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Conditions for entanglement purification with general two-qubit states

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We present the convergence study of a recurrence entanglement purification protocol using arbitrary two-qubit initial states. The protocol is based on a rank 2 projector in the Bell basis which serves as a two-qubit operation replacing the usual controlled-NOT gate. We show that the whole space of two-qubit density matrices is mapped onto an invariant subspace characterized by seven real parameters. By analyzing this type of density matrices we are able to find general conditions for entanglement purification in the form of two inequalities between pairs of diagonal elements and pairs of coherences. We show that purifiable initial states do not necessary require a fidelity larger than one half with respect to any maximally entangled pure state. Furthermore, we find a family of states parametrized by their concurrence that can be perfectly converted into a Bell state in just one step of the protocol with probability proportional to the square of the concurrence.

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I. INTRODUCTION

Entanglement purification protocols [1–3] generate Bell states from an ensemble of noisy entangled states. They have found application in many areas of quantum information theory [4] and prominently in proposed quantum communication technologies [5]. From a mathematical perspective, the effect of these protocols can be described as a nonlinear map of the parameters defining the input density matrices. The first of the so-called recurrence protocols [4] was introduced by Bennett and collaborators [1]. This protocol relies on Werner states [6], which are described by one parameter and therefore the convergence of the corresponding map can be studied in a simple way [1]. A more efficient protocol was introduced by Deutsch *et al.* [2] based on Bell diagonal states involving a three-dimensional parameter space. The convergence analysis is more intricate in this case and was studied separately in detail by Macchiavello [7].

There are two essential steps in recurrence entanglement purification protocols [4], which are reconsidered in this work: First, the application of random local unitary rotations transforming any initial density matrix into a Werner [2] or a Bell-diagonal state [3]. Second, the application of bilateral controlled-NOT gates on the spatially separated quantum systems. The latter step has already been modified in our proposed purification protocol [8] based on Refs. [1,2]. The modification was motivated by the design of a multiphoton-assisted quantum repeater and instead of controlled-NOT gates it is based on a rank 2 projection in the Bell basis which can be implemented in a one-atom maser setup [9]. We have studied in Ref. [8] the performance of the modified protocol using Werner and other Bell-diagonal states. A natural question arises of whether this protocol can still work for arbitrary input states making the step of random local unitary rotations ("twirling") superfluous. This problem holds great significance, because random operations can waste important entanglement and their implementation is not straightforward. Therefore, a solution to this problem could offer a less sophisticated purification protocol and a better compatibility with certain experimental settings.

In this paper we show that our protocol, first introduced in Ref. [8], can exploit the entanglement of arbitrary initial states without using random unitary operations. After one iteration of the protocol, any input state is mapped onto a density matrix described by seven real parameters. This form is invariant under further iterations of the protocol. We exploit this feature to analyze the convergence properties for any input state of this form. Based on this analysis, we generalize to the case of arbitrary two-qubit states and find conditions that allow purification of Bell states. We demonstrate that in contrast to previous schemes [1,2], a purifiable state does not require an initial fidelity larger than one half with respect to any Bell state. In particular, we find a class of states with overlap less than one half with any maximally entangled pure state that can be purified in just one step of the protocol.

The paper is organized as follows. In Sec. II we reintroduce and explain the steps of our protocol. In Sec. III we study the set of density matrices characterized by seven real parameters and derive the conditions for a successful entanglement purification. In Sec. IV we extend the analysis to density matrices characterized by 15 real parameters. Examples which obey the newly found conditions are presented in Sec. V. In the appendix we present the stability analysis of the fixed points for the seven-dimensional parameter space.

