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Random unitary dynamics of quantum networks

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Abstract
We investigate the asymptotic dynamics of quantum networks under repeated applications of random unitary operations. It is shown that in the asymptotic limit of large numbers of iterations this dynamics is generally governed by a typically low dimensional attractor space. This space is determined completely by the unitary operations involved and it is independent of the probabilities with which these unitary operations are applied. Based on this general feature analytical results are presented for the asymptotic dynamics of arbitrarily large cyclic qubit networks whose nodes are coupled by randomly applied controlled-NOT operations.

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In recent years specific features of classical networks have been investigated intensively [1] because they exhibit generic organizing principles shared by rather different systems, such as living cells or the Internet. In view of these developments and in view of recent significant progress in the control and manipulation of quantum systems [2] it is natural to extend these investigations into the quantum domain and to explore characteristic phenomena of quantum networks.

In a natural generalization of its classical analog, the nodes of a typical quantum network are formed by spatially localized distinguishable quantum systems, and couplings between different nodes originating from interactions or communication are described in general by a completely positive quantum operation [3]. Iterative applications of this quantum operation give rise to a dynamical evolution of this quantum network. So, in general, contrary to typical interacting many-body quantum systems [4] the couplings between different parts of a quantum network cannot be described by infinitesimal generators, such as Hamiltonians or Lindblad operators. Quantum networks can model rather different physical quantum systems, such as interacting gases or in the context of quantum information processing a quantum-communication-based internet. Contrary to their classical counterparts they are not only able to manipulate classical correlations but also to distribute entanglement which is a characteristic quantum resource. As strong entanglement cannot be shared freely between several quantum systems [5] the distribution of entanglement over a quantum network follows rules which are fundamentally different from those governing the distribution of classical correlations.

In this paper we explore characteristic features of the intricate interplay between entanglement and decoherence in quantum networks. Whereas unitary evolution can generate entanglement, i.e. genuine quantum correlations, decoherence may destroy it again. Thus, the competition between these two counteracting tendencies is expected to produce interesting dynamical features. One of the simplest transformations which can be considered in this context is a random unitary transformation which acts repeatedly on a quantum network. Thereby, in the spirit of similar models studied in the context of random matrix theory [6] the classical randomness involved models uncontrolled external degrees of freedom originating from couplings to a reservoir. Furthermore, iterated random unitary transformations are also useful for implementing various tasks relevant for quantum information processing. In quantum state purification protocols, for example, they can provide reliable approximations for twirling operations which are difficult to implement in other ways [7]. In large quantum networks constituted by large numbers of coupled elementary quantum systems repeatedly applied random unitary transformation typically give rise to complicated dynamical features. In particular, in the asymptotic limit of large numbers of iterations interesting characteristic dynamical features are expected to arise. The complications involved are already apparent in simple paradigms, such as quantum networks whose nodes are formed by elementary two-level quantum systems (qubits) coupled by randomly chosen unitary controlled-NOT operations. Even for moderately large numbers of qubits, say between 10 and 30, a detailed description of such quantum networks becomes computationally intractable. Therefore, alternative theoretical methods have to be developed for exploring their characteristic dynamical features.

In the following it is shown that despite these complications the asymptotic dynamics of arbitrary quantum systems originating from iterated random unitary quantum operations exhibits very regular patterns. It turns out that it is governed by a typically low dimensional attractor space which is determined completely by the unitary transformations involved and its structure is independent of the probability distributions with which these unitary transformations are selected. This basic property of random unitary quantum operations allows us to investigate systematically the asymptotic dynamics even of arbitrarily large quantum networks. As a particular example we present first results exploring the characteristic features of cyclic qubit networks whose dynamics is determined by random unitary controlled-NOT operations. The motivation for this particular choice is the fact that the controlled NOT is the most striking example of an entanglement generating quantum gate and we find the question whether the randomness in its application will lead ultimately to a complete elimination of entanglement in the system particularly interesting.

