs only when the linear polarizers are in the *crossed polarizer* configuration. Furthermore, the null Mueller matrix always arises whenever the polarizers are crossed, regardless of the angle of the transmission axis of the first polarizer.

5.3. THE MUELLER MATRIX OF A RETARDER

A retarder is a polarizing element which changes the phase of the optical beam. Strictly speaking, its correct name is phase shifter. However, historical usage has led to the alternative names retarder, wave plate, and compensator. Retarders introduce a phase shift of ϕ between the orthogonal components of the incident field. This is accomplished by causing a phase shift of $+\phi/2$ along the x axis and a phase shift of $-\phi/2$ along the y axis. In optics these axes are referred to as the *fast* and *slow* axes, respectively. In Figure 4 we show the incident and emerging beam and the retarder. The components of the emerging beam are related to the incident beam by

$$E_x'(z,t) = e^{+i\phi/2}E_x(z,t) \tag{24a}$$

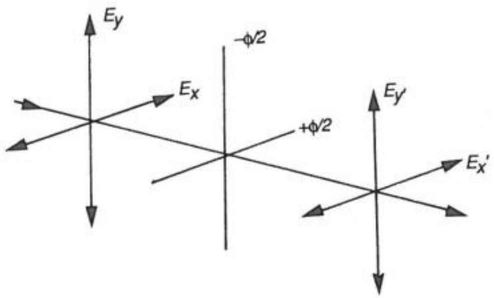

Figure 4 Propagation of a polarized beam through a retarder.

$$E_y'(z,t) = e^{-i\phi/2}E_y(z,t) \tag{24b}$$

Referring again to the definition of the Stokes parameters (6) and (7) and substituting (24a) and (24b) into these equations, we find that

$$S_0' = S_0 \tag{25a}$$

$$S_1' = S_1 \tag{25b}$$

$$S_2' = S_2\cos\phi - S_3\sin\phi \tag{25c}$$

$$S_3' = S_2\sin\phi + S_3\cos\phi \tag{25d}$$

Equation 25 can be written in matrix form as

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{26}$$

Note that for an ideal phase shifter (retarder) there is no loss in intensity; that is, $S_0' = S_0$.

The Mueller matrix for a retarder with a phase shift ϕ is, from (26),

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{27}$$

There are two special cases of (27) which appear often in polarizing optics. These are the cases for quarter-wave retarders ($\phi = 90°$) and half-wave retarders ($\phi = 180°$), respectively. For a quarter-wave retarder (27) becomes

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{28}$$

The quarter-wave retarder has the property that it transforms a linearly polarized beam with its axis at $+45°$ or $-45°$ into a right or left circularly polarized beam, respectively. To show this property, consider the Stokes vector for a linearly polarized $\pm 45°$ beam,

$$S = I_0\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix} \tag{29}$$

Matrix-multiplying (29) with (28) yields

$$S' = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix} \tag{30}$$

which is the Stokes vector for right (left) circularly polarized light. The transformation of linearly polarized light to circularly polarized light is an important application of quarter-wave retarders. However, circularly polarized light is obtained only if the incident linearly polarized light is oriented at $\pm 45°$.

On the other hand, if the incident light is right (left) circularly polarized light, then matrix-multiplying (30) by (28) yields

$$S' = I_0 \begin{pmatrix} 1 \\ 0 \\ \mp 1 \\ 0 \end{pmatrix} \tag{31}$$

which is the Stokes vector for linear $-45°$ or $+45°$ polarized light. The quarter-wave retarder can be used to transform linearly polarized light to circularly polarized light or circularly polarized light to linearly polarized light.

The other important type of wave retarder is the half-wave retarder ($\phi = 180°$). For this condition (28) reduces to

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{32}$$

A half-wave retarder is characterized by a diagonal matrix. The terms $m_{22} = m_{33} = -1$ reverse the ellipticity and orientation of the polarization state of the incident beam. To show this formally, we have initially

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{33}$$

We also saw previously that the orientation angle ψ and the ellipticity angle χ is given in terms of the Stokes parameters:

$$\tan 2\psi = \frac{S_2}{S_1} \tag{34a}$$

$$\sin 2\chi = \frac{S_3}{S_0} \tag{34b}$$

Matrix-multiplying (33) and (32) gives

$$S' = \begin{pmatrix} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{pmatrix} = \begin{pmatrix} S_0 \\ S_1 \\ -S_2 \\ -S_3 \end{pmatrix} \tag{35}$$

where

$$\tan 2\psi' = \frac{S'_2}{S'_1} \tag{36a}$$

$$\sin 2\chi' = \frac{S'_3}{S'_0} \tag{36b}$$

Substituting (35) into (36) yields

$$\tan 2\psi' = \frac{-S_2}{S_1} = -\tan 2\psi \tag{37a}$$

$$\sin 2\chi' = \frac{-S_3}{S_0} = -\sin 2\chi \tag{37b}$$

Hence,

$$\psi' = 90° - \psi \tag{38a}$$

$$\chi' = 90° + \chi \tag{38b}$$

Half-wave retarders also possess the property that they can rotate the polarization ellipse. This important property shall be discussed later in the chapter on the rotation of a polarizer and retarder.

5.4. THE MUELLER MATRIX OF A ROTATOR

The final way to change the polarization state of an optical field is to allow a beam to propagate through a polarizing element which rotates the orthogonal field components $E_x(z,t)$ and $E_y(z,t)$ through an angle θ. In order to derive the Mueller matrix for rotation, we consider Figure 5. The angle θ describes the rotation of E_x to E'_x and of E_y to E'_y. Similarly, the angle β is the angle between E and E_x. In the figure the point P is described in the E'_x, E'_y coordinate system by

$$E'_x = E\cos(\beta - \theta) \tag{39a}$$

$$E'_y = E\sin(\beta - \theta) \tag{39b}$$

In the E_x, E_y coordinate system we have

$$E_x = E\cos\beta \tag{40a}$$

$$E_y = E\sin\beta \tag{40b}$$

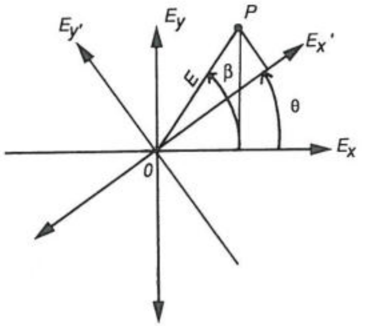

Figure 5 Rotation of the optical field components by a rotator.

Expanding the trigonometric functions in (39) gives

$$E_x' = E(\cos\beta\cos\theta + \sin\beta\sin\theta) \tag{41a}$$

$$E_y' = E(\sin\beta\cos\theta - \sin\theta\cos\beta) \tag{41b}$$

Collecting terms in (41) using (40) then gives

$$E_x' = E_x\cos\theta + E_y\sin\theta \tag{42a}$$

$$E_y' = -E_x\sin\theta + E_y\cos\theta \tag{42b}$$

Equations (42a) and (42b) are the amplitude equations for rotation. In order to find the Mueller matrix we form the Stokes parameters for (42) as before and find the Mueller matrix for rotation:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{43}$$

We note that a physical rotation of θ leads to the appearance of 2θ in (43) rather than θ because we are working in the intensity domain; in the amplitude domain we would expect just θ.

Rotators are primarily used to change the orientation angle of the polarization ellipse. To see this behavior, suppose the orientation angle of an incident beam is ψ. Then we can write

$$\tan 2\psi = \frac{S_2}{S_1} \tag{44}$$

For the emerging beam we have a similar expression with the variables in (44) replaced with primed variables. Using (43) we see that the orientation angle ψ'

is then

$$\tan 2\psi' = \frac{-S_1\sin 2\theta + S_2\cos 2\theta}{S_1\cos 2\theta + S_2\sin 2\theta} \tag{45}$$

Equation (44) is now written as

$$S_2 = S_1\tan 2\psi \tag{46}$$

Substituting (46) into (45), we readily find that

$$\tan 2\psi' = \tan(2\psi - 2\theta) \tag{47}$$

so

$$\psi' = \psi - \theta \tag{48}$$

Equation (48) shows that a rotator merely rotates the polarization ellipse of the incident beam; the ellipticity remains unchanged. The sign is negative in (48) because the rotation is clockwise. If the rotation is counterclockwise, that is, θ is replaced by $-\theta$ in (43), then we find

$$\psi' = \psi + \theta \tag{49}$$

In the derivation of the Mueller matrices for a polarizer, retarder, and rotator, we have assumed that the axes of these devices are aligned along the E_x and E_y (or x,y axes), respectively. In practice, we find that the polarization elements are often rotated. Consequently, it is also necessary for us to know the form of the Mueller matrices for the rotated polarizing elements. We now consider this problem.

5.5. THE MUELLER MATRICES FOR ROTATED POLARIZING COMPONENTS

To derive the Mueller matrix for rotated polarizing components, we refer to Figure 6. The axes of the polarizing component are seen to be rotated through an angle θ to the x' and y' axes. We must, therefore, also consider the components of the incident beam along the x' and y' axes. In terms of the Stokes vector of the incident beam, S, we then have

$$S' = M_R(2\theta)\cdot S \tag{50}$$

where $M_R(2\theta)$ is the Mueller matrix for rotation and S' is the Stokes vector of the beam whose axes are along x' and y'.

The S' beam now interacts with the polarizing element characterized by its Mueller matrix M. The Stokes vector S'' of the beam emerging from the rotated polarizing component is

$$S'' = M\cdot S' = M\cdot M_R(2\theta)\cdot S \tag{51}$$

where we have used (50). Finally, we must take the components of the emerging beam along the original x and y axes as seen in Figure 6. This can be described

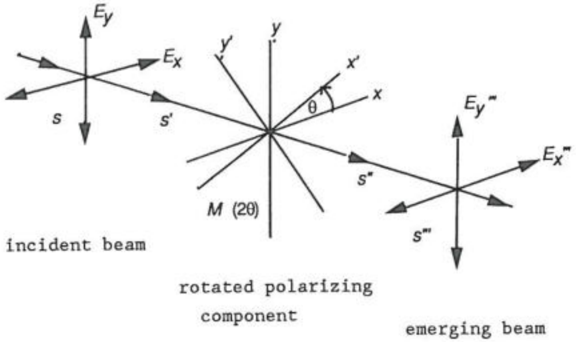

Figure 6 Derivation of the Mueller matrix for rotated polarizing components.

by a counterclockwise rotation of S'' through $-\theta$ and back to the original x,y axes, so

$$S''' = M_R(-2\theta)\cdot S''$$
$$= [M_R(-2\theta)\cdot M\cdot M_R(2\theta)]\cdot S \qquad (52)$$

where $M_R(-2\theta)$ is, again, the Mueller matrix for rotation and S''' is the Stokes vector of the emerging beam. Equation (52) can be written as

$$S''' = M(2\theta)\cdot S \qquad (53)$$

where

$$M(2\theta) = M_R(-2\theta)\cdot M\cdot M_R(2\theta) \qquad (54)$$

Equation (54) is the Mueller matrix of a rotated polarizing component. We recall that the Mueller matrix for rotation $M_R(2\theta)$ is given by

$$M_R(2\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (55)$$

The rotated Mueller matrix expressed by (54) appears often in the treatment of polarized light. Of particular interest are the Mueller matrices for a rotated polarizer and a rotated retarder. The Mueller matrix for a rotated "rotator" is also interesting, but in a different way. We recall that a rotator rotates the polarization ellipse by an amount θ. If the rotator is now rotated through an angle α, then one discovers, using (54), that $M(2\theta) = M_R(2\theta)$; that is, the rotator is unaffected by a mechanical rotation. Thus, the polarization ellipse cannot be rotated by rotating a rotator! The rotation comes about only by the intrinsic behavior of the rotator.

It is possible, however, to rotate the polarization ellipse mechanically by rotating a half-wave plate, as we shall soon demonstrate.

The Mueller matrix for a rotated polarizer is most conveniently found by expressing the Mueller matrix of a polarizer in angular form, namely,

$$M = \frac{p^2}{2}\begin{pmatrix} 1 & \cos 2\alpha & 0 & 0 \\ \cos 2\alpha & 1 & 0 & 0 \\ 0 & 0 & \sin 2\alpha & 0 \\ 0 & 0 & 0 & \sin 2\alpha \end{pmatrix} \qquad (56)$$

Carrying out the matrix multiplication according to (54) and using (55), the Mueller matrix for a rotated polarizer is

$$M = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\alpha\cos 2\theta & \cos 2\alpha \sin 2\theta & 0 \\ \cos 2\alpha\cos 2\theta & \cos^2 2\theta + \sin 2\alpha \sin^2 2\theta & (1-\sin 2\alpha)\sin 2\theta\cos 2\theta & 0 \\ \cos 2\alpha \sin 2\theta & (1-\sin 2\alpha)\sin 2\theta\cos 2\theta & \sin^2 2\theta + \sin 2\alpha\cos^2 2\theta & 0 \\ 0 & 0 & 0 & \sin 2\alpha \end{pmatrix} \qquad (57)$$

In (57) we have set p^2 to unity. We note that a linear vertical polarizer, a neutral density filter, and a linear vertical polarizer correspond to $\alpha = 0°$, 45°, and 90°, respectively.

The most common form of (57) is the Mueller matrix for an ideal linear horizontal polarizer ($\alpha = 0°$). For this value (57) reduces to

$$M_P(2\theta) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta\cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (58)$$

In (58) we have written $M_P(2\theta)$ to indicate that this is the Mueller matrix for a rotated ideal linear polarizer. The form of (58) can be checked immediately by setting $\theta = 0$ (no rotation). Upon doing this, we obtain the Mueller matrix of a linear horizontal polarizer:

$$M_P(0°) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (59)$$

One can readily see that for $\theta = 45°$ and 90° Eq. (58) reduces to the Mueller matrix for an ideal linear +45° and vertical polarizer, respectively. The Mueller matrix for a rotated ideal linear polarizer, (58), appears often in the generation and analysis of polarized light.

Next we turn to determining the Mueller matrix for a retarder or wave plate. We recall that the Mueller matrix for a wave plate is given by

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{60}$$

From (54) the Mueller matrix for the rotated wave plate, (60) is found to be

$$M_c(\phi, 2\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \cos\phi \sin^2 2\theta & (1-\cos\phi)\sin 2\theta \cos 2\theta & \sin\phi \sin 2\theta \\ 0 & (1-\cos\phi)\sin 2\theta \cos 2\theta & \sin^2 2\theta + \cos\phi \cos^2 2\theta & -\sin\phi \cos 2\theta \\ 0 & -\sin\phi \sin 2\theta & \sin\phi \cos 2\theta & \cos\phi \end{pmatrix} \tag{61}$$

For $\theta = 0°$, (61) reduces to (60) as expected. There is a particularly interesting form of (61). This is for a phase shift of $\phi = 180°$, a so-called half-wave plate. For $\phi = 180°$ (61) reduces to

$$M_c(180°, 4\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\theta & \sin 4\theta & 0 \\ 0 & \sin 4\theta & -\cos 4\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{62}$$

Equation (62) looks very similar to the Mueller matrix for rotation $M_R(2\theta)$, (55), which we write simply as M_R,

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{63}$$

However, (62) differs from (63) in some essential ways. The first is the ellipticity. The Stokes vector of an incident beam is

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{64}$$

Matrix-multiplying (64) with (63) yields the Stokes vector S':

$$S' = \begin{pmatrix} S_0 \\ S_1 \cos 2\theta + S_2 \sin 2\theta \\ -S_1 \sin 2\theta + S_2 \cos 2\theta \\ S_3 \end{pmatrix} \tag{65}$$

The ellipticity angle χ' is

$$\sin 2\chi' = \frac{S_3'}{S_0'} \qquad \frac{S_3}{S_0} = \sin 2\chi \tag{66}$$

Thus, the ellipticity is not changed under true rotation. Matrix-multiplying (64) and (62), however, yields a Stokes vector S':

$$S' = \begin{pmatrix} S_0 \\ S_1 \cos 4\theta + S_2 \sin 4\theta \\ -S_1 \sin 4\theta - S_2 \cos 4\theta \\ -S_3 \end{pmatrix} \tag{67}$$

The ellipticity angle χ' is now

$$\sin 2\chi' = \frac{S_3'}{S_0'} = \frac{-S_3'}{S_0} = -\sin 2\chi \tag{68}$$

Thus,

$$\chi' = \chi + 90° \tag{69}$$

so the ellipticity angle χ of the incident beam is advanced 90° by using a rotated half-wave plate.