II. ENTANGLEMENT PURIFICATION PROTOCOL

In this section we review the entanglement purification protocol introduced in Ref. [8], which is based on the protocols of Refs. [1,2]. Let us consider as initial condition the product state of two-qubit pairs

$$\boldsymbol{\rho} = \rho^{A_1,B_1}\rho^{A_2,B_2}. \tag{1}$$

Both pairs are assumed to start in the same state ρ with a certain degree of entanglement and their qubit components are assumed to be in distant locations labeled by A and B. The aim of an entanglement purification protocol is to trade two pairs for one pair with a larger degree of entanglement using only local operations in laboratories A and B and classical communication between them. This can be achieved

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by increasing the fidelity with respect to any of the Bell states,

$$|\Psi^{\pm}\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \quad |\Phi^{\pm}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle). \tag{2}$$

The purification protocol that we consider in this work consists of the following steps:

(i) The two-qubit quantum operation M (a rank 2 projector) is applied locally in locations A and B, where

$$M = |\Psi^-\rangle\langle\Psi^-| + |\Phi^-\rangle\langle\Phi^-|. \tag{3}$$

After a successfully applied quantum operation the four-qubit system attains the state

$$\boldsymbol{\rho}' = \frac{\Pi\boldsymbol{\rho}\Pi^\dagger}{\mathrm{Tr}\{\Pi^\dagger\Pi\boldsymbol{\rho}\}}, \quad \Pi = M^{A_1,A_2}M^{B_1,B_2}. \tag{4}$$

(ii) One of the pairs is then locally measured in the computational basis, say pair (A_2,B_2). The choice of the measured pair is unimportant as the operation is symmetric and the initial qubit pairs are identical. There are four possible states in which one can find pair (A_2,B_2): $|jk\rangle^{A_2,B_2} \equiv |j\rangle^{A_2}|k\rangle^{B_2}$ with $j,k \in \{0,1\}$. A successful measurement of one of the states $|jk\rangle^{A_2,B_2}$ results in the two-qubit state

$$\tilde{\rho}^{A_1,B_1} = \mathrm{Tr}_{A2,B2}\{|jk\rangle\langle jk|^{A_2,B_2}\boldsymbol{\rho}'\}. \tag{5}$$

(iii) Depending on the result of the measurement, the quantum gate $V_j^{A_1}V_{k+1}^{B_1}$ is applied to the remaining qubit pair, where

$$V_j = (|1\rangle\langle 1| + i|0\rangle\langle 0|)\sigma_x{}^j \tag{6}$$

with the Pauli operator $\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$. The final two-qubit state is then given by

$$\rho'^{A_1,B_1} = \big(V_j^{A_1}V_{k+1}^{B_1}\big)\tilde{\rho}^{A_1,B_1}\big(V_j^{A_1}V_{k+1}^{B_1}\big)^\dagger. \tag{7}$$

We remark again that there is a free choice of the qubit pair to be measured as our quantum operation M is symmetric in contrast to the controlled-NOT gate used in the seminal protocols [1,2]. As the entangled pairs are assumed to start in the same state ρ, in what follows we will omit the labels and study the two-qubit map $\rho \to \rho'$.

Now that we have explained the key elements of the protocol, we will study the resulting map between initial and output states of the qubit pairs. Consider a general two-qubit density matrix that can be written as

$$\rho = \sum_{i,j=1}^{4} r_{ij}|i\rangle\langle j|, \quad r_j \equiv r_{jj} \tag{8}$$

where we have chosen to work in the Bell basis labeled in the following way:

$$|1\rangle \equiv |\Psi^-\rangle, \quad |2\rangle \equiv |\Phi^-\rangle, \quad |3\rangle \equiv |\Phi^+\rangle, \quad |4\rangle \equiv |\Psi^+\rangle. \tag{9}$$

Due to the conditions $\mathrm{Tr}\{\rho\} = 1$ and $\rho^\dagger = \rho$ we have the following relations:

$$r_1 + r_2 + r_3 + r_4 = 1, \quad r_{ij} = (r_{ji})^*.$$

The probabilistic part of this protocol takes place in step (i), i.e., in the application of the bilateral two-qubit quantum operation M. The success probability of the quantum operation is $N/2$, with

$$N = (r_1 + r_2)^2 + (r_3 + r_4)^2 - (r_{12} + r_{21})^2 - (r_{34} + r_{43})^2. \tag{10}$$