In order to put the problem into perspective consider an arbitrary quantum system whose dynamics is described by the iterated application of a random unitary operation

$$\Phi(\hat{\rho}) = \sum_{i\in I} p_i \hat{U}_i \hat{\rho} \hat{U}_i^{\dagger}. \tag{1}$$

Thus, a single application of this quantum operation consists of selecting a unitary linear operator $\hat{U}_i \in \mathcal{U} = \{\hat{U}_l | l \in I\}$ randomly according to the normalized probability distribution $\{0 < p_l \leqslant 1, i \in I\}$ and of applying it onto an initially prepared quantum state $\hat{\rho} \in \mathcal{B}(\mathcal{H})$. ($\mathcal{B}(\mathcal{H})$ denotes the Hilbert space of linear operators over the d-dimensional Hilbert space $\mathcal{H}$ describing the quantum system under consideration.) Correspondingly, after n iterations of this random unitary operation the quantum system is in the state $\Phi^{(n)}(\hat{\rho}) \equiv \Phi(\Phi(\Phi(\cdots(\Phi(\hat{\rho})))))$.

The dynamics of (1) can model various physical situations. For example, if the quantum system consists of a large number of distinguishable few-level systems it can model colliding distinguishable low-dimensional systems whose collisions are well separated in time. In this case each $\hat{U}_i \in \mathcal{U}$ describes the unitary evolution of a completed collision which occurs with probability p_i. Alternatively the iterated dynamics of (1) can also describe a quantum walk [8] whose ideal unitary dynamics is perturbed by random imperfections. In the context of quantum information processing this dynamics may model the probabilistic exchange of quantum information between different nodes within a quantum internet in which different nodes entangle each other by unitary operations selected randomly by its users.

Random unitary operations have a number of characteristic properties. Obviously they belong to the class of unital quantum operations [3, 9] which leave the maximally mixed quantum state $\hat{\rho} = \mathbf{1}/d^2$ invariant. Defining the Hilbert–Schmidt scalar product $(\hat{A}, \hat{B}) = \mathrm{Tr}(\hat{A}^\dagger B)$ in the Hilbert space of linear operators $\mathcal{B}(\mathcal{H})$ the adjoint of Φ of (1) is given by

$$\Phi^\dagger(\hat{\rho}) = \sum_{i \in I} p_i \hat{U}_i^\dagger \hat{\rho} \hat{U}_i. \tag{2}$$

Thus, in general Φ is not normal, i.e. $[\Phi^\dagger, \Phi] \neq 0$, so that it cannot be diagonalized. Nevertheless, its properties can still be analyzed systematically with the help of Jordan normal forms [10]. We show in the following that despite the resulting complications the asymptotic dynamics of iterated random unitary operations $\Phi^{(n)}$ is governed by surprisingly regular patterns which considerably simplify their description especially in the large-n limit. In special cases this asymptotic dynamics can even be determined analytically.

For a discussion of this asymptotic dynamics we use the fact that Φ (1) cannot decrease the von Neuman entropy S, i.e.,

$$S(\Phi(\hat{\rho})) \geqslant \sum_{i \in I} p_i S\big(\hat{U}_i \hat{\rho} \hat{U}_i^\dagger\big) = S(\hat{\rho}). \tag{3}$$

The resulting monotony of $S(\Phi^{(n)}(\hat{\rho}))$ and its boundedness for the finite-dimensional quantum systems at hand imply that in the limit of large numbers of iterations $\Phi^{(n)}(\hat{\rho})$ leads to a constant von Neuman entropy. Thus, the relations [11]

$$\lim_{n\to\infty} \big\| \hat{U}_i \hat{\rho}_n \hat{U}_i^\dagger - \hat{U}_{i'} \hat{\rho}_n \hat{U}_{i'}^\dagger \big\| = 0 \tag{4}$$

have to be fulfilled for $\hat{\rho}_n = \Phi^{(n)}(\hat{\rho})$ and for all unitary operations with $i, i' \in I$. For sufficiently large values of n therefore the asymptotic quantum states $\hat{\rho}_n$ are restricted by the requirements

$$\hat{U}_i \hat{\rho}_n \hat{U}_i^\dagger = \hat{U}_{i'} \hat{\rho}_n \hat{U}_{i'}^\dagger. \tag{5}$$