The next difference is for the orientation angle ψ'. For a rotator, (63), the orientation angle associated with the incident beam, ψ, is

$$\tan 2\psi = \frac{S_2}{S_1} \tag{70}$$

so we immediately find from (70) and (65) that

$$\tan 2\psi' = \frac{\sin 2\psi \cos 2\theta - \sin 2\psi \cos 2\psi}{\cos 2\psi \cos 2\theta + \sin 2\psi \sin 2\theta} \tag{71}$$

whence

$$\psi' = \psi - \theta \tag{72}$$

Equation (72) shows that a mechanical rotation in θ increases ψ by the same amount and in the same direction (by definition, a clockwise rotation of θ increases). On the other hand, for a half-wave plate the orientation angle ψ', (67), is

$$\tan 2\psi' = \frac{S_1 \sin 4\theta - S_2 \cos 4\theta}{S_1 \cos 4\theta + S_2 \sin 4\theta} \tag{73}$$

Substituting (71) into (73), we find

$$\tan 2\psi' = \frac{\cos 2\psi \sin 4\theta - \sin 2\psi \cos 4\theta}{\cos 2\psi \cos 4\theta + \sin 2\psi \sin 4\theta} \tag{74}$$

so

$$\psi' = 2\theta - \psi \tag{75a}$$

and

$$\psi' = -(\psi - 2\theta) \tag{75b}$$

Comparing (75b) with (72), we see that rotating the half-wave plate clockwise causes ψ' to rotate counterclockwise by an amount twice that of a rotator. Because the rotation of a half-wave plate is opposite to a true rotator, it is called a *pseudorotator*. When a mechanical rotation of θ is made using a half-wave plate, the polarization ellipse is rotated by 2θ and in a direction opposite to the direction of the mechanical rotation. For a true mechanical rotation of θ the polarization ellipse is rotated by an amount θ and in the same direction as the rotation.

This discussion of rotation of half-wave plate is more than academic, however. Very often manufacturers sell half-wave plates as polarization rotators. Strictly speaking, this belief is quite correct. However, one must realize that the use of a half-wave plate rather than a true rotator requires a mechanical mount with twice the resolution. That is, if we use a rotator in a mount with, say, 2′ of resolution, then in order to obtain the same resolution with a half wave plate a mechanical mount with 1′ of resolution is required. The simple fact is that *doubling* the resolution of a mechanical mount can be very expensive in comparison with using a true rotator. The cost for doubling the resolution of a mechanical mount can easily double, whereas the cost increase between a quartz rotator and a half-wave plate is usually much less. In general, if the objective is to rotate the polarization ellipse by a known fixed amount, it is better to use a rotator rather than a half-wave plate.

The use of a half-wave plate as a "rotator" is very useful. Another important property of half-wave plates is that it can be used to reverse the polarization state. In order to illustrate this behavior, consider that we have an incident beam which is right or left circularly polarized. Its Stokes vector is

$$S = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix} \tag{76}$$

Matrix-multiplying (76) with (62) yields (we set $\theta = 0°$ in (62)),

$$S' = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \mp 1 \end{pmatrix} \tag{77}$$

We see that we again obtain circularly polarized light but opposite to its original state; that is, right circularly polarized light is transformed to left circularly polarized light, and vice versa. Similarly, if we have incident linear +45° polarized light, the emerging beam is linear −45° polarized light. It is this property of reversing the ellipticity and the orientation, manifested by the negative sign in the m_{22} and m_{33}, that also makes half-wave plates very useful.

Finally, we consider the Mueller matrix of a rotated quarter-wave plate. We set $\phi = 90°$ in (62) and we have

$$M_c(90°, 2\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta & \sin 2\theta \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta \cos 2\theta & \sin^2 2\theta & -\cos 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \end{pmatrix} \tag{78}$$

Consider that we have an incident linearly horizontally polarized beam, so its Stokes vector is ($I_0 = 1$)

$$S = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{79}$$

We multiply (79) by (78), and we find the Stokes vector S' is

$$S' = \begin{pmatrix} 1 \\ \cos^2 2\theta \\ \sin 2\theta \cos 2\theta \\ -\sin 2\theta \end{pmatrix} \tag{80}$$

We see immediately from (80) that the orientation angle ψ' and the ellipticity angle χ' of the emerging beam are given by

$$\tan 2\psi' = \tan 2\theta \tag{81a}$$

$$\sin 2\chi' = -\sin 2\theta \tag{81b}$$

Thus, the rotated quarter-wave plate has the property that it can be used to generate any desired orientation and ellipticity starting with an incident linearly horizontally polarized beam. However, we can only select one of these parameters; we have no control over the other parameter. We also note that if we initially have right or left circularly polarized light the Stokes vector of the output beam is

$$S' = \begin{pmatrix} 1 \\ \sin 2\theta \\ \mp\cos 2\theta \\ 0 \end{pmatrix} \tag{82}$$

which is the Stokes vector for linearly polarized light. While it is well known that a quarter-wave retarder can be used to create linearly polarized light, (82) shows that an additional variation is possible by rotating the retarder, namely, the orientation can be controlled.

Equation (80) shows that we can generate any desired orientation or ellipticity of a beam, but not both. This leads to the question of how we can generate an elliptically polarized beam of any desired orientation and ellipticity regardless of the polarization state of an incident beam.

5.6. THE GENERATION OF ELLIPTICALLY POLARIZED LIGHT

In the previous section we derived the Mueller matrices for a rotated polarizer and a rotated retarder. We now apply these matrices to generating an elliptically polarized beam of any desired orientation and ellipticity. In order to do this we refer to Figure 7. In the figure we show an incident beam of arbitrary polarization. The beam propagates first through an ideal polarizer rotated through an angle θ and then through a retarder, with its fast axis along the x axis. The Stokes vector

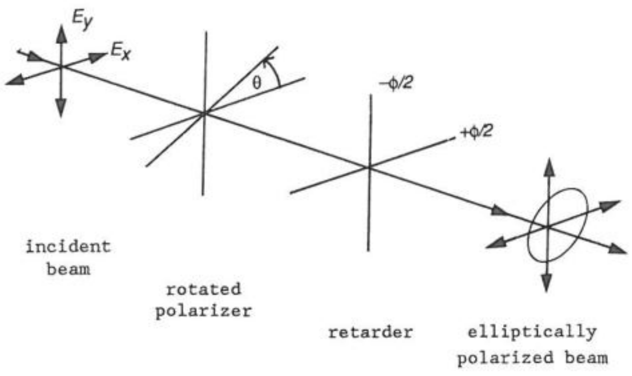

Figure 7 The generation of elliptically polarized light.

of the incident beam is

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{83}$$

It is important that we consider the optical source to be arbitrarily polarized. At first sight, for example, we might wish to use unpolarized light or linearly polarized light. However, unpolarized light is surprisingly difficult to generate, and the requirement to generate ideal linearly polarized light calls for an excellent linear polarizer. We can avoid this problem if we consider that the incident beam is of unknown but arbitrary polarization. Our objective is to create an elliptically polarized beam of any desired ellipticity and orientation and which is totally independent of the polarization state of the incident beam.

The Mueller matrix of a rotated ideal linear polarizer is

$$M_P(2\theta) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta\cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta\cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{84}$$

Matrix-multiplying (83) by (84) yields

$$S' = \frac{1}{2}(S_0 + S_1\cos 2\theta + S_2\sin 2\theta)\begin{pmatrix} 1 \\ \cos 2\theta \\ \sin 2\theta \\ 0 \end{pmatrix} \tag{85}$$

The Mueller matrix of the retarder or compensator (nonrotated) is

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{86}$$

Matrix-multiplying (85) by (86) then gives the Stokes vector of the beam emerging from the retarder,

$$S'' = I(\theta)\begin{pmatrix} 1 \\ \cos 2\theta \\ \cos\phi\sin 2\theta \\ \sin\phi\sin 2\theta \end{pmatrix} \tag{87a}$$

where

$$I(\theta) = \frac{1}{2}(S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta) \tag{87b}$$

Equation (87a) is the Stokes vector of an elliptically polarized beam. We immediately find from (87a) that the orientation angle ψ (we drop the double prime) is

$$\tan 2\psi = \cos\phi \tan 2\theta \tag{88a}$$

and the ellipticity angle χ is

$$\sin 2\chi = \sin\phi \sin 2\theta \tag{88b}$$

We must now determine the θ and ϕ which will generate the desired values of ψ and χ. We divide (88a) by $\tan 2\theta$ and (88b) by $\sin 2\theta$, square the equations, and add. The result is

$$\cos 2\theta = \pm \cos 2\chi \cos 2\psi \tag{89}$$

To determine the required phase shift ϕ, we divide (88b) by (88a):

$$\frac{\sin 2\chi}{\tan 2\psi} = \tan\phi \cos 2\theta \tag{90}$$

Solving for $\tan\phi$ and using (89), we easily find that

$$\tan\phi = \frac{\tan 2\chi}{\sin 2\psi} \tag{91}$$

Thus, (89) and (91) are the equations for the angles θ and ϕ to which the polarizer and the retarder must be set in order to obtain the desired ellipticity and orientation angles χ and ψ.

We have thus shown that using only a rotated ideal linear polarizer and a retarder we can generate any state of elliptically polarized light. There is a final interesting fact about (89) and (91). We write (89) and (91) as a pair in the form

$$\cos 2\theta = \pm \cos 2\chi \cos 2\psi \tag{92a}$$

$$\tan 2\chi = \sin 2\psi \tan\phi \tag{92b}$$

Equations (92a) and (92b) are recognized as equations arising from spherical trigonometry for a right spherical triangle. In Figure 8 we have drawn a right spherical triangle. The orientation angle 2ϕ and the ellipticity angle 2χ are plotted on the equator (zero latitude) and the longitude, respectively.

The angle 2ψ (the orientation of the polarization ellipse) is plotted on the equator, and the angle 2χ (the ellipticity of the polarization ellipse) is plotted on the longitude. If a great circle is drawn from point A to point B, the length of the arc $\overline{AB}$ is given by (92a) and corresponds to 2θ as shown in the figure. Similarly, the phase ϕ is the angle between the arc $\overline{AB}$ and the equator; its value is given by (92b). We see from Figure 8 that we can easily determine θ and ϕ

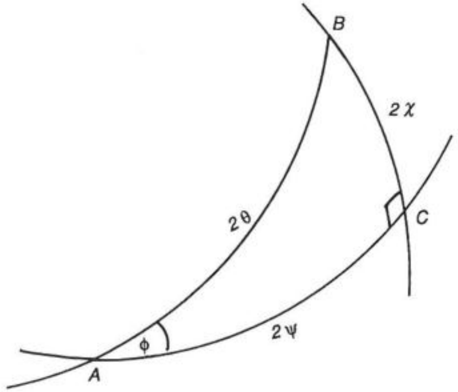

Figure 8 A right spherical triangle drawn on the surface of a sphere.

by (1) measuring the length of the arc $\overline{AB}$ and (2) the angle between the arc $\overline{AB}$ and the equator on a sphere.

The polarization equations (92a) and (92b) are intimately associated with spherical trigonometry and a sphere. Furthermore, we recall from Section 4.3 that when the Stokes parameters were expressed in terms of the orientation angle and the ellipticity angle they led directly to the Poincaré sphere. In fact, (92a) and (92b) describe a spherical triangle which plots directly onto the Poincaré sphere. Thus, we see that even at this early stage in our study of polarized light there is a strong connection between the equations of polarized light and its representation on a sphere. In fact, one of the most remarkable properties of polarized light is that there is such a close relation between these equations and the equations of spherical trigonometry. In Chapter 11, the Poincaré sphere, these relations will be discussed in depth. Finally, in order to provide the reader with background material on right spherical triangles a brief discussion of the fundamentals of spherical trigonometry is presented at the end of Section 11.2.

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6 Methods for Measuring the Stokes Polarization Parameters

6.1. INTRODUCTION

We now turn our attention to the important problem of measuring the Stokes polarization parameters. In Chapter 7 we shall also discuss the measurement of the Mueller matrices. The first method for measuring the Stokes parameters is due to Stokes and is probably the best known method; this method was discussed in Section 4.4. There are other methods for measuring the Stokes parameters. However, we have refrained from discussing these methods until we had introduced the Mueller matrices for a polarizer, a retarder, and a rotator. The formalism of the Mueller matrix calculus and the Stokes vector allows us to treat all of these measurement problems in a very simple and direct manner. While, of course, the problems could have been treated using the amplitude formulation, the use of the Mueller matrix calculus greatly simplifies the analysis.

In theory, the measurement of the Stokes parameters should be quite simple. However, in practice there are difficulties. This is due, primarily, to the fact that while the measurement of S_0, S_1, and S_2 is quite straightforward, the measurement of S_3 is more difficult. In fact, as we pointed out, before the advent of optical detectors it was not even possible to measure the Stokes parameters using Stokes' measurement method (Section 4.4). It is possible, however, to measure the Stokes parameter using the eye as a detector by using a so-called null method; this is discussed in Section 6.4. In this chapter we discuss Stokes' method along with other methods, which includes the circular polarizer method, the null-intensity method, the Fourier analysis method, and the method of Kent and Lawson.

6.2. THE CLASSICAL MEASUREMENT METHOD—THE QUARTER-WAVE RETARDER POLARIZER METHOD

The Mueller matrices for the polarizer (diattenuator), retarder (phase shifter), and rotator can now be used to analyze various methods for measuring the Stokes parameters. A number of methods are known. We first consider the application of the Mueller matrices to the classical measurement of the Stokes polarization parameters using a quarter-wave retarder and a polarizer. This is the same problem that was treated in Section 4.4; it is the problem originally considered by Stokes (1852). The result is identical, of course, with that obtained by Stokes. However, the advantage of using the Mueller matrices (or, as it is sometimes called, the Mueller calculus) is that a formal method can be used to treat not only this type of problem but other polarization problems as well.