Provided both implementations of M are successful, the steps (ii) and (iii) can be taken for granted as they involve unitary gates and none of the qubit measurements is discarded. The resulting two-qubit density matrix ρ' in the same basis as (8) has the nonzero entries

$$r_1' = \frac{r_1^2 + r_2^2 - r_{12}^2 - r_{21}^2}{N}, \quad r_2' = 2\frac{r_3 r_4 - |r_{34}|^2}{N},$$
$$r_4' = \frac{r_3^2 + r_4^2 - r_{34}^2 - r_{43}^2}{N}, \quad r_3' = 2\frac{r_1 r_2 - |r_{12}|^2}{N},$$
$$r_{14}' = \frac{r_{14}^2 + r_{23}^2 - r_{13}^2 - r_{24}^2}{N}, \quad r_{23}' = 2\frac{r_{23}^* r_{14}^* - r_{13}^* r_{24}^*}{N}. \tag{11}$$

Together with $r_{41}' = (r_{14}')^*$ and $r_{32}' = (r_{23}')^*$, these are the only nonvanishing elements of ρ' after a single iteration of the protocol. The constant N plays here the role of a normalization factor and the resulting density matrix has eight vanishing entries. Therefore, only seven parameters characterize these type of states, in contrast to the 15 parameters needed to describe a general two-qubit state. This simplification motivates the study of the purification protocol for density matrices described by the seven primed parameters of Eq. (11).

It is worth noting that the protocol in Ref. [2] is also able to reduce the 15-dimensional parameter space to a seven-dimensional one. This happens technically in a rather different way, as in our protocol. After measuring the target pair during the procedure, one keeps the control pair only if the target pair was found in the state $|11\rangle$. The result can be obtained by a straightforward application of the protocol in Ref. [2] on general two-qubit density matrices and to our knowledge it has never been reported.

III. ENTANGLEMENT PURIFICATION OF X STATES IN THE BELL BASIS

In this section we concentrate our study on initial two-qubit states described by seven free parameters, which have been delineated in Eq. (11). In the Bell basis (9) they have the matrix representation

$$\rho = \begin{pmatrix} r_1 & 0 & 0 & r_{14} \\ 0 & r_2 & r_{23} & 0 \\ 0 & r_{32} & r_3 & 0 \\ r_{41} & 0 & 0 & r_4 \end{pmatrix}, \tag{12}$$

and therefore we refer to them them as X states. The diagonal elements $r_1,r_2,r_3,r_4 \in [0,1]$ and the off-diagonal elements $r_{23}, r_{14} \in \mathbb{C}$ fulfill the conditions $r_1 + r_2 + r_3 + r_4 = 1$, $r_{32} = r_{23}^*$ and $r_{41} = r_{14}^*$. The four eigenvalues can be represented as the two pairs $\lambda_{23}^{\pm}$ and $\lambda_{14}^{\pm}$ which can be obtained from the quadratic formula

$$\lambda_{jk}^{\pm} = \frac{r_j + r_k \pm \sqrt{(r_j - r_k)^2 + 4|r_{jk}|^2}}{2}. \tag{13}$$

As the eigenvalues of ρ have to be real non-negative numbers, the radical in Eq. (13) has to be positive, and this imposes the following restriction to the absolute value of the coherences:

$$|r_{23}| \leqslant \sqrt{r_2 r_3}, \quad |r_{14}| \leqslant \sqrt{r_1 r_4}. \tag{14}$$

In the case of input states in the form of Eq. (12), the output state ρ' remains in the same form and according to (11) with the new coefficients,

$$r_1' = \frac{r_1^2 + r_2^2}{N}, \quad r_2' = 2\frac{r_3 r_4}{N}, \quad r_{14}' = \frac{r_{14}^2 + r_{23}^2}{N},$$
$$r_4' = \frac{r_3^2 + r_4^2}{N}, \quad r_3' = 2\frac{r_1 r_2}{N}, \quad r_{23}' = 2\frac{r_{23}^* r_{14}^*}{N}, \tag{15}$$

and the normalization factor

$$N = (r_1 + r_2)^2 + (1 - r_1 - r_2)^2. \tag{16}$$

In order to find the convergence properties of this map, we will first study the behavior of the off-diagonal elements after one iteration by considering the norm of the pair (r_{14}, r_{23}). It turns out that this quantity does not increase, but typically decreases after iteration of the map. This can be noted by analyzing the ratio