From (5) we note that for a given set of unitary transformations $\mathcal{U}$ the set of all its solutions forms a linear space, the attractor space $\mathcal{A} \subset \mathcal{B}$ and that Φ (1) acts on this attractor space unitarily. As unitary transformations are normal the restriction of Φ onto the attractor space $\mathcal{A}$ can be diagonalized with the help of a complete set of orthonormal eigenvectors $\hat{X}_\lambda \in \mathcal{A}$. These eigenvectors fulfil the eigenvalue equations and orthonormality constraints

$$\hat{U}_i \hat{X}_\lambda \hat{U}_i^\dagger = \lambda \hat{X}_\lambda, \mathrm{Tr}\big(X_\lambda^\dagger X_{\lambda'}\big) = \delta_{\lambda\lambda'} \tag{6}$$

for all $i \in I$. From (6) it is apparent that the linear operators of the attractor space $\mathcal{A}$ also form a C^*-algebra. This property implies a number of useful relations, such as

$$|\lambda| = 1, \qquad \hat{X}_\lambda^\dagger = \hat{X}_{\lambda^*}, \qquad \lambda^n \lambda'^m \neq 1 \longrightarrow \mathrm{Tr}\left(\hat{X}_\lambda^n \hat{X}_{\lambda'}^m\right) = 0.$$

Thus, the attractor space $\mathcal{A}$ is defined by all linear operators $\hat{X}_\lambda$ fulfilling (6) with the additional requirement $|\lambda| = 1$. Furthermore, all asymptotically accessible quantum states $\hat{\rho}_n = \Phi^{(n)}(\hat{\rho})$ are contained in this attractor space, i.e. $\lim_{n\to\infty} \hat{\rho}_n \in \mathcal{A}$. Let us now introduce the (continuous) projection operator

$$\mathcal{P}(.) = \sum_{|\lambda|=1} \mathrm{Tr}\big(\hat{X}_\lambda^\dagger(.)\big)\hat{X}_\lambda, \tag{7}$$

which projects the linear space $\mathcal{B}$ onto the attractor space $\mathcal{A}$. In view of the relation $1/\lambda^* = \lambda$ for $|\lambda| = 1$ it commutes with the random unitary operations of the random unitary channel (1), i.e. $[\mathcal{P}, \Phi] = 0$, so that we can conclude with the help of (6) and (7)

$$\begin{aligned}\lim_{n\to\infty} \Phi^{(n)}(\hat{\rho}) &= \mathcal{P}(\lim_{n\to\infty} \Phi^{(n)}(\hat{\rho})) = \lim_{n\to\infty} \Phi^{(n)}\,(\hat{\rho}_{\mathrm{out}}) \\ &= \lim_{n\to\infty} \hat{U}_{l_0}^{(n)}\hat{\rho}_{\mathrm{out}}\hat{U}_{l_0}^{(n)\dagger} = \lim_{n\to\infty} \sum_{|\lambda|=1} \lambda^n\, \mathrm{Tr}\big(\hat{X}_\lambda^\dagger(\hat{\rho})\big)\hat{X}_\lambda\end{aligned} \tag{8}$$

with $l_0 \in I$ arbitrary and with the projected initially prepared quantum state $\hat{\rho}_{\mathrm{out}} = \mathcal{P}(\hat{\rho})$. Equation (8) is a main result of our work. It shows explicitly how the asymptotic dynamics of Φ (1) is determined by the structure of the attractor space $\mathcal{A}$. Of course, (8) could also have been derived in a direct but more cumbersome way with the help of a Jordan normal form decomposition of Φ [12].

According to (8) the determination of the asymptotic dynamics of $\Phi^{(n)}$ can be divided into two main steps. In the first step one determines the set of eigenvalues λ of unit modulus and the associated orthonormal basis $\{\hat{X}_\lambda\}$ of the linear space $\mathcal{A}$ using (6). Typically, this is a difficult task which may be facilitated by symmetries. In the second step one evaluates the unitary asymptotic action of $\Phi^{(n)}$ onto the projected quantum state $\hat{\rho}_{\mathrm{out}}$ according to (8). Typically, the attractor space is expected to be of sufficiently low dimensions so that (8) implies significant simplifications.