The Stokes parameters can be measured as shown in Figure 1. An optical beam is characterized by its four Stokes parameters S_0, S_1, S_2, and S_3. The Stokes vector of this beam is represented by

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{1}$$

The Mueller matrix of a retarder with its fast axis at 0° is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{2}$$

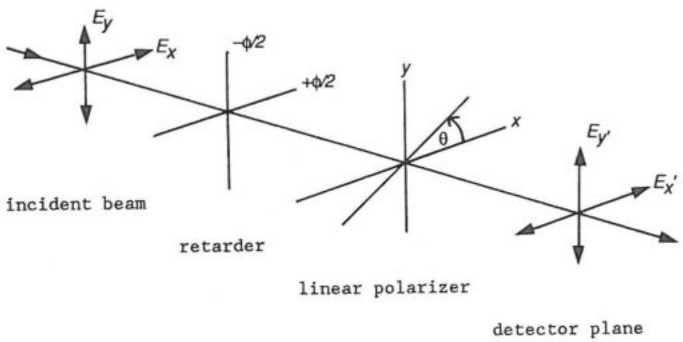

Figure 1 Classical measurement of the Stokes parameters.

The Stokes vector S' of the beam emerging from the retarder is obtained by matrix multiplication of (2) and (1), so

$$S' = \begin{pmatrix} S_0 \\ S_1 \\ S_2\cos\phi - S_3\sin\phi \\ S_2\sin\phi + S_3\cos\phi \end{pmatrix} \tag{3}$$

The Mueller matrix of an ideal linear polarizer with its transmission axis set at an angle θ is

$$M = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta\cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta\cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

The Stokes vector S'' of the beam emerging from the linear polarizer is found by matrix multiplication of (4) and (3). However, we are only interested in the intensity I'', which is the first Stokes parameters S_0'' of the beam incident on the optical detector shown in Figure 1. Multiplying the first row of (4) with (3), we then find the intensity of the beam emerging from the quarter-wave retarder-polarizer combination to be

$$I(\theta,\phi) = \frac{1}{2}[S_0 + S_1\cos 2\theta + \sin 2\theta(S_2\cos\phi - S_3\sin\phi)]$$

$$= \frac{1}{2}[S_0 + S_1\cos 2\theta + S_2\sin 2\theta\cos\phi - S_3\sin 2\theta\sin\phi] \tag{5}$$

Equation (5) is Stokes' famous intensity relation for the Stokes parameters. The Stokes parameters are then found from the following conditions on θ and ϕ:

$$S_0 = I(90°, 0°) + I(90°, 0°) \tag{6a}$$

$$S_1 = I(90°, 0°) - I(90°, 0°) \tag{6b}$$

$$S_2 = 2I(45°, 0°) - S_0 \tag{6c}$$

$$S_3 = S_0 - 2I(45°, 90°) \tag{6d}$$

In practice, S_0, S_1, and S_2 are easily measured by removing the quarter-wave plate ($\phi = 90°$) from the optical train. In order to measure S_3, however, the wave plate must be reinserted into the optical train with the linear polarizer set at $\theta = 45°$. This immediately raises a problem because the wave plate absorbs some optical energy. In order to obtain an accurate measurement of the Stokes parameters the absorption factor must be introduced, *ab initio*, into the Mueller matrix for the wave plate. The absorption factor which we write as p must be determined from a separate measurement and will then appear in (5) and (6). We can easily derive the Mueller matrix for an absorbing wave plate as follows.

The field components E_x and E_y of a beam emerging from an absorbing retarder in terms of the incident field components E_x and E_y are

$$E'_x = E_x e^{+i\phi/2} e^{-\alpha_x} \tag{7a}$$

$$E'_y = E_y e^{-i\phi/2} e^{-\alpha_y} \tag{7b}$$

where α_x and α_y are the absorption coefficients. We can also express the exponential absorption factors in (7) as

$$p_x = e^{-\alpha_x} \tag{8a}$$

$$p_y = e^{-\alpha_y} \tag{8b}$$

Using (7) and (8) in the defining equations for the Stokes parameters and the Mueller matrix in Section 5.2, we find the Mueller matrix for an anisotropic absorbing retarder:

$$M = \frac{1}{2}\begin{pmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_x^2 & 0 & 0 \\ 0 & 0 & 2p_xp_y\cos\phi & -2p_xp_y\sin\phi \\ 0 & 0 & 2p_xp_y\sin\phi & 2p_xp_y\cos\phi \end{pmatrix} \tag{9}$$

Thus, we see that an absorbing retarder behaves simultaneously as a polarizer and a retarder. If we again use the angular representation for the polarizer behavior, then we can write (9) as

$$M = \frac{p^2}{2}\begin{pmatrix} 1 & \cos 2\gamma & 0 & 0 \\ \cos 2\gamma & 1 & 0 & 0 \\ 0 & 0 & \sin 2\gamma\cos\phi & -\sin 2\gamma\sin\phi \\ 0 & 0 & \sin 2\gamma\sin\phi & \sin 2\gamma\cos\phi \end{pmatrix} \tag{10}$$

where $p_x^2 + p_y^2 = p^2$. We note that for $\gamma = 45°$ we have an isotropic retarder; that is, the absorption is equal along both axes. If p^2 is also unity, then (9) reduces to an ideal phase retarder.

The intensity of the emerging beam $I(\theta, \phi)$ is obtained by multiplying the first row of (4) by (10), and the result is

$$I(\theta,\phi) = \frac{p^2}{2}[(1+\cos 2\gamma)S_0 + (\cos 2\gamma + \cos 2\theta)S_1 + (\sin 2\gamma\cos\phi\sin 2\theta)S_2 + (\sin 2\gamma\sin\phi\sin 2\theta)S_3] \tag{11}$$

If we were now to make all four intensity measurements with a quarter-wave retarder in the optical train, then (11) would reduce for each of the four combinations of θ and $\phi = 90°$ to

$$S_0 = \frac{1}{p^2}[I(0°,0°) + I(90°,0°)] \tag{12a}$$

$$S_1 = \frac{1}{p^2}[I(0°,0°) - I(90°,0°)] \tag{12b}$$

$$S_2 = \frac{2}{p^2}I(45°,0°) - S_0 \tag{12c}$$

$$S_3 = S_0 - \frac{2}{p^2}I(45°,90°) \tag{12d}$$

Thus, each of the intensities in (12) are reduced by p^2, and this has no effect on the final value of the Stokes parameters with respect to each other. Furthermore, if we are interested in the ellipticity and the orientation, then we take ratios of the Stokes parameters S_3/S_0 and S_2/S_1 and the absorption factor p^2 cancels out. However, this is not exactly the way the measurement is made. Usually, the first three intensity measurements are made *without* the wave plate present, so the first three parameters are measured according to (6). The last measurement is done *with* a quarter-wave retarder in the optical train, (12d), so the equations are

$$S_0 = I(90°,0°) + I(90°,0°) \tag{13a}$$

$$S_1 = I(90°,0°) - I(90°,0°) \tag{13b}$$

$$S_2 = 2I(45°,0°) - S_0 \tag{13c}$$

$$S_3 = S_0 - \frac{2}{p^2}I(45°,90°) \tag{13d}$$

Thus, (13d) shows that the absorption fact p^2 enters in the measurement of the fourth Stokes parameters S_3. It is therefore necessary to measure the absorption factor p^2. The easiest way to do this is to place a linear polarizer between an optical source and a detector and measure the intensity; this is called I_0. Next, the retarder with its fast axis in the horizontal x direction is inserted between the linear polarizer and the detector. The intensity is then measured with the polarizer generating linearly horizontal and linear vertical polarized light, respectively (see (11)). Dividing each of these measured intensities by I_0 and adding the results gives p^2. Thus, we see that the measurement of the first three Stokes Parameters is very simple, but the measurement of the fourth parameter S_3 requires a considerable amount of additional effort.

It would therefore be preferable if a method could be devised whereby the absorption measurement could be eliminated. A method for doing this can be devised, and we now consider this method.

6.3. THE MEASUREMENT OF THE STOKES PARAMETERS USING A CIRCULAR POLARIZER

The problem of absorption by a retarder can be completely overcome by using a *single* polarizing element, namely, a circular polarizer; this will be defined shortly. The beam is allowed to enter one side of the circular polarizer, whereby the first three parameters can be measured. The circular polarizer is then flipped 180°, and the final Stokes parameter is measured. A circular polarizer is made by cementing a quarter-wave retarder to a linear +45° polarizer. This ensures that the retarder and polarizer axes are always fixed with respect to each other. Furthermore, because the same optical path is used in all four measurements, the problem of absorption vanishes; the four intensities are reduced by the same amount.

To measure all four Stokes parameters with a circular polarizer, we first construct a circular polarizer. This is done by using a linear polarizer with its transmission axis set at +45°, followed by a quarter-wave plate with its axis at 0°. This configuration is shown in Figure 2.

The Mueller matrix for the polarizer–wave plate combination is

$$M = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14a}$$

and

$$M = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \tag{14b}$$

Equation (14b) is the Mueller matrix of a circular polarizer. The reason for calling (14b) a circular polarizer is that regardless of the polarization state of the incident beam the emerging beam is always circularly polarized. This is easily shown by assuming the Stokes vector of an incident beam is

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{15}$$

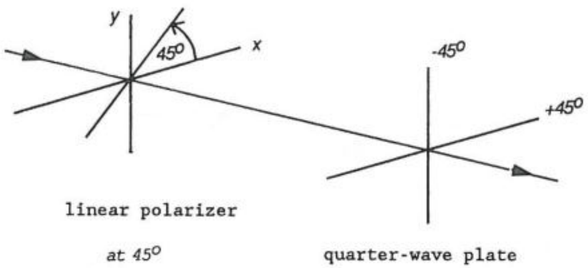

Figure 2 Construction of a circular polarizer using a linear polarizer and a quarter-wave plate.

Matrix multiplication of (15) and (14b) then yields

$$S' = \frac{1}{2}(S_0 + S_2)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{16}$$

which is the Stokes vector for right circularly polarized light (RCP). Thus, regardless of the polarization state of the incident beam, the output beam is *always* right circularly polarized. Hence, the name *circular* polarizer. Equation (14b) defines a circular polarizer.

Next, consider that the quarter-wave plate–polarizer combination is "flipped"; that is, the linear polarizer now follows the quarter-wave plate. The Mueller matrix for this combination is obtained with the Mueller matrices in (14b) interchanged; we note that the axis of the linearly polarizer when it is flipped causes a sign change in the Mueller matrix (see Figure 1). Then

$$M = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{17a}$$

so

$$M = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{17b}$$

Equation (17b) is the matrix of a linear polarizer. That (17b) is a linear polarizer can be easily seen by matrix-multiplying (17b) and (15):

$$S' = \frac{1}{2}(S_0 - S_3)\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{18}$$

which is the Stokes vector for linearly −45° polarized light. Regardless of the polarization state of the incident beam, the final beam is always linear +45° polarized. It is of interest to note that in the case of the "circular" side of the polarizer configuration, (16), the intensity varies only with the linear component, S_2, in the incident beam. On the other hand, for the "linear" side of the polarizer, (18), the intensity varies only with S_3, the circular component in the incident beam.

The circular polarizer is now placed in a rotatable mount. We saw earlier that the Mueller matrix for a rotated polarizing component, M, is given by the relation

$$M(2\theta) = M_R(-2\theta)MM_R(2\theta) \tag{19}$$

where $M_R(2\theta)$ is the rotation Mueller matrix,

$$M_R(2\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

and $M(2\theta)$ is the Mueller matrix of the rotated polarizing element. The Mueller matrix for the circular polarizer with its axis rotated through an angle θ is then found by substituting (14b) into (19). The result is

$$M_c(2\theta) = \frac{1}{2}\begin{pmatrix} 1 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -\sin 2\theta & \cos 2\theta & 0 \end{pmatrix} \tag{21}$$

where the subscript C refers to the fact that (21) describes the circular side of the polarizer combination. We see immediately that the Stokes vector emerging from the beam of the rotated circular polarizer is, using (21) and (15),

$$S_C = \frac{1}{2}(S_0 - S_1 \sin 2\theta + S_2 \cos 2\theta)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{22}$$

Thus, as the circular polarizer is rotated, the intensity varies but the polarization state remains unchanged, that is, circular. We note again that the total intensity depends only on the linear components, S_1 and S_2, in the incident beam.

The Mueller matrix when the circular polarizer is flipped to its linear side is, from (17b) and (19),

$$M_L(2\theta) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ \sin 2\theta & 0 & 0 & \sin 2\theta \\ -\cos 2\theta & 0 & 0 & -\cos 2\theta \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

where the subscript L refers to the fact that (23) describes the linear side of the polarizer combination. The Stokes vector of the beam emerging from the rotated linear side of the polarizer, (23) and (15), is

$$S_L = \frac{1}{2}(S_0 + S_3)\begin{pmatrix} 1 \\ \sin 2\theta \\ -\cos 2\theta \\ 0 \end{pmatrix} \tag{24}$$

Under a rotation of the circular polarizer on the linear side, (24) shows that the polarization is always linear. The total intensity is constant and depends only on the circular component S_3 in the incident beam.

The intensities detected on the circular and linear sides are, respectively, from (22) and (24),

$$I_C(\theta) = \frac{1}{2}(S_0 - S_1 \sin 2\theta + S_2 \cos 2\theta) \tag{25a}$$

$$I_L(\theta) = \frac{1}{2}(S_0 + S_3) \tag{25b}$$

The intensity on the linear side, (25b), is seen to be independent of the rotation of polarizer. This fact allows a simple check when the measurement is being made. If the circular polarizer is rotated and the intensity does not vary, then one knows the measurement is being made on I_L, the linear side.

In order to obtain the Stokes parameters, we first use the circular side of the polarizing element and rotate it to $\theta = 0°$, 45°, and 90°, respectively, and then flip it to the linear side. The measured intensities are then

$$I_C(0°) = \frac{1}{2}(S_0 + S_2) \tag{26a}$$

$$I_C(45°) = \frac{1}{2}(S_0 - S_1) \tag{26b}$$

$$I_C(90°) = \frac{1}{2}(S_0 - S_2) \tag{26c}$$

$$I_L(0°) = \frac{1}{2}(S_0 + S_3) \tag{26d}$$

The I_L value is conveniently taken to be $\theta = 0°$. Solving (26) for the Stokes parameters finally yields

$$S_0 = I_C(0°) + I_C(90°) \tag{27a}$$

$$S_1 = S_0 - 2I_C(45°) \tag{27b}$$

$$S_2 = I_C(0°) - I_C(90°) \tag{27c}$$

$$S_3 = 2I_L(0°) - S_0 \tag{27d}$$

Equation (27) is similar to the classical equations for measuring the Stokes parameters, (6), but the intensity combinations are distinctly different. The use of a circular polarizer to measure the Stokes parameter is simple and accurate because (1) only a single rotating mount is used, (2) the polarizing beam propagates through the same optical path so the problem of absorption losses can be ignored, and (3) the axes of the wave plate and polarizer are permanently fixed with respect to each other.

6.4. THE NULL-INTENSITY METHOD

In previous sections the Stokes parameters were expressed in terms of measured intensities. These measurement methods, however, are suitable only for use with quantitative detectors. We pointed out earlier that before the advent of solid-state detectors and photomultipliers the only available detector was the human eye. It can only measure the presence of light or no light (a null intensity). It is possible, as we shall now show, to measure the Stokes parameters from the condition of a null-intensity state. This can be done by using a variable retarder (phase shifter) followed by a linear polarizer in a rotatable mount. Devices are manufactured which can change the phase between the orthogonal components of an optical beam. They are called Babinet–Soleil compensators, and they are usually placed in a rotatable mount. Following the compensator is a linear polarizer which is also placed in a rotatable mount. This arrangement can be used to obtain a null intensity. In order to carry out the analysis, the reader is referred to Figure 3.