$$\frac{|r_{14}'| + |r_{23}'|}{|r_{14}| + |r_{23}|} \leqslant \frac{|r_{14}| + |r_{23}|}{N} \leqslant 1, \tag{17}$$

where we have used the normalization factor in Eq. (16). The first inequality can be obtained by direct evaluation of r_{14}' and r_{23}' using Eq. (15) and by taking into account the triangle inequality $|r_{14}^2 + r_{23}^2| \leqslant |r_{14}|^2 + |r_{23}|^2$. The second inequality in Eq. (17) follows from considering the ratio between the maximum value of $|r_{14}| + |r_{23}|$ and the minimum value of N, both of which are 1/2. To show this, first note that according to Eq. (16), N attains its minimum value 1/2 when $r_1 + r_2 = 1/2$. Second, we take into account the conditions in (14) that combined with the normalization condition for ρ and the fact that for any pair of real numbers $2xy \leqslant x^2 + y^2$ implies the relation $|r_{14}| + |r_{23}| \leqslant 1/2$ between the coherences [10]. Taking the equality case in (14) one can note that the equality in Eq. (17) is achieved only with the initial conditions

$$|r_{14}| = r_1 = r_4 = 0.5, \quad |r_{23}| = r_2 = r_3 = 0,$$
$$|r_{14}| = r_1 = r_4 = 0, \quad |r_{23}| = r_2 = r_3 = 0.5, \tag{18}$$
$$|r_{14}| = |r_{23}| = r_1 = r_2 = r_3 = r_4 = 0.25.$$

Only in these three cases does the norm of the coherences remain constant after repeated iteration of the map in (15). For all other values of the coherences, the ratio in Eq. (17) is strictly less than one and therefore r_{14} and r_{23} tend to zero by repeated iterations of the map (15). We remark that in all these three cases, all the diagonal values are less or equal than one half.

Now we turn our attention to the diagonal elements. The aim is to find the initial conditions that after iteration of the map tend to one of the Bell states. The protocol we have presented here is not the same as the protocol in Ref. [2]; however, the map (15) for the diagonal elements r_1, r_2, r_3, r_4 has a similar structure. Therefore, we follow the idea of the proof in Ref. [7], where Bell diagonal density matrices were studied. The same treatment is justified in our case as the diagonal elements in Eq. (15) do not depend on the off-diagonal ones.

We start this discussion by noting that $r_1' \geqslant r_3'$ and $r_4' \geqslant r_2'$, which indicates that neither r_2 or r_3 can reach unit value, actually not even above 1/2. With this fact we recognize that only r_1 or r_4 can potentially increase their value to unity after repeated iterations of the map (15). Noting the symmetry of the map under interchange of $r_1 \leftrightarrow r_4$ and $r_2 \leftrightarrow r_3$ allows us to consider only the maximization of one of them, say, r_1. Indeed the case $r_1 = 1$ and all other parameters equal to zero is a fixed point of the map (15). Now, let us rewrite the value of r_1' as

$$r_1' = \frac{r_1^2 + r_2^2}{2(r_1^2 + r_2^2) - F(r_1, r_2)},$$

were we have introduced the quadratic form

$$F(r_1, r_2) = (2r_1 - 1)(1 - 2r_2). \tag{19}$$

Whenever $F(r_1, r_2)$ is positive, r_1' attains values larger than 1/2. This actually happens whenever r_1 or r_2 are larger than 1/2 and in this case the function F increases after iteration of the map (15), as can be noted from the relation

$$\frac{F(r_1', r_2')}{F(r_1, r_2)} = \frac{N - 4r_3 r_4}{N^2} = \frac{(1-x)^2 + y^2}{[(1-x)^2 + x^2]^2} \geqslant 1, \tag{20}$$

with $x = r_3 + r_4$ and $y = r_4 - r_3$. The inequality follows from the fact that $1 - x \geqslant (1-x)^2 + x^2$ for $x \in [0, 1/2)$, which holds true if $r_1 > 1