Equation (8) hints at some remarkable general features of the asymptotic dynamics of $\Phi^{(n)}$. First of all, the space of attractors $\mathcal{A}$ is determined completely by the set of unitary operators $\mathcal{U}$. In particular, this implies that the asymptotic dynamics is independent of the classical probability distribution characterizing Φ. Nevertheless, in general this probability distribution still influences the rate of convergence. Furthermore, if one of the unitary operations from the set $\mathcal{U}$ is the unit operation (apart from a global phase) the only possible eigenvalue is given by $\lambda = 1$. The resulting asymptotic dynamics is thus stationary. Another general consequence can be derived for unitary operations $\mathcal{U}$ which generate a ray representation of a finite group. Because the unit-element of this group is a product of the generators contained in $\mathcal{U}$, for each eigenvalue λ there is a natural number n_λ with $\lambda^{n_\lambda} = 1$. Thus, the resulting asymptotic dynamics is periodic. However, (8) also applies to the most general non-stationary and non-periodic cases of asymptotic dynamics.

Let us now apply the general results (8) to the description of the asymptotic dynamics of a particular class of quantum networks consisting of qubits which are coupled by randomly selected controlled-NOT operations. A controlled-NOT operation between qubits i and j is defined by $C_{i,j}|a\rangle_i \otimes |b\rangle_j = |a\rangle_i \otimes |b \oplus a\rangle_j$. It is known to entangle qubits effectively because it produces pure maximally entangled Bell states from separable states of the form $(|0\rangle_i + |1\rangle_i)/\sqrt{2} \otimes |b\rangle_j$, for example. Here, the pure states $|a\rangle$ with $a \in \{0, 1\}$ denote orthonormal basis states of the computational basis of a qubit and $a \oplus b$ with $b \in \{0, 1\}$ denotes addition modulo 2. Let us further assume that our network has a one-dimensional cyclic topology so that nodes i and $i + 1$ with $i = 1, \ldots, N$ are coupled by controlled-NOT operations $C_{i,i+1}$ and that in view of the cyclic topology qubits $N + 1$ and 1 are identical (see figure 1).

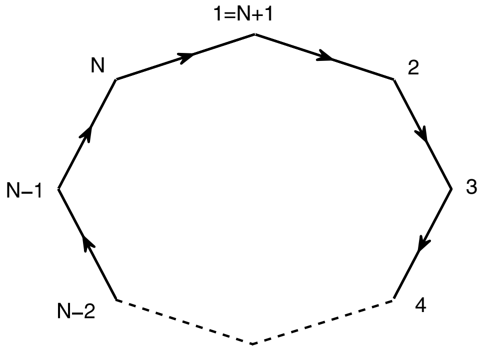

Figure 1. Network with one-dimensional cyclic topology. Vertices correspond to nodes (qubits), oriented edges visualize which nodes are coupled by controlled-NOT operations.

In order to determine the asymptotic limit of the corresponding random unitary operation one has to solve the eigenvalue problem (6). The possible eigenvalues can be determined easily by noting that controlled-NOT operations have the property $C_{i,i+1}^2 = \mathbf{1}$. Therefore, the only possible eigenvalues (6) are given by $\lambda = \pm 1$. The determination of all eigenvectors with eigenvalue $\lambda = 1$ is facilitated by the observation that the pure quantum states $|0\rangle$ and $|\Phi\rangle = \sum_{z=1}^{2^N-1} |z\rangle/\sqrt{2^N-1}$ (with $|z\rangle \equiv |j_N\rangle|j_{N-1}\rangle\cdots|j_2\rangle|j_1\rangle$, $z = \sum_{i=1}^N 2^{i-1} j_i$, and $j_i \in \{0, 1\}$) are invariant under all controlled-NOT operations of the cyclic network. Using the additional fact that the unit operator is a solution of (6) with eigenvalue $\lambda = 1$ we find the following orthonormal eigenvectors for $\lambda = 1$

$$\begin{aligned} &\hat{X}_1 = |0\rangle\langle 0|, \qquad \hat{X}_2 = |0\rangle\langle\Phi|, \qquad \hat{X}_3 = |\Phi\rangle\langle 0|, \\ &\hat{X}_4 = |\Phi\rangle\langle\Phi|, \qquad \hat{X}_5 = (\mathbf{1} - |0\rangle\langle 0| - |\Phi\rangle\langle\Phi|)/\sqrt{2^N - 2}. \end{aligned} \tag{9}$$