The Stokes vector of the incident beam to be measured is

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{28}$$

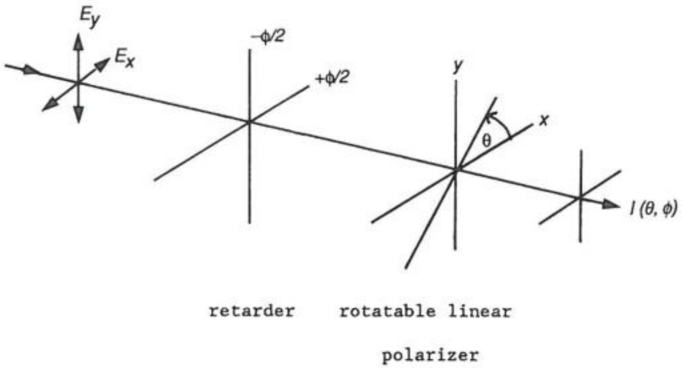

Figure 3 Null intensity measurement of the Stokes parameters.

The analysis is simplified considerably if the α, δ form of the Stokes vector derived in Section 4.3 is used:

$$S = I_0 \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos \delta \\ \sin 2\alpha \sin \delta \end{pmatrix} \tag{29}$$

The axis of the Babinet–Soleil compensator is set at 0°. The Stokes vector of the beam emerging from the compensator is found by multiplying the matrix of the nonrotated compensator (Section 5.3) with (29):

$$S' = I_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos \delta \\ \sin 2\alpha \sin \delta \end{pmatrix} \tag{30}$$

Carrying out the matrix multiplication in (30) and using the well-known trigonometric sum formulas, we readily find

$$S' = I_0 \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos(\phi + \delta) \\ \sin 2\alpha \sin(\phi + \delta) \end{pmatrix} \tag{31}$$

Two important observations on (31) can be made. The first is that (31) can be transformed to linearly polarized light if S_3' can be made to be equal to zero. This can be done by setting $\phi + \delta$ to 180°. If we then analyze S' with a linear polarizer, we see that a null intensity can be obtained by rotating the polarizer; at the null setting we can then determine α. This method is the procedure that is almost always used to obtain a null intensity. The null-intensity method works because δ in (30) is simply transformed to $\phi + \delta$ in (31) after the beam propagates through the compensator (retarder). For the moment we shall retain the form of (31) and not set $\phi + \delta$ to 180° (π radians). The function of the Babinet–Soleil compensator in this case is to transform elliptically polarized light (28) to linearly polarized light.

Next, the beam represented by (31) is incident on a linear polarizer with its transmission axis at an angle θ. The Stokes vector S'' of the beam emerging from the rotated polarizer is now

$$S'' = \frac{I_0}{2} \begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos(\phi + \delta) \\ \sin 2\alpha \sin(\phi + \delta) \end{pmatrix} \tag{32}$$

where we have used the Mueller matrix of a rotated linear polarizer, Section 5.5. We are interested only in the intensity of the beam emerging from the rotated polarizer; that is, $S_0'' = I(\theta, \phi)$. Carrying out the matrix multiplication with the first row in the Mueller matrix and the Stokes vector in (32) yields

$$I(\theta, \phi) = \frac{I_0}{2}[1 + \cos 2\theta \cos 2\alpha + \sin 2\theta \sin 2\alpha \cos(\phi + \delta)] \tag{33}$$

We now set $\phi + \delta = \pi$ in (33) and find

$$I(\theta, \pi - \delta) = \frac{I_0}{2}[1 + \cos 2\theta \cos 2\alpha - \sin 2\theta \sin 2\alpha] \tag{34a}$$

which reduces to

$$I(\theta, \pi - \delta) = \frac{I_0}{2}[1 + \cos 2(\theta + \alpha)] \tag{34b}$$

The linear polarizer is rotated until a null intensity is observed. At this angle $\theta + \alpha = \pi/2$, and we have

$$I\left(\frac{\pi}{2} - \alpha, \pi - \delta\right) = 0 \tag{35}$$

The angles δ and α associated with the Stokes vector of the incident beam are thus found from the conditions

$$\delta = \pi - \phi \tag{36a}$$

$$\alpha = \frac{\pi}{2} - \theta \tag{36b}$$

Equations (36a) and (36b) are the required relations between α and δ of the Stokes vector (31) and ϕ and θ, the phase setting on the Babinet–Soleil compensator and the angle of rotation of the linear polarizer, respectively.

From the values obtained for α and δ we can determine the corresponding values for the orientation angle ψ and the ellipticity χ of the incident beam. We saw in Eq. (42) in Section 4.3 that ψ and χ could be expressed in terms of α and δ, namely,

$$\tan 2\psi = \tan 2\alpha \cos \delta \tag{37a}$$

$$\sin 2\chi = \sin 2\alpha \sin \delta \tag{37b}$$

Substituting (36) into (37), we see that ψ and χ can be expressed in the terms of the measured values of θ and ϕ:

$$\tan 2\psi = \tan 2\theta \cos \phi \tag{38a}$$

$$\sin 2\chi = \sin 2\theta \sin \phi \tag{38b}$$

Remarkably, (38) is identical to (37) in form. It is only necessary to take the measured values of θ and ϕ and insert them into (38) to obtain ψ and χ. Equations (37a) and (37b) can be solved in turn for α and δ following the derivation

given in Section 5.6, and we have

$$\cos 2\alpha = \pm \cos 2\chi \cos 2\psi \tag{39a}$$

$$\tan \delta = \frac{\tan 2\chi}{\sin 2\psi} \tag{39b}$$

The procedure to find the null-intensity angles θ and ϕ is first to set the Babinet–Soleil compensator with its fast axis to 0° and its phase angle to 0°. The phase is then adjusted until the intensity is observed to be a minimum. At this point in the measurement the intensity will not necessarily be zero, only a minimum, as we see from (34b),

$$I(\theta, \pi - \delta) = \frac{I_0}{2}[1 + \cos 2(\theta + \alpha)] \tag{34b}$$

Next, the linear polarizer is rotated through an angle θ until a null intensity is observed; the setting at which this angle occurs is then measured. In theory this completes the measurement. In practice, however, one finds that a small adjustment in phase of the compensator and rotation angle of the linear polarizer are almost always necessary to obtain a null intensity. Substituting the observed angular settings on the compensator and the polarizer into Eqs. (38) and (39), we then find the Stokes vector (29) of the incident beam. We note that (29) is a normalized representation of the Stokes vector if I_0 is set to unity.

6.5. FOURIER ANALYSIS USING A ROTATING QUARTER-WAVE RETARDER

Another method for measuring the Stokes parameters is to allow a beam to propagate through a rotating quarter-wave retarder followed by a linearly horizontal polarizer; the retarder rotates at an angular frequency of ω. This arrangement is shown in Figure 4.

The Stokes vector of the incident beam to be measured is

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{40}$$

The Mueller matrix of the rotating quarter-wave retarder, (78) in Section 5.5 is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta & \sin 2\theta \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta \cos 2\theta & \sin^2 2\theta & -\cos 2\theta \\ 0 & \sin 2\theta & \cos 2\theta & 0 \end{pmatrix} \tag{41}$$

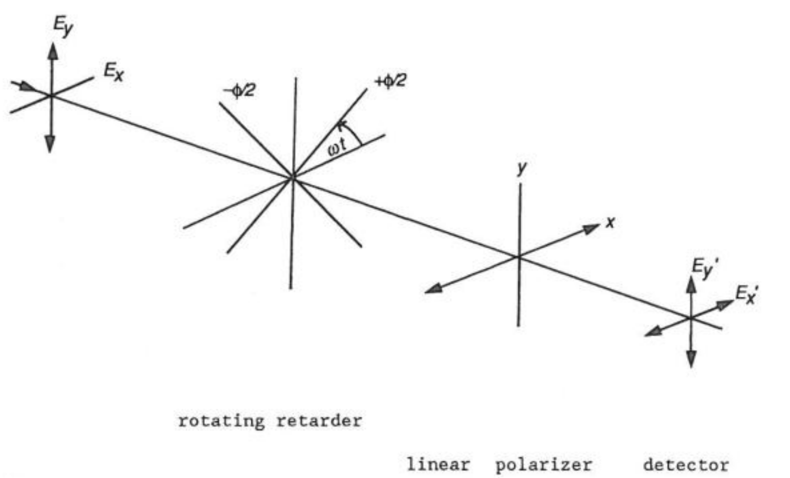

Figure 4 Measurement of the Stokes parameters using a rotating quarter-wave retarder and a linear polarizer.

where $\theta = \omega t$. Multiplying (41) with (40) yields

$$S' = \begin{pmatrix} S_0 \\ S_1\cos^2 2\theta + S_2 \sin 2\theta \cos 2\theta + S_3 \sin 2\theta \\ S_1 \sin 2\theta \cos 2\theta + S_2 \sin^2 2\theta - S_3 \cos 2\theta \\ S_1 \sin 2\theta + S_2 \cos 2\theta \end{pmatrix} \tag{42}$$

The Mueller matrix of the linear horizontal polarizer is

$$M = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{43}$$

The Stokes vector of the beam emerging from the rotating quarter-wave plate–polarizer combination is then found from (42) and (43) to be

$$S' = \frac{1}{2}(S_0 + S_1 \cos^2 2\theta + S_2 \sin 2\theta \cos 2\theta + S_3 \sin 2\theta)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{44}$$

The intensity $S_0' = I(\theta)$ is

$$I(\theta) = \frac{1}{2}(S_0 + S_1 \cos^2 2\theta + S_2 \sin 2\theta \cos 2\theta + S_3 \sin 2\theta) \tag{45}$$

Equation 45 can be rewritten by using the trigonometric half-angle formulas:

$$I(\theta) = \frac{1}{2}\left[\left(S_0 + \frac{S_1}{2}\right) + \frac{S_1}{2}\cos 4\theta + \frac{S_2}{2}\sin 4\theta + S_3 \sin 2\theta\right] \tag{46}$$

Replacing θ with ωt, (46) can be written as

$$I(\theta) = \frac{1}{2}[A + B \sin 2\omega t + C \cos 4\omega t + D \sin 4\omega t] \tag{47a}$$

where

$$A = S_0 + \frac{S_1}{2} \tag{47b}$$

$$B = S_3 \tag{47c}$$

$$C = \frac{S_1}{2} \tag{47d}$$

$$D = \frac{S_2}{2} \tag{47e}$$

Equation (47a) describes a truncated Fourier series. It shows that we have a dc term (A), a double frequency term (B), and two quadruple frequency terms in quadrature (C and D). The coefficients are found by carrying out a Fourier analysis of (47a). We easily find that ($\theta = \omega t$)

$$A = \frac{1}{\pi}\int_0^{2\pi} I(\theta)\, d\theta \tag{48a}$$

$$B = \frac{2}{\pi}\int_0^{2\pi} I(\theta) \sin 2\theta \, d\theta \tag{48b}$$

$$C = \frac{2}{\pi}\int_0^{2\pi} I(\theta) \cos 4\theta \, d\theta \tag{48c}$$

$$D = \frac{2}{\pi}\int_0^{2\pi} I(\theta) \sin 4\theta \, d\theta \tag{48d}$$

Solving (47) for the Stokes parameters gives

$$S_0 = A - C \tag{49a}$$

$$S_1 = 2C \tag{49b}$$

$$S_2 = 2D \tag{49c}$$

$$S_3 = B \tag{49d}$$

In practice, the quarter-wave retarder is placed in a fixed mount which can be rotated and driven by a stepper motor through N steps. Equation (47a) then becomes, with $\omega t = n\theta_j$ (θ_j is the step size),

$$I_n(\theta_j) = \frac{1}{2}\left[A + B\sin 2n\theta_j + C\cos 4n\theta_j + D\sin 4n\theta_j\right] \tag{50a}$$

and

$$A = \frac{2}{N}\sum_{n=1}^{N} I(n\theta_j) \tag{50b}$$

$$B = \frac{4}{N}\sum_{n=1}^{N} I(n\theta_j)\sin 2n\theta_j \tag{50c}$$

$$C = \frac{4}{N}\sum_{n=1}^{N} I(n\theta_j)\cos 4n\theta_j \tag{50d}$$

$$D = \frac{4}{N}\sum_{n=1}^{N} I(n\theta_j)\sin 4n\theta_j \tag{50e}$$

As an example of (50), consider the rotation of a quarter-wave retarder that makes a complete rotation in 16 steps, so $N = 16$. Then the step size is $\theta_j = 2\pi/N = 2\pi/16 = \pi/8$. Equation (50) is then written as

$$A = \frac{1}{8}\sum_{n=1}^{16} I\left(n\frac{\pi}{8}\right) \tag{51a}$$

$$B = \frac{1}{4}\sum_{n=1}^{16} I\left(n\frac{\pi}{8}\right)\sin\left(n\frac{\pi}{4}\right) \tag{51b}$$

$$C = \frac{1}{4}\sum_{n=1}^{N} I\left(n\frac{\pi}{8}\right)\cos\left(n\frac{\pi}{2}\right) \tag{51c}$$

$$D = \frac{1}{4}\sum_{n=1}^{16} I\left(n\frac{\pi}{8}\right)\sin\left(n\frac{\pi}{2}\right) \tag{51d}$$

Thus, the data array consists of 16 measured intensities I_1 through I_{16}. We have written each intensity value as $I(n\pi/8)$ to indicate that the intensity is measured at intervals of $\pi/8$; we observe that when $n = 16$ we have $I(2\pi)$ as expected. At each step the intensity is stored to form (51a), multiplied by $\sin(n\pi/4)$ to form B, $\cos(n\pi/2)$ to form C and $\sin(n\pi/2)$ to form D. The sums are then performed according to (51), and we obtain A, B, C, and D. The Stokes parameters are then found from (49) using these values.

6.6. THE METHOD OF KENT AND LAWSON

In Section 6.4 we saw that the null-intensity condition could be used to determine the Stokes parameters and, hence, the polarization state of an optical beam. The null-intensity method remained the only practical way to measure the polarization state of an optical beam before the advent of photodetectors. It is fortunate that the eye is so sensitive to light and can easily detect its presence or absence. Had this not been the case, the progress made in polarized light would surely not have been as rapid as it was. One can obviously use a photodetector as well as the eye, using the null-intensity method described in Section 6.4. However, the existence of photodetectors allows one to consider an extremely interesting and novel method for determining the polarization state of an optical beam.

In 1937, C. V. Kent and J. Lawson proposed a new method for measuring the ellipticity and orientation of a polarized optical beam using a Babinet–Soleil compensator and a photomultiplier tube (PMT). They noted that it was obvious that a photomultiplier could simply replace the human eye as a detector, and then used to determine the null condition. However, Kent and Lawson went beyond this and made several importation observations. The first was that the use of the PMT could obviously overcome the problem of eye fatigue. They also noted that, in terms of sensitivity (at least in 1937) for weak illuminations, determining the null intensity was as difficult with a photomultiplier tube as with the human eye. They observed that the PMT really operated best with full illumination. In fact, because the incident light at a particular wavelength is usually much greater than the laboratory illumination the measurement could be done with the room lights on. They now noted that this property of the PMT could be exploited fully if the incident optical beam whose polarization was to be determined was transformed not to linearly polarized light but to circularly polarized light. By then analyzing the beam with rotating linear polarizer, a constant intensity would be obtained when the condition of circularly polarized light was obtained or, as they said, "no modulation." From this condition of "no modulation" the ellipticity and orientation angles of the incident beam could then be determined. Interestingly, they detected the circularly polarized light by converting the optical signal to an audio signal and then used a headphone set to determine the constant-intensity condition.

It is worthwhile to study this method because it enables us to see how photodetectors provide an alternative method for measuring the Stokes parameters and how they can be used to their optimum, that is, in the measurement of polarized light at high intensities. The measurement is described by the experimental configuration in Figure 5. In Figure 5 the Stokes vector of the incident elliptically polarized beam to be measured is represented by

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{52}$$

The primary use of a Babinet–Soleil compensator is to create an arbitrary state of elliptically polarized light. This is accomplished by changing the phase

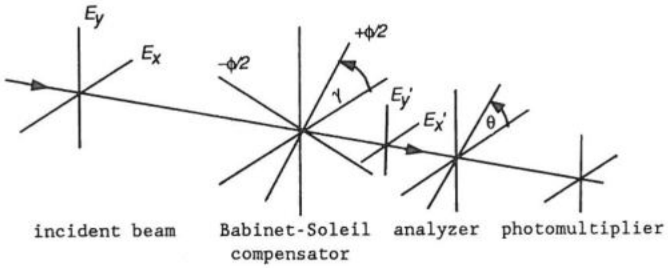

Figure 5 The measurement of the ellipticity and orientation of an elliptically polarized beam using a compensator and a photodetector.

and orientation of the incident beam. We recall from Section 5.5 that the Mueller matrix for a rotated retarder is

$$M_c(2\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\gamma + \cos\phi \sin^2 2\gamma & (1-\cos\phi)\sin 2\gamma \cos 2\gamma & \sin\phi \sin 2\gamma \\ 0 & (1-\cos\phi)\sin 2\gamma\cos 2\gamma & \sin^2 2\gamma + \cos\phi\cos^2 2\gamma & -\sin\phi\cos 2\gamma \\ 0 & -\sin\phi\sin 2\gamma & \sin\phi\cos 2\gamma & \cos\phi \end{pmatrix} \tag{53}$$

where γ is the angle that the fast axis makes with the horizontal x axis and ϕ is the phase shift. In terms of a matrix equation we see from Figure 5 and (52) and (53) that

$$S' = M_c(2\gamma)S \tag{54}$$

The beam emerging from the Babinet–Soleil compensator is then found by multiplying (52) by (53)

$$S = \begin{pmatrix} S_0 \\ S_1(\cos^2 2\gamma + \cos\phi\sin^2 2\gamma) + S_2(1-\cos\phi)\sin^2\gamma\cos 2\gamma + S_3\sin\phi\sin 2\gamma \\ S_1(1-\cos\phi)\sin 2\gamma\cos 2\gamma + S_2(\sin^2 2\gamma + \cos\phi\cos^2 2\gamma) - S_3\sin\phi\cos 2\gamma \\ -S_1\sin\phi\sin 2\gamma + S_2\sin\phi\cos 2\gamma + S_3\cos\phi \end{pmatrix} \tag{55}$$

For the moment let us assume that we have elliptically polarized light incident on a rotating ideal linear polarizer. The Stokes vector of the beam incident on the rotating linear polarizer is represented by

$$S = \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha\cos\delta \\ \sin 2\alpha \sin\delta \end{pmatrix} \tag{56}$$

The Mueller matrix of the rotating linear polarizer is

$$M = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta\cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta\cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{57}$$

The Stokes vector of the beam emerging from the rotating analyzer is found by multiplying (56) and (57):

$$S' = \frac{1}{2}[1 + \cos 2\alpha\cos 2\theta + \sin 2\alpha\cos\delta\sin 2\theta]\begin{pmatrix} 1 \\ \cos 2\theta \\ \sin 2\theta \\ 0 \end{pmatrix} \tag{58}$$

Thus, as the analyzer is rotated we see that the intensity varies, that is, modulated. We now note that if the intensity is to be independent of the rotation, that is, the angle θ, then we must have

$$\cos 2\alpha = 0 \tag{59a}$$

$$\sin 2\alpha\cos\delta = 0 \tag{59b}$$

We immediately see that (59a) and (59b) are satisfied if $2\alpha = 90°$ (or 270°). Substituting these values in (56), we have

$$S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{60}$$

which is the Stokes vector for right circularly polarized light. There are only two states of polarized light which lead to a constant intensity when a linear polarizer is rotated, namely, incident unpolarized light and circularly polarized light. In fact, rotating a linear polarizer is an excellent test for determining if the incident light is circularly polarized. In order to test for unpolarized light we need only rotate a polarizer followed by a wave plate; if the intensity remains constant, then the light is unpolarized. To test further to see if the light is circularly polarized, we see from (58) that if the light is circularly polarized ($2\alpha = 90°$ and $\delta = 90°$) the intensity remains constant as the linear polarizer is rotated.

In order to obtain circularly polarized light, the Stokes parameters in (55) must satisfy the conditions

$$S_0' = S_0 \tag{61a}$$

$$S_1^1 = S_1(\cos^2 2\gamma + \cos\phi\sin^2 2\gamma) + S_2(1-\cos\phi)\sin 2\gamma\cos 2\gamma + S_3\sin\phi\sin 2\gamma = 0 \tag{61b}$$

$$S_1(1-\cos\phi)\sin 2\gamma\cos 2\gamma + S_2(\sin^2 2\gamma + \cos\phi\cos^2 2\gamma) - S_3(\sin\phi\cos 2\gamma) = 0 \tag{61c}$$

$$-S_1(\sin\phi\sin 2\gamma) + S_2(\sin\phi\cos 2\gamma) + S_3\cos\phi = S_0 \tag{61d}$$

We must now solve these equations for S_1, S_2, and S_3 in terms of γ and ϕ (S_0 is unaffected by the wave plate). While it is straightforward to solve (61), the algebra is surprisingly tedious and complicated. Fortunately, the problem can be solved in another way, because we know the transformation equation for describing a rotated compensator.

To solve this problem, we take the following approach. According to Figure 5, the Stokes vector of the emerging beam S' is related to the Stokes vector of the incident beam S by the equation

$$S' = M_c(2\gamma)S \tag{62}$$

where $M_c(2\gamma)$ is given by (53) above. We recall that $M_c(2\gamma)$ is the rotated Mueller matrix for a retarder, so (69) can also be written as

$$S' = [M(-2\gamma)\cdot M_c\cdot M(2\gamma)]\cdot S \tag{63a}$$

where

$$M(2\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & \sin 2\gamma & 0 \\ 0 & -\sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{63b}$$

and

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{63c}$$

We now transform (63a) to right circularly polarized light and write (63a) with the Stokes vector of the incident beam written out as

$$S' = M(-2\gamma)\cdot M_c\cdot M(2\gamma)\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{64}$$

We have chosen S' to be the right circularly polarized light. While we could immediately invert (64) to find the Stokes vector of the incident beam, it is simplest to find S in steps. Multiplying both sides of (64) by $M(2\gamma)$, we have

$$M_c\cdot M(2\gamma)\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = M(2\gamma)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{65}$$

Next, we multiply (65) by M_c^{-1} to find

$$M(2\gamma)\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = M_c^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \sin\phi \\ \cos\phi \end{pmatrix} \tag{66}$$

Finally, (66) is multiplied by $M(-2\gamma)$, and we have

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = M(-2\gamma)\begin{pmatrix} 1 \\ 0 \\ \sin\phi \\ \cos\phi \end{pmatrix} = \begin{pmatrix} 1 \\ -\sin 2\gamma\sin\phi \\ \cos 2\gamma\sin\phi \\ \cos\phi \end{pmatrix} \tag{67}$$

Thus, the Stokes parameters of the incident beam in terms of the settings γ and ϕ are, according to (67),

$$S_0 = 1 \tag{68a}$$

$$S_1 = -\sin 2\gamma\sin\phi \tag{68b}$$

$$S_2 = \cos 2\gamma\sin\phi \tag{68c}$$

$$S_3 = \cos\phi \tag{68d}$$

We can check to see if (67) is correct. We know that if $\phi = 0°$, that is, the retarder is not present, then the only way S' can be right circularly polarized is if the incident beam S is right circularly polarized. Substituting $\phi = 0°$ into (67), we find

$$S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{69}$$

which is the Stokes vector for right circularly polarized light.

The numerical value of the Stokes parameters can be determined directly from (68). However, we can also express the Stokes parameters in terms of α and δ in (56) or in terms of the orientation and ellipticity angles ψ and χ. Thus, we can equate (56) to (67) and write

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos \delta \\ \sin 2\alpha \sin \delta \end{pmatrix} = \begin{pmatrix} 1 \\ -\sin 2\gamma \sin \phi \\ \cos 2\gamma \sin \phi \\ \cos \phi \end{pmatrix} \tag{70}$$

or, in terms of the orientation and ellipticity angles,

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{pmatrix} = \begin{pmatrix} 1 \\ -\sin 2\gamma \sin \phi \\ \cos 2\gamma \sin \phi \\ \cos \phi \end{pmatrix} \tag{71}$$

We now solve for S in terms of the measured values of γ and ϕ. Let us first consider (70) and equate the matrix elements

$$\cos 2\alpha = \pm \sin 2\gamma \sin \phi \tag{72a}$$

$$\sin 2\alpha \cos \delta = \cos 2\gamma \sin \phi \tag{72b}$$

$$\sin 2\alpha \sin \delta = \cos \phi \tag{72c}$$

In (72a) we have written $\pm$ to include left circularly polarized light. We divide (72b) by (72c) and find

$$\cot \delta = \cos 2\gamma \tan \phi \tag{73a}$$

Similarly, we divide (72b) by (72a) and find

$$\cos \delta = \pm \cot 2\gamma \cot 2\alpha \tag{73b}$$

We can group the results by renumbering (72a) and (73) and write

$$\cos 2\alpha = \pm \sin 2\gamma \sin \phi \tag{74a}$$

$$\cot \delta = \cos 2\gamma \tan \phi \tag{74b}$$

$$\cos \delta = \pm \cot 2\gamma \cot 2\alpha \tag{74c}$$

Equations 74 are the equations of Kent and Lawson. However, our results appear to differ from theirs only because of notation. Kent and Lawson expressed the incident field components by

$$E_x = \cos \gamma e^{-i\Delta} \tag{75a}$$

$$E_y = \sin \gamma \tag{75b}$$

The Stokes vector corresponding to (75) is then

$$S_{KL} = \begin{pmatrix} 1 \\ \cos 2\gamma \\ \sin 2\gamma \cos \Delta \\ \sin 2\gamma \sin \Delta \end{pmatrix} \tag{76}$$

where S_{KL} indicates that this is the Stokes vector in the notation of Kent and Lawson. Another difference in the notation of Kent and Lawson is that they used α to represent the phase shift and ϕ to represent the azimuth of the compensator. Thus, we see that our notation is transformed to their notation in the following way:

$$\begin{aligned} \alpha &\to \gamma \\ \delta &\to \Delta \\ \gamma &\to \phi \\ \phi &\to \alpha \end{aligned} \tag{77}$$

Using these transformation equations, we see that (74) transforms to

$$\cos 2\gamma = \pm \sin 2\phi \sin \alpha \tag{78a}$$

$$\cot \Delta = \cos 2\phi \tan \alpha \tag{78a}$$

$$\cos \Delta = \cot 2\phi \cot 2\gamma \tag{78b}$$

which are the equations actually given by Kent and Lawson. We shall, however, remain with the notation used in the development of this chapter in order to remain consistent throughout this book.

Thus, by measuring γ and ϕ, the angular rotation and phase of the Babinet–Soleil compensator, respectively, we can determine the azimuth α and phase δ of the incident beam. We also pointed out that we can use γ and ϕ to determine the ellipticity χ and orientation ψ of the incident beam from (71). Equating terms in (71) we have

$$\cos 2\chi \cos 2\psi = \pm \sin 2\gamma \sin \phi \tag{79a}$$

$$\cos 2\chi \sin 2\psi = \cos 2\gamma \sin \phi \tag{79b}$$

$$\sin 2\chi = \cos \phi \tag{79c}$$

Dividing (79b) by (79a), we find

$$\tan 2\psi = \pm \cot 2\gamma \tag{80}$$

Squaring (79a) and (79b) and adding gives

$$\cos 2\chi = \sin \phi \tag{81}$$

Dividing (79c) by (81) then gives

$$\tan 2\chi = \cot\phi \tag{82}$$

We renumber (80) and (81) as the pair

$$\tan 2\psi = \pm\cot 2\gamma \tag{83a}$$

$$\tan 2\chi = \cot\phi \tag{83b}$$

We can rewrite (83a) and (83b) as

$$\tan 2\psi = \pm\tan(90^\circ - 2\gamma) \tag{84a}$$

$$\tan 2\chi = \tan(90^\circ - \phi) \tag{84b}$$

so

$$\psi = 45^\circ - \gamma \tag{85a}$$

$$\chi = 45^\circ - \frac{\phi}{2} \tag{85b}$$

We can check (85a) and (85b). We know that a linear +45° polarized beam of light is transformed to right circularly polarized light if we have a quarter-wave retarder. In terms of the incident beam $\psi = 45^\circ$ and $\chi = 0^\circ$. Substituting these values in (85a) and (85b), respectively, we find that $\gamma = 0^\circ$ and $\phi = 90^\circ$ for the retarder. This is exactly what we would expect using a quarter-wave retarder with its fast axis in the x direction (see (63c)).

While nulling techniques for determining the elliptical parameters are very common, we see that the method of Kent and Lawson provides a very interesting alternative. We emphasize that nulling techniques were developed long before the appearance of photodetectors. Nulling techniques continue to be used because they are extremely sensitive and require, in principle, only an analyzer. Nevertheless, the method of Kent and Lawson has a number of advantages, foremost of which is that it can be used in ambient light and with high optical intensities. The method of Kent and Lawson requires the use of a Babinet–Soleil compensator and a rotatable polarizer. However, the novelty and potential of the method and its full exploitation of the quantitative nature of photodetectors should not be overlooked.

6.7. SIMPLE TESTS TO DETERMINE THE STATE OF POLARIZATION OF AN OPTICAL BEAM

In the laboratory one often has to determine if an optical beam is unpolarized, partially polarized, or completely polarized. If it is completely polarized, then we must determine if it is elliptically polarized or linearly or circularly polarized. In this section we consider this problem. Stokes' method for determining the Stokes parameters is a very simple and direct way of carrying out these tests (Sections 4.4).

We recall that the polarization state can be measured using a linear polarizer and a quarter-wave retarder. If a polarizer made of calcite is used, then it

transmits satisfactorily from 0.2 micron to 2.0 micron, more than adequate for visual work and into the near infrared. Quarter-wave retarders, on the other hand, are designed to transmit at a single wavelength, e.g., HeNe laser radiation at 0.628 micron. Therefore, the quarter-wave retarder should be matched to the wavelength of the polarizing radiation. In Figure 6 we show the experimental configuration for determining the state of polarization. We emphasize that we are not trying to determine the Stokes parameters quantitatively but merely determining the polarization state of the light.