By a somewhat lengthy calculation one can prove from (6) by induction that there are no additional eigenvectors. Thus, for any number of qubits N the eigenspace associated with $\lambda = 1$ is five dimensional and is given by (9). By induction one can also demonstrate that for $N > 2$ solutions of (6) with eigenvalue $\lambda = -1$ do not exist. It is only in the special case of $N = 2$ that a non-trivial normalized eigenvector exists. Explicitly it is given by [12]

$$\hat{X}_6 = -|0\rangle|1\rangle\langle 1|\langle 0| + |0\rangle|1\rangle\langle 1|\langle 1| + |1\rangle|0\rangle\langle 0|\langle 1| - |1\rangle|0\rangle\langle 1|\langle 1| - |1\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|1\rangle\langle 1|\langle 0|. \tag{10}$$

Having determined an orthonormal basis of the attractor space $\mathcal{A}$, the general form of the asymptotic dynamics of the quantum network can be determined with the help of (8). Thus, for any number of qubits with $N > 2$ the projected quantum state $\hat{\rho}_{\text{out}}$ is given by

$$\hat{\rho}_{\text{out}} = p\frac{\hat{P}_2\hat{\rho}\hat{P}_2}{p} + (1-p)\frac{\mathbf{1} - \hat{P}_2}{2^N - 2}. \tag{11}$$

This state is stationary because it is invariant under all controlled-NOT operations under consideration. Here, the projection operator $\hat{P}_2 = |0\rangle\langle 0| + |\Phi\rangle\langle\Phi|$ projects onto the two-dimensional subspace $\mathcal{H}_2 \subset \mathcal{H}$ spanned by the pure states $|0\rangle$ and $|\Phi\rangle$ which are invariant under all controlled-NOT operations under consideration. The probability $p = \text{Tr}\{\hat{P}_2\hat{\rho}\}$ measures the overlap of the initially prepared quantum state $\hat{\rho}$ with this invariant subspace. Equation (11) implies that any initially prepared quantum state $\hat{\rho}$ which is contained completely in subspace $\mathcal{B}(\mathcal{H}_2)$ is not affected by the randomly applied controlled-NOT operations. Thus, $\mathcal{B}(\mathcal{H}_2)$ forms a decoherence-free subspace [13, 14].

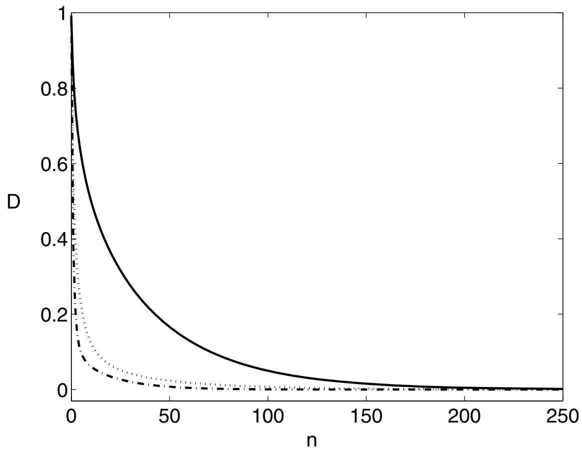

Figure 2. Distance D between the nth iterate $\hat{\rho}_n$ and the projected quantum state $\hat{\rho}_{\text{out}}$ for different pure six-qubit states $\hat{\rho} = |\psi\rangle\langle\psi|$: $|\psi\rangle = |000001\rangle$(——), $|111111\rangle$(— · —), $|101010\rangle$(· · · · · ·).

Let us finally discuss the convergence towards the asymptotic dynamics. A convenient measure for the distance D between the quantum state after n iterations $\hat{\rho}_n$ and the corresponding asymptotic quantum state $\hat{U}^n_{l_0}\hat{\rho}_{\text{out}}\hat{U}^{n\dagger}_{l_0}$ can be based on the Hilbert–Schmidt norm by the relation $D^2 = (\hat{\rho}_n - \hat{U}^n_{l_0}\hat{\rho}_{\text{out}}\hat{U}^{n\dagger}_{l_0}, \hat{\rho}_n - \hat{U}^n_{l_0}\hat{\rho}_{\text{out}}\hat{U}^{n\dagger}_{l_0})$.