We recall from Section 6.2 that the intensity $I(\theta,\phi)$ of the beam emerging from the retarder-polarizer combination shown in Figure 6 is

$$I(\theta,\phi) = \frac{1}{2}[S_0 + S_1\cos 2\theta + S_2\cos\phi\sin 2\theta - S_3\sin\phi\sin 2\theta] \tag{86}$$

where θ is the angle of rotation of the polarizer and ϕ is the phase shift of the retarder. In our tests we shall set ϕ to 0° (no retarder in the optical train) or 90° (a quarter-wave plate in the optical train). The respective intensities according to (86) are then

$$I(\theta,0^\circ) = \frac{1}{2}[S_0 + S_1\cos 2\theta + S_2\sin 2\theta] \tag{87a}$$

$$I(\theta,90^\circ) = \frac{1}{2}[S_0 + S_1\cos 2\theta - S_3\sin 2\theta] \tag{87b}$$

The first test we wish to perform is to determine if the light is unpolarized or completely polarized. In order to determine if it is unpolarized, the retarder is removed ($\phi = 0^\circ$), so we use (87a). The polarizer is now rotated through 180°. If the intensity remains constant throughout the rotation, then we must have

$$S_1 = S_2 = 0 \quad \text{and} \quad S_0 \neq 0 \tag{88}$$

If the intensity varies so (88) is not satisfied, then we know that we do not have unpolarized light. If, however, the intensity remains constant, then we are still not certain if we have unpolarized light because the parameter S_3 may be present.

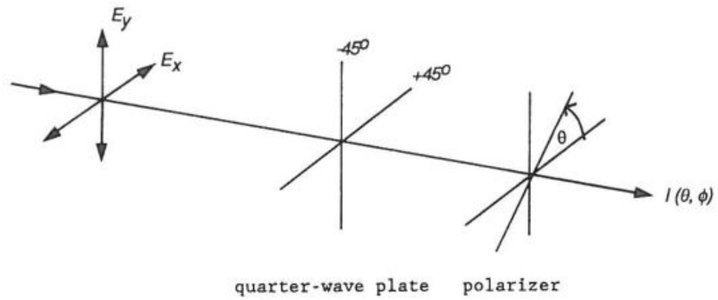

Figure 6 Experimental configuration to determine the state of polarization of an optical beam.

We must, therefore, test for its presence. The retarder is now reintroduced into the optical train, and we use (87b):

$$I(\theta, 90°) = \frac{1}{2}[S_0 + S_1 \cos 2\theta - S_3 \sin 2\theta] \tag{87b}$$

The polarizer is now rotated. If the intensity remains constant, then

$$S_1 = S_3 = 0 \quad \text{and} \quad S_0 \neq 0 \tag{89}$$

Thus, from (88) and (89) we see that (86) becomes

$$I(\theta, \phi) = \frac{1}{2}S_0 \tag{90}$$

which is the condition for unpolarized light.

If neither (88) or (89) is satisfied, we then assume that the light is elliptically polarized; the case of partially polarized light is excluded for the moment. Before we test for elliptically polarized light, however, we test for linear or circular polarization. In order to test for linearly polarized light, the retarder is removed from the optical train and so the intensity is again given by (87a)

$$I(\theta, 0°) = \frac{1}{2}[S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta] \tag{87a}$$

We recall that the Stokes vector for elliptically polarized light is

$$S = I_0 \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos \delta \\ \sin 2\alpha \sin \delta \end{pmatrix} \tag{91}$$

Substituting S_1 and S_2 in (91) into (87a) gives

$$I(\theta, 0°) = \frac{1}{2}[1 + \cos 2\alpha \cos 2\theta + \sin 2\alpha \cos \delta \sin 2\theta] \tag{92}$$

The polarizer is again rotated. If we obtain a null intensity, then we know that we have linearly polarized light because (87a) can only become a null if $\delta = 0°$ or 180°, a condition for linearly polarized light. For this condition we can write (92) as

$$I(\theta, 0°) = \frac{1}{2}[1 + \cos(2\alpha - 2\theta)] \tag{93}$$

which can only be zero if the incident beam is linearly polarized light. However, if we do not obtain a null intensity, we can have elliptically polarized light or circularly polarized light. To test for these possibilities, the quarter-wave retarder is reintroduced into the optical train so the intensity is again given by (87b):

$$I(\theta, 90°) = \frac{1}{2}[S_0 + S_1 \cos 2\theta - S_3 \sin 2\theta] \tag{87b}$$

Now, if we have circularly polarized light, then S_1 must be zero so (87b) will become

$$I(\theta, 90°) = \frac{1}{2}[S_0 - S_3 \sin 2\theta] \tag{94}$$

The polarizer is again rotated. If a null intensity is obtained, then we must have circularly polarized light. If, on the other hand, a null intensity is not obtained, then we must have a condition described by (87b), which is elliptically polarized light.

To summarize, if a null intensity is not obtained with either the polarizer by itself or with the combination of the polarizer and the quarter-wave retarder, then we must have elliptically polarized light.

Thus, by using a polarizer-quarter-wave retarder combination, we can test for the polarization states. The only state remaining is partially polarized light. If none of these tests described above is successful, we then assume that the incident beam is partially polarized.

To be completely confident of the tests, it is best to use a high-quality calcite polarizer and a quartz quarter-wave plate. It is, of course, possible to make these tests with Polaroid and mica quarter-wave plates. However, these materials are not as good, in general, as calcite and quartz and there is less confidence in the results.

If we are certain that the light is elliptically polarized, then we can consider (86) further. Equation (86) is

$$I(\theta, \phi) = \frac{1}{2}[S_0 + S_1 \cos 2\theta + S_2 \cos \phi \sin 2\theta - S_3 \sin \phi \sin 2\theta] \tag{86}$$

We can express (86) as

$$I(\theta, \phi) = \frac{1}{2}[S_0 + S_1 \cos 2\theta + (S_2 \cos \phi - S_3 \sin \phi) \sin 2\theta] \tag{95}$$

or

$$I(\theta, \phi) = [A + B \cos 2\theta + C \sin 2\theta] \tag{96a}$$

where

$$A = \frac{S_0}{2} \tag{96b}$$

$$B = \frac{S_1}{2} \tag{96c}$$

$$C = \frac{S_2 \cos \phi - S_3 \sin \phi}{2} \tag{96d}$$

For an elliptically polarized beam given by (91), I_0 is normalized to 1, and we write

$$S = \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \cos \delta \\ \sin 2\alpha \sin \delta \end{pmatrix} \tag{97}$$

so from (96) we see that

$$A = \frac{1}{2} \tag{98a}$$

$$B = \frac{\cos 2\alpha}{2} \tag{98b}$$

$$C = \frac{\cos(\phi + \delta)\sin 2\alpha}{2} \tag{98c}$$

The intensity (96a) can then be written as

$$I = \frac{1}{2}\,[1 + \cos 2\alpha \cos 2\theta + \sin 2\alpha \cos(\phi + \delta) \sin 2\theta] \tag{98d}$$

We now find the maximum and minimum intensities of (98d) by differentiating (98d) with respect to θ and setting $dI(\theta)/d\theta = 0$. The angles where the maximum and minimum intensities occur are then found to be

$$\tan 2\beta = \frac{C}{B} = \frac{-C}{-B} \tag{99}$$

Substituting (99) into (96a), the corresponding maximum and minimum intensity are, respectively,

$$I(\max) = A + \sqrt{B^2 + C^2} \tag{100a}$$

$$I(\min) = A - \sqrt{B^2 + C^2} \tag{100b}$$

From (88) we see that we can then write (100) as

$$I(\max, \min) = \frac{1}{2}\left[1 \pm \sqrt{\cos^2 2\alpha + \sin^2 2\alpha \cos^2(\phi + \delta)}\right] \tag{101}$$

Let us now remove the wave late from the optical train so that $\phi = 0°$; we then have only a linear polarizer which can be rotated through θ. Equation (101) then reduces to

$$I(\max, \min) = \frac{1}{2}\left[1 \pm \sqrt{\cos^2 2\alpha + \sin^2 2\alpha \cos^2 \delta}\right] \tag{102}$$

For linearly polarized light $\delta = 0°$ or 180°, so (97) becomes

$$S = \begin{pmatrix} 1 \\ \cos 2\alpha \\ \pm \sin 2\alpha \\ 0 \end{pmatrix} \tag{103}$$

and (102) becomes

$$I(\max, \min) = \frac{1}{2}[1 \pm 1] \tag{104}$$

The corresponding angles for the maximum and minimum intensities are seen from (98) and (99) to be $\theta = \alpha$ and $\theta = -\alpha$, respectively. Thus, linearly polarized light always gives a maximum intensity of unity and a minimum intensity of zero (null).

Next, if we have circularly polarized light, $\delta = 90°$ or 270° and $\alpha = 45°$, as is readily shown by inspecting (97). For this condition (102) reduces to

$$I(\max, \min) = \frac{1}{2}[1 \pm 0] = \frac{1}{2} \tag{105}$$

so the intensity is always constant and reduced to 1/2. We also see that if we have *only* the condition $\delta = 90°$ or 270°, then (97) becomes

$$S = \begin{pmatrix} 1 \\ \cos 2\alpha \\ 0 \\ \pm \sin 2\alpha \end{pmatrix} \tag{106}$$

which is the Stokes vector of an elljpse in a standard form, that is, unrotated. The corresponding intensity is, from (101),

$$I(\max, \min) = \frac{1}{2}[1 \pm \cos 2\alpha] \tag{107}$$

Similarly, if $\alpha = \pm 45°$ and δ is not equal to either 90° or 270°, then (97) becomes

$$S = \begin{pmatrix} 1 \\ 0 \\ \cos \delta \\ \sin \delta \end{pmatrix} \tag{108}$$

and (102) reduces to

$$I(\max, \min) = \frac{1}{2}[1 \pm \cos \delta] \tag{109}$$

This final analysis confirms the earlier results given in the first part of this chapter. We see that if we rotate a linear polarizer and we observe a null intensity

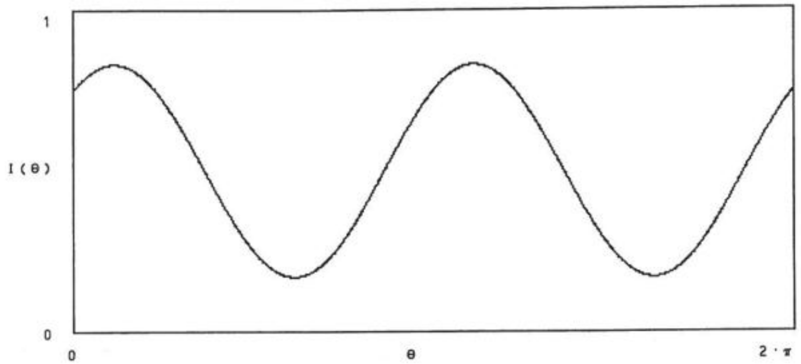

Figure 7 Intensity plot of an elliptically polarized beam for $\alpha = \pi/6$ and $\delta = \pi/3$.

at two angles over a single rotation, we have linearly polarized light; if we observe a constant intensity, we have circularly polarized light; and if we observe maximum and minimum (non-null) intensities, we have elliptically polarized light.

In Figures 7 and 8 we have plotted the intensity as a function of the rotation angle of the analyzer. Specifically, in Figure 7 we show the intensity for the condition where the parameters of the incident beam described by (97) are $\alpha = \pi/6$ (30°) and $\delta = \pi/3$ (60°); the compensator is not in the wave train, so $\phi = 0$.

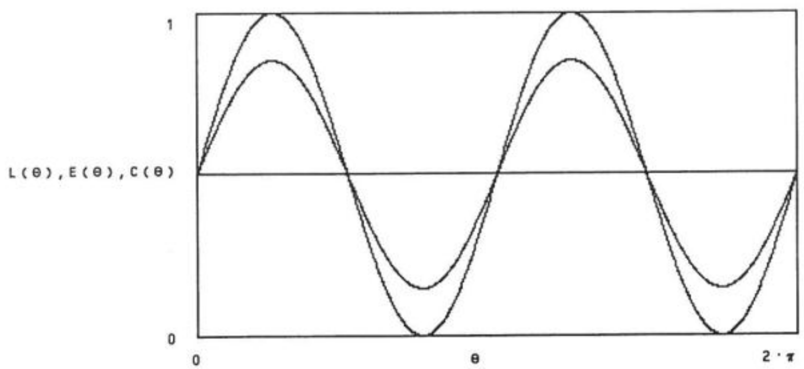

Figure 8 Plot of the intensity for a linearly polarized beam, an elliptically polarized beam, and a circularly polarized beam.

According to (97), the Stokes vector is

$$S = \begin{pmatrix} 1 \\ 1/2 \\ \sqrt{3/4} \\ 3/4 \end{pmatrix} \tag{110}$$

The intensity expected for (110) is seen from (98d) to be

$$I(\theta) = \frac{1}{2}\left[1 + \frac{1}{2}\cos 2\theta + \sqrt{\frac{3}{4}}\sin 2\theta\right] \tag{111}$$

The plot of (111) is given in Figure 7.

We see from (110) that the square root of the sum of the squares S_1, S_2, and S_3 is equal to unity as expected. Inspecting Figure 8, we see that there is a maximum intensity and a minimum intensity. However, because there is no null intensity we know that the light is elliptically polarized, which agrees, of course, with (110).

In Figure 8 we consider an elliptically polarized beam such that $\alpha = \pi/4$ and of arbitrary phase δ. This beam is described by the Stokes vector given by (108):

$$S = \begin{pmatrix} 1 \\ 0 \\ \cos\delta \\ \sin\delta \end{pmatrix} \tag{108}$$

The corresponding intensity for (108), according to (98d), is

$$I = \frac{1}{2}[1 + \cos\delta \sin 2\theta] \tag{112}$$

Specifically, we now consider (108) for $\delta = 0$, $\pi/4$, and $\pi/2$. The Stokes vectors corresponding to these conditions are, respectively,

$$S(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad S\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad S\left(\frac{\pi}{2}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{113}$$

The Stokes vectors in (113) correspond to linear +45° polarized light, elliptically polarized light, and right circularly polarized light, respectively. Inspection of Figure 8 shows the corresponding plot for the intensities given by (112) for each of the Stokes vectors in (113). The linearly polarized beam gives a null intensity, the elliptically polarized beam gives maximum and minimum intensities, and the circularly polarized beam yields a constant intensity of 0.5.

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7 The Measurement of the Characteristics of Polarizing Elements

7.1. INTRODUCTION

In the previous chapter we described a number of methods for measuring and characterizing polarized light and, especially, the Stokes polarization parameters. We now turn our attention to measuring the characteristics of the three major polarizing components, namely, the polarizer (diattenuator), wave plate and rotator. For a polarizer it is necessary to measure the attenuation coefficients of the orthogonal axes, for a wave plate the relative phase shift, and for a rotator the angle of rotation. It is of practical importance to make these measurements. Before proceeding with any experiment in which polarizing elements are to be used, it is good practice to determine if they are performing according to their specifications. This characterization is also necessary because over time polarizing components change: e.g., the optical coatings deteriorate, and in the case of Polaroid the material becomes discolored. In addition, one finds that, in spite of one's best laboratory controls, quarter-wave and half-wave retarders which operate at different wavelengths become mixed up. Finally, the quality control of manufacturers of polarizing components is not perfect, and imperfect components are sold.

The characteristics of all three types of polarizing elements can be determined by using a pair of high-quality calcite polarizers which are placed in high-resolution angular mounts; the polarizing element being tested is placed between these two polarizers. A practical angular resolution is 0.1° (6′ of arc) or less. High-quality calcite polarizers and mounts are expensive, but in a laboratory where polarizing components are used continually their cost is well justified.

7.2. THE MEASUREMENT OF THE ATTENUATION COEFFICIENTS OF A POLARIZER (DIATTENUATOR)

A linear polarizer is characterized by its attenuation coefficients p_x and p_y along its orthogonal x and y axes. We now describe the experimental procedure for measuring these coefficients. The measurement configuration is shown in Figure 1. In the experiment the polarizer to be tested is inserted between the two polarizers as shown. The reason for using two polarizers is that the same configuration can also be used to test wave plates (retarders) and rotators. Thus, we can have a single, permanent, test configuration for measuring all three types of polarizing components.

The Mueller matrix of a polarizer (diattenuator) with its axes along the x and y directions is

$$M_p = \frac{1}{2}\begin{pmatrix} p_x^2+p_y^2 & p_x^2-p_y^2 & 0 & 0 \\ p_x^2-p_y^2 & p_x^2+p_y^2 & 0 & 0 \\ 0 & 0 & 2p_xp_y & 0 \\ 0 & 0 & 0 & 2p_xp_y \end{pmatrix} \qquad 0 \le p_{x,y} \le 1 \tag{1}$$

It is convenient to rewrite (1) as

$$M_p = \begin{pmatrix} A & B & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{pmatrix} \tag{2a}$$

where

$$A = \frac{1}{2}(p_x^2+p_y^2) \tag{2b}$$

$$B = \frac{1}{2}(p_x^2-p_y^2) \tag{2c}$$

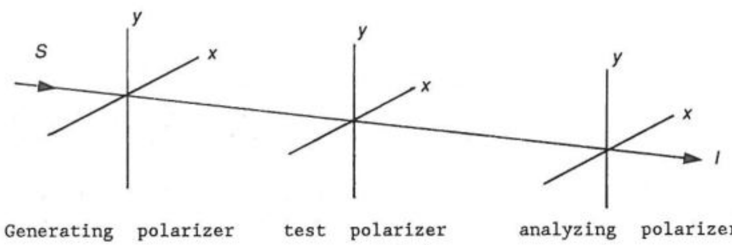

Figure 1 Experimental configuration to measure the attenuation coefficients p_x and p_y of a polarizer (diattenuator).

$$C = \frac{1}{2}(2p_xp_y) \tag{2d}$$

In practice, while we are interested only in determining p_x^2 and p_y^2, it is useful to measure p_xp_y as well, because a polarizer satisfies the relation

$$A^2 = B^2 + C^2 \tag{3}$$

as the reader can easily show from (2). Equation (3) serves as a useful check on the measurements. The optical source emits a beam characterized by a Stokes vector

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{4}$$

In the measurement the first polarizer, which is often called *the generating polarizer*, is set to +45°. The Stokes vector of the beam emerging from the generating polarizer is then

$$S = I_0\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

where $I_0 = (1/2)(S_0+S_2)$ is the intensity of the emerging beam. The Stokes vector of the beam emerging from the test polarizer is found to be, after multiplying (2a) and (5),

$$S' = I_0\begin{pmatrix} A \\ B \\ C \\ 0 \end{pmatrix} \tag{6}$$

The polarizer before the optical detector is often called *the analyzing polarizer* or simply *the analyzer*. The analyzer is mounted so that it can be rotated to an angle α. The Mueller matrix of the rotated analyzer is

$$M_A = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\alpha & \sin\alpha & 0 \\ \cos 2\alpha & \cos^2 2\alpha & \sin 2\alpha\cos 2\alpha & 0 \\ \sin 2\alpha & \sin 2\alpha\cos 2\alpha & \sin^2 2\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

The Stokes vector of the beam incident on the optical detector is then seen from multiplying (7) by (6) to be

$$S' = \frac{I_0}{2}(A + B\cos 2\alpha + C\sin 2\alpha)\begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix} \tag{8}$$

and the intensity of the beam is

$$I(\alpha) = \frac{I_0}{2}(A + B\cos 2\alpha + C\sin 2\alpha) \tag{9}$$

By rotating the analyzer to $\alpha = 0°$, $45°$, and $90°$, (9) yields the following equations:

$$I(0°) = \frac{I_0}{2}(A + B) \tag{10a}$$

$$I(45°) = \frac{I_0}{2}(A + C) \tag{10b}$$

$$I(90°) = \frac{I_0}{2}(A - B) \tag{10c}$$

Solving for A, B, and C, we then find

$$A = \frac{I(0°) + I(90°)}{I_0} \tag{11a}$$

$$B = \frac{I(0°) - I(90°)}{I_0} \tag{11b}$$

$$C = \frac{2I(45°) - I(0°) - I(90°)}{I_0} \tag{11c}$$

which are the desired relations. From (2) we also see that

$$p_x^2 = A + B \tag{12a}$$

$$p_y^2 = A - B \tag{12b}$$

so that we can write (10a) and (10c) as

$$p_x^2 = \frac{2I(0°)}{I_0} \tag{13a}$$

$$p_y^2 = \frac{2I(90°)}{I_0} \tag{13b}$$

Thus, it is only necessary to measure $I(0°)$ and $I(90°)$, the intensities in the x and y directions, respectively, to obtain p_x^2 and p_y^2. The intensity I_0 of the beam emerging from the generating polarizer is measured without the polarizer under test and the analyzer in the optical train.

It is not necessary to measure C. Nevertheless, experience shows that the additional measurement of $I(45°)$ enables one to use (3) as a check on the measurements.

In order to determine p_x^2 and p_y^2 in (13) it is necessary to know I_0. However, a relative measurement of p_y^2/p_x^2 is just as useful. We divide (12b) by (12a) and we obtain

$$\frac{p_y^2}{p_x^2} = \frac{I(90°)}{I(0°)} \tag{14}$$

We see that this type of measurement does not require a knowledge of I_0. Thus, measuring $I(0°)$ and $I(90°)$ and forming the ratio (14) yields the relative value of the absorption coefficients of the polarizer.

In order to obtain A, B, and C and then p_x^2 and p_y^2 in the method described above, an optical detector is required. However, the magnitude of p_x^2 and p_y^2 can also be obtained using a null-intensity method. To show this we write (3) again:

$$A^2 = B^2 + C^2 \tag{3}$$

This suggests that we can write

$$B = A\cos\gamma \tag{15a}$$

$$C = A\sin\gamma \tag{15b}$$

Substituting (15a) and (15b) into (9), we then have

$$I(\alpha) = \frac{I_0 A}{2}[1 + \cos(2\alpha - \gamma)] \tag{16a}$$

and

$$\tan\gamma = \frac{C}{B} \tag{16b}$$

where (16b) has been obtained by dividing (15a) by (15b).

We see that $I(\alpha)$ leads to a null intensity at

$$\alpha_{\text{null}} = 90° + \frac{\gamma}{2} \tag{17}$$

where α_{null} is the angle at which the null is observed. Substituting (17) into (16b) then yields

$$\frac{C}{B} = \tan 2\alpha_{\text{null}} \tag{18}$$

Thus by measuring γ from the null-intensity condition, we can find B/A and C/A from (15a) and (15b), respectively. For convenience we set $A = 1$, respectively. Then we see from (12) that

$$p_x^2 = 1 + B \tag{19a}$$

$$p_y^2 = 1 - B \tag{19b}$$

The ratio C/B in (18) can also be used to determine the ratio p_y/p_x, which we can then square to form p_y^2/p_x^2. From (2)

$$B = \frac{1}{2}(p_x^2 - p_y^2) \tag{2c}$$

$$C = \frac{1}{2}(2p_xp_y) \tag{2d}$$

Substituting (2b) and (2c) into (18) gives

$$\tan 2\alpha_{null} = \frac{2p_xp_y}{p_x^2 - p_y^2} \tag{20}$$

The form of (20) suggests that we set

$$p_x = p\cos\beta \qquad p_y = p\sin\beta \tag{21a}$$

so

$$\tan 2\alpha_{null} = \frac{\sin 2\beta}{\cos 2\beta} = \tan 2\beta \tag{21b}$$

and

$$\beta = \alpha_{null} \tag{21c}$$

This leads immediately to

$$\frac{p_y}{p_x} = \tan\beta = \tan(\alpha_{null}) \tag{22a}$$

or, using (17),

$$\frac{p_y^2}{p_x^2} = \cot^2\left(\frac{\gamma}{2}\right) \tag{22b}$$

Thus, the shift in the intensity, (16a), enables us to determine p_y^2/p_x^2 directly from γ. We always assume that $p_y^2/p_x^2 \leq 1$. A neutral density filter is described by $p_x^2 = p_y^2$ so the range on p_y^2/p_x^2 limits γ to

$$90° \leq \gamma \leq 180° \tag{22c}$$

For $p_y^2/p_x^2 = 0$, an ideal polarizer, $\gamma = 180°$, whereas for $p_y^2/p_x^2 = 1$, a neutral density filter $\gamma = 90°$ as shown by (22b). We see that the closer the value of γ is to 180°, the better is the polarizer. As an example, for commercial Polaroid HN22 at 0.550 microns $p_y^2/p_x^2 = 2 \times 10^{-6}/0.48 = 4.2 \times 10^{-6}$ so from (22b) we see that $\gamma = 179.77°$ and $\alpha_{null} = 179.88°$, respectively; the nearness of γ to 180° shows that it is an excellent polarizing material.

The parameters A, B, and C can also be obtained by Fourier-analyzing (9), assuming that the analyzing polarizer can be continuously rotated over a half or full cycle. Equation (9) is

$$I(\alpha) = \frac{I_0}{2}(A + B\cos 2\alpha + C\sin 2\alpha) \tag{9}$$

From the point of view of Fourier analysis A describes a dc term, and B and C describe second-harmonic terms in quadrature. It is only necessary to integrate over half a cycle, that is, from 0° to π, in order to determine A, B, and C. We easily find that

$$A = \frac{2}{\pi I_0}\int_0^\pi I(\alpha)\,d\alpha \tag{23a}$$

$$B = \frac{4}{\pi I_0}\int_0^\pi I(\alpha)\cos 2\alpha\,d\alpha \tag{23b}$$

$$C = \frac{4}{\pi I_0}\int_0^\pi I(\alpha)\sin 2\alpha\,d\alpha \tag{23c}$$

Throughout this analysis we have assumed that the axes of the polarizer being measured lie along the x and y directions. If this is not the case, then the polarizer under test should be rotated to its x and y axes in order to make the measurement. The simplest way to determine rotation angle θ is to remove the polarizer under test and rotate the generating polarizer to 0° and the analyzing polarizer to 90°, respectively.

Finally, another method to determine A, B, and C is to place the test polarizer in a rotatable mount between polarizers in which the axes of both are in the y direction. The test polarizer is then rotated until a minimum intensity is observed from which A, B, and C can be found. The Stokes vector emerging from the generating y generating polarizer is

$$S = \frac{I_0}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{24}$$

The Mueller matrix of the rotated test polarizer (2a) is

$$M = \begin{pmatrix} A & B\cos 2\theta & B\sin 2\theta & 0 \\ B\cos 2\theta & A\cos^2 2\theta + C\sin^2 2\theta & (A-C)\sin 2\theta\cos 2\theta & 0 \\ B\sin 2\theta & (A-C)\sin 2\theta & A\sin^2 2\theta + C\cos^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

The intensity of the beam emerging form the y analyzing polarizer is

$$I(\theta) = \frac{I_0}{4}[(A + C) - 2B\cos 2\theta + (A - C)\cos^2 2\theta] \tag{26}$$

Equation (26) can be solved for its maximum and minimum values by differentiating $I(\theta)$ with respect to θ and setting $dI(\theta)/d\theta = 0$. We then find

$$\sin 2\theta[B - (A - C)\cos 2\theta] = 0 \tag{27}$$

The solutions of (27) are

$$\sin 2\theta = 0 \tag{28a}$$

and

$$\cos 2\theta = \frac{B}{A-C} \tag{28b}$$

For (28a) we have $\theta = 0°$ and $90°$. The corresponding value of the intensities are then, (26),

$$I(0°) = \frac{I_0}{2}[A - B] \tag{29a}$$

$$I(90°) = \frac{I_0}{2}[A + B] \tag{29b}$$

The second solution (28b), upon substitution into (26), leads to $I(\theta) = 0$. Thus, the minimum intensity is given by (29a) and the maximum intensity by (29b). In view of the fact that both the generating and analyzing polarizer are in the y direction, this is exactly what one would expect. We also note in passing that at $\theta = 45°$, (26) reduces to

$$I(45°) = \frac{I_0}{4}[A + C] \tag{29c}$$

We can again divide (29) through by I_0 and then solve (29) for A, B, and C.

We see that several methods can be used to determine the absorption coefficients of the orthogonal axes of a polarizer. In the first method we generate a linear +45° polarized beam and then rotate the analyzer to obtain A, B, and C of the polarizer being tested. This method requires a quantitative optical detector. However, if an optical detector is not available, it is still possible to determine A, B, and C by using the null-intensity method; rotating the analyzer until a null is observed leads to A, B, and C. On the other hand, if the analyzer can be mounted in a rotatable mount which can be stepped (electronically), then a Fourier analysis of the signal can be made and we can again find A, B, and C. Finally, if the transmission axes of the generating and analyzing polarizers are parallel to one another, conveniently chosen to be in the y direction, and the test polarizer is rotated, then we can also determine A, B, and C by rotating the test polarizer to 0°, 45°, and 90°.

7.3. THE MEASUREMENT OF THE PHASE SHIFT OF A RETARDER

There are numerous occasions when it is important to know the phase shift of a retarder. The most common types of retarders or wave plates are quarter-wave plates and half-wave plates. These two types are most often used to create circularly polarized light and to rotate or reverse the polarization ellipse, respectively.

Two methods can be used for measuring the phase shift using two linear polarizers following the experimental configuration given in the previous section.

In the first method a wave plate is placed between the two linear polarizers mounted in the "crossed" position. The transmission axes of the first polarizer and second polarizer are in the x and y directions, respectively. By rotating the wave plate, the direction (angle) of the fast axis is rotated and, as we shall soon see, the phase can be found. The second method is very similar to the first except that the wave plate is rotated to = 45°. In this position the phase can also be found. We now consider both methods.

For the first method we refer to Figure 2. It is understood that the correct wavelength must be used; that is, if the wave plate is specified for, say 6328 Å, then the optical source should emit this wavelength. In the visible domain calcite polarizers are, as usual, best. However, high-quality Polaroid is also satisfactory, but its optical bandpass is much more restricted. In Figure 2 the transmission axes of the polarizers (or diattentuators) are in the x and y directions, respectively. The Mueller matrix for the wave plate rotated through an angle θ is

$$M(\phi,\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \cos\phi \sin^2 2\theta & (1-\cos\phi)\sin 2\theta\cos 2\theta & \sin\phi\sin 2\theta \\ 0 & (1-\cos\phi)\sin 2\theta\cos 2\theta & \sin^2 2\theta + \cos\phi\cos^2 2\theta & -\sin\phi\cos 2\theta \\ 0 & -\sin\phi\sin 2\theta & \sin\phi\cos 2\theta & \cos\phi \end{pmatrix} \tag{30}$$

where the phase shift ϕ is to be determined. The Mueller matrix for an ideal linear polarizer is

$$M_{xy} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ \pm 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{31}$$

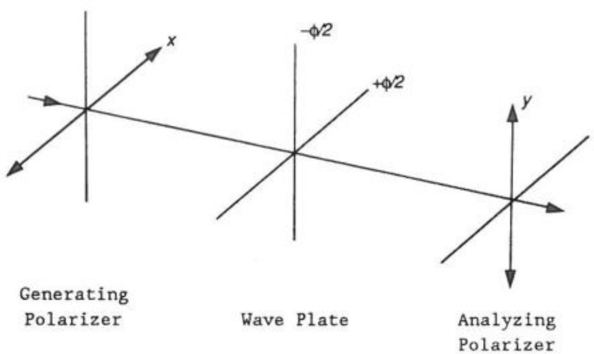

Figure 2 Crossed polarizer method to measure the phase of a wave plate.

where x and y correspond to + and − in (31). The Mueller matrix for Figure 2 is then

$$M = M_y M(\phi, \theta) M_x \tag{32}$$

Carrying out the matrix multiplication in (32) using (30) and (31) then yields

$$M = \frac{I_0}{8}(1 - \cos\phi)(1 - \cos 4\theta)\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{33}$$

Equation (33) shows that the polarizing train behaves as a pseudopolarizer. The intensity of the optical beam on the detector is then

$$I(\theta, \phi) = \frac{I_0}{4}(1 - \cos\phi)(1 - \cos 4\theta) \tag{34}$$

where I_0 is the intensity of the optical source.

Equation (34) immediately allows us to determine the direction of the fast axis of the wave plate; ideally it should be at $\theta = 0°$. When the wave plate is inserted between the crossed polarizers, the intensity on the detector should be zero, according to (34), at $\theta = 0°$. If it is not zero, the wave plate should be rotated until a null intensity is observed. After this angle has been found, the wave plate is rotated 45° according to (34) to obtain the maximum intensity. In order to determine ϕ, it is necessary to know I_0. The easiest way to do this is to rotate the x polarizer (the first polarizer) to the y position and remove the wave plate; both linear polarizers are then in the y direction. The intensity I_D on the detector is then

$$I_D = \frac{I_0}{4} \tag{35}$$

so (34) can be written as

$$I(\theta, \phi) = I_D(1 - \cos\phi)(1 - \cos 4\theta) \tag{36}$$

The wave plate is now reinserted into the polarizing train. The maximum intensity, $I(\theta, \phi)$, takes place when the wave plate is rotated to $\theta = 45°$. At this angle (36) is solved for ϕ, and we have

$$\phi = \cos^{-1}\left[1 - \frac{I(45°, \phi)}{2I_D}\right] \tag{37}$$

The disadvantage of using the cross-polarizer method is that it requires that we know the intensity of the beam, I_0, entering the polarizing train. This problem can be overcome by another method, namely, rotating the analyzing polarizer and fixing the wave plate at 45°. We now consider this second method.

The experimental configuration is identical to the first method except that the analyzer can be rotated through an angle α. The Stokes vector of the beam

emerging from the generating polarizer is

$$S = \frac{I_0}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{38}$$

Matrix multiplication of (38) and (30) yields

$$S' = \frac{I_0}{2}\begin{pmatrix} 1 \\ \cos^2 2\theta + \cos\phi \sin^2 2\theta \\ (1 - \cos\phi)\sin 2\theta \cos 2\theta \\ -\sin\phi \sin 2\theta \end{pmatrix} \tag{39}$$

We assume that the fast axis of the wave plate is at $\theta = 0°$. If it is not, the wave plate should be adjusted to $\theta = 0°$ by using the crossed polarizer method described in the first method; we note that at $\theta = 0°$, (39) reduces to

$$S' = \frac{I_0}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{40}$$

so that the analyzing polarizer should give a null intensity when it is in the y direction. Assuming the wave plate's fast axis is now properly adjusted, we rotate the wave plate counterclockwise to $\theta = 45°$. Then (39) reduces to

$$S' = \frac{I_0}{2}\begin{pmatrix} 1 \\ \cos\phi \\ 0 \\ -\sin\phi \end{pmatrix} \tag{41}$$

This is a Stokes vector for elliptically polarized light. The conditions $\phi = 90°$ and 180° correspond to right circularly polarized light and linearly vertically polarized, respectively. We note that the LVP state arises because for $\phi = 180°$ the wave plate behaves as a pseudorotator. The Mueller matrix of the analyzing polarizer is

$$M(\phi) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\alpha & \sin 2\alpha & 0 \\ \cos 2\alpha & \cos^2 2\alpha & \sin 2\alpha \cos 2\alpha & 0 \\ \sin 2\alpha & \sin 2\alpha \cos 2\alpha & \sin^2 2\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{42}$$

The Stokes vector of the beam emerging from the analyzer is then

$$S = \frac{I_0}{4}(1 + \cos\phi\cos 2\alpha)\begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix} \tag{43}$$

so the intensity is

$$I(\alpha, \phi) = \frac{I_0}{4}(1 + \cos\phi\cos 2\alpha) \tag{44}$$

In order to find ϕ, (44) is evaluated at $\alpha = 0^\circ$ and 90°, respectively, so

$$I(0^\circ, \phi) = \frac{I_0}{4}(1 + \cos\phi) \tag{45a}$$

$$I(90^\circ, \phi) = \frac{I_0}{4}(1 - \cos\phi) \tag{45b}$$

Equation (45a) is divided by (45b) and solved for $\cos\phi$:

$$\cos\phi = \frac{I(0^\circ, \phi) - I(90^\circ, \phi)}{I(0^\circ, \phi) + I(90^\circ, \phi)} \tag{46}$$

We note that in this method the source intensity need not be known.

We can also determine the direction of the fast axis of the wave plate in a "dynamic" fashion. The intensity of the beam emerging from the analyzer when it is in the y position is (see (39) and (42))

$$I_y = \frac{I_0}{4}[1 - (\cos^2 2\theta + \cos\phi\sin^2 2\theta)] \tag{47a}$$

where θ is the angle of the fast axis measured from the horizontal x axis. We now see that when the analyzer is in the x position

$$I_x = \frac{I_0}{4}[1 + (\cos^2 2\theta + \cos\phi\sin^2 2\theta)] \tag{47b}$$

Adding (47a) and (47b) yields

$$I_x + I_y = \frac{I_0}{2} \tag{48a}$$

Next, subtracting (47a) from (47b) yields

$$I_x - I_y = \frac{I_0}{2}(\cos^2 2\theta + \cos\phi\sin^2 2\theta) \tag{48b}$$

We see that when $\theta = 0$ the sum and difference intensities (48) are equal. Thus, one can measure I_x and I_y continuously as the wave plate is rotated and the analyzer is flipped between the horizontal and vertical directions until (48a) equals (48b). When this occurs, the amount of rotation which has taken place determines the magnitude of the rotation angle of the fast axis from the x axis.

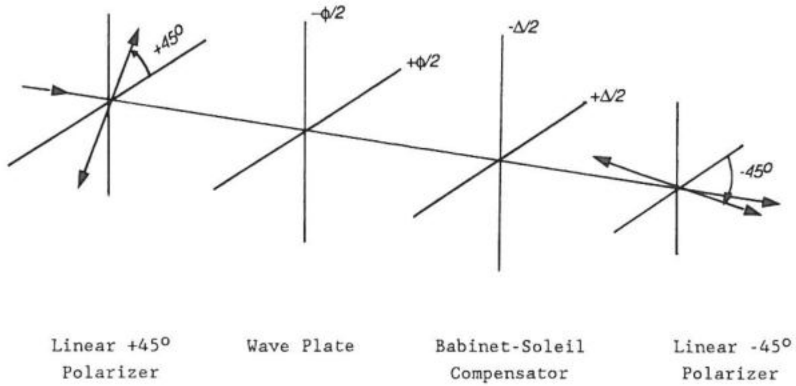

Figure 3 Measurement of the phase shift of a wave plate using a Babinet–Soleil compensator.

Finally, if a compensator is available, the phase shift can be measured as follows. Figure 3 shows the measurement method. The compensator is placed between the wave plate under test and the analyzer. The transmission axes of the generating and analyzing polarizers are set at $+45^\circ$ and $+135^\circ$, that is, in the crossed position, respectively.

The Stokes vector of the beam incident on the test wave plate is

$$S = \frac{I_0}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{49}$$

The Mueller matrix of the test wave plate is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{50}$$

Matrix-multiplying (50) by (49) yields

$$S = \frac{I_0}{2}\begin{pmatrix} 1 \\ 0 \\ \cos\phi \\ \sin\phi \end{pmatrix} \tag{51}$$

The Mueller matrix of the Babinet-Soleil compensator is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Delta & -\sin\Delta \\ 0 & 0 & \sin\Delta & \cos\Delta \end{pmatrix} \tag{52}$$

Multiplying (52) by (51) yields the Stokes vector of the beam incident on the linear $-45°$ polarizer:

$$S = \frac{I_0}{2}\begin{pmatrix} 1 \\ 0 \\ \cos(\Delta + \phi) \\ \sin(\Delta + \phi) \end{pmatrix} \tag{53}$$

Finally, the Mueller matrix for the ideal linear polarizer with its transmission axis at $-45°(+135°)$ is

$$M = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{54}$$

Multiplying the first row of (54) by (53) gives the intensity on the detector, namely,

$$I(\Delta + \phi) = \frac{I_0}{4}[1 - \cos(\Delta + \phi)] \tag{55}$$

We see that a null intensity is found at

$$\Delta = 360° - \phi \tag{56}$$

from which we then find ϕ.

There are still other methods to determine the phase of the wave plate, and the techniques developed here can provide a useful starting point. However, the methods described here should suffice for most problems.

7.4. THE MEASUREMENT OF THE ROTATION ANGLE OF A ROTATOR

The final type of polarizing element which we wish to characterize is a rotator. The Mueller matrix of a rotator is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{57}$$

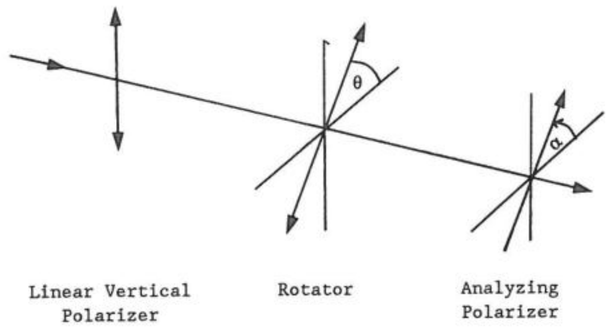

Figure 4 Measurement of the rotation angle θ of a rotator.

The angle θ can be determined by inserting the rotator between a pair of polarizers in which the generating polarizer is fixed in the y position and the analyzing polarizer can be rotated. This configuration is shown in Figure 4.

The Stokes vector of the beam incident on the rotator is

$$S = \frac{I_0}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{58}$$

The Stokes vector of the beam incident on the analyzer is then found by matrix-multiplying (57) by (58):

$$S' = \frac{I_0}{2}\begin{pmatrix} 1 \\ -\cos 2\theta \\ \sin 2\theta \\ 0 \end{pmatrix} \tag{59}$$

The Mueller matrix of the analyzer is

$$M = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\alpha & \sin 2\alpha & 0 \\ \cos 2\alpha & \cos^2 2\alpha & \sin 2\alpha\cos 2\alpha & 0 \\ \sin 2\alpha & \sin 2\alpha \cos 2\alpha & \sin^2 2\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$

The intensity of the beam emerging from the analyzer is then seen from (59) and (60) to be

$$I(\alpha) = \frac{I_0}{4}[1 - \cos(2\alpha + 2\theta)] \tag{61}$$

The analyzer is rotated and, according to (61), a null intensity will be observed at

$$\alpha = 180° - \theta \tag{62a}$$

or, simply,

$$\theta = 180° - \alpha \tag{62b}$$

Another method for determining the angle θ is to rotate the generating polarizer sequentially to 0°, 45°, 90° and 135°. The rotator and the analyzing polarizer are fixed with their axes in the horizontal direction. The intensities of the beam emerging from the analyzing polarizer for these four angles are then

$$I(0°) = \frac{I_o}{4}(1 + \cos 2\theta) \tag{63a}$$

$$I(45°) = \frac{I_o}{4}(1 + \sin 2\theta) \tag{63b}$$

$$I(90°) = \frac{I_o}{4}(1 - \cos 2\theta) \tag{63c}$$

$$I(135°) = \frac{I_o}{4}(1 - \sin 2\theta) \tag{63d}$$

Subtracting (63c) from (63a) and (63d) from (63b) yields

$$\left(\frac{I_o}{2}\right)\cos 2\theta = I(0°) - I(90°) \tag{64a}$$

$$\left(\frac{I_o}{2}\right)\sin \theta = I(45°) - I(135°) \tag{64b}$$

Dividing (64b) by (64a) then yields the angle of rotation θ,

$$\theta = \tan^{-1}[(I(45°) - I(135°))/(I(0°) - I(90°))] \tag{65}$$

In the null-intensity method an optical detector is not required, whereas in this second method a photodetector is needed. However, one soon discovers that even a null measurement can be improved by several orders of magnitude below the sensitivity of the eye by using an optical detector–amplifier combination.

Finally, as with the measurement of wave plates other configurations can be considered. However, the two methods described here should, again, suffice for most problems.

REFERENCES

Books

1. Clark, D., and Grainger, J. F., *Polarized Light and Optical Measurement*, Pergamon Press, Oxford, 1971.

8
Mueller Matrices for Reflection and Transmission

8.1. INTRODUCTION

In previous chapters the Mueller matrices were introduced in a very formal manner. The Mueller matrices were derived for a polarizer, retarder, and rotator in terms of their fundamental behavior; their relation to actual physical problems was not emphasized. In this chapter we apply the Mueller matrix formulation to a number of problems of great interest and importance in the physics of polarized light. One of the major reasons for discussing the Stokes parameters and the Mueller matrices in these earlier chapters is that they provide us with an excellent tool for treating many physical problems in a much simpler way than is usually done in optical textbooks. In fact, one quickly discovers that many of these problems are sufficiently complex that they preclude any but the simplest to be considered without the application of the Stokes parameters and the Mueller matrix calculus.

One of the earliest problems encountered in the study of optics is the behavior of light which is reflected and transmitted at an air-glass interface. Around 1808, E. Malus discovered, quite by accident, that unpolarized light became polarized when it was reflected from glass. Further investigations were made shortly afterward by D. Brewster, who was led to enunciate his famous law relating the polarization of the reflected light and the refractive index of the glass to the incident angle now known as the Brewster angle; the practical importance of this discovery was immediately recognized by Brewster's contemporaries. Consequently, the study of the interaction of light with glass and its reflection and transmission as well as its polarization is a topic of great importance.