In figure 2 numerical results are presented depicting the distance D between the states $\hat{\rho}_n$ and $\hat{\rho}_{\text{out}} = \hat{U}^n_{l_0}\hat{\rho}_{\text{out}}\hat{U}^{n\dagger}_{l_0}$ for a cyclic quantum network consisting of $N = 6$ qubits and for various initial quantum states $\hat{\rho}$ with a uniform probability distribution. It is apparent that convergence is achieved in a strictly monotonic way. Nevertheless, the rate of convergence depends on the initially prepared quantum state.

In figure 3 analogous numerical results are depicted for a cyclic quantum network consisting of two qubits and for the distance D between the states $\hat{\rho}_n$ and $\hat{\rho}_{\text{out}}$. Contrary to the previously considered case, now the asymptotic dynamics is not always stationary, i.e. typically $\hat{\rho}_{\text{out}} \neq \hat{U}^n_{l_0}\hat{\rho}_{\text{out}}\hat{U}^{n\dagger}_{l_0}$, because according to (10) there is also a non-trivial eigenspace with eigenvalue $\lambda = -1$. Depending on whether the initially prepared quantum state $\hat{\rho}$ overlaps with this eigenspace or not the asymptotic dynamics of $\hat{\rho}_n$ is non-stationary or stationary.

In conclusion, we have presented a general method which allows us to determine the asymptotic dynamics of arbitrary quantum systems under the influence of iterated random unitary operations. It is based on the determination of the associated asymptotic attractor space. This attractor space is independent of the probability distribution characterizing the classical randomness and is typically low dimensional thus simplifying the asymptotic dynamical description considerably. Although this probability distribution influences the rate of convergence the asymptotic dynamics itself is robust against perturbations of the classical randomness involved. As a particular application of this method results have been presented for cyclic qubit networks. From these results we can conclude that the classical randomness involved in random unitary transformations may not always lead to a stationary asymptotic state. Furthermore, in cases in which it does the resulting asymptotic stationary state need not be a maximally mixed state, i.e., the dynamics does not lead to complete thermalization. The asymptotic dynamics may involve a decoherence-free subspace. In such cases, all quantum coherences and entanglement within this decoherence free subspace will be preserved but

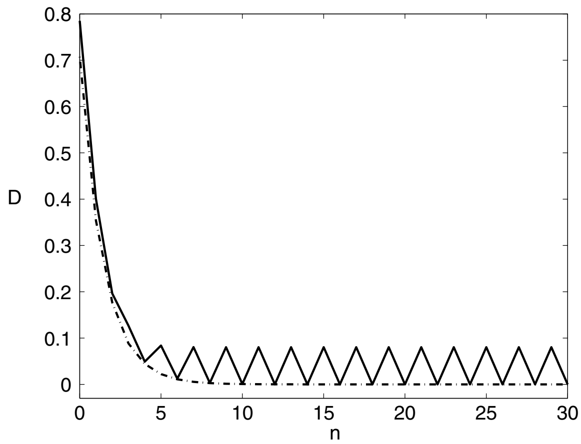

Figure 3. Distance D between the nth iterate $\hat{\rho}_n$ and the projected quantum state $\hat{\rho}_{\text{out}}$ for different pure two-qubit states $\hat{\rho} = |\psi\rangle\langle\psi|/\sqrt{|\langle\psi|\psi\rangle|}$: $|\psi\rangle = 0.3\,e^{i\pi/5}|10\rangle + e^{i\pi/7}|11\rangle$(——), $|10\rangle + |11\rangle$(— · —).

all quantum coherences between quantum states inside and outside this decoherence-free subspace will be destroyed completely in the asymptotic limit.

It is expected that the exploration of the structure of attractor spaces associated with the dynamics of more general networks will shed light also onto other open problems of network dynamics in the quantum domain, such as the connection between network topology and the resulting asymptotic dynamics.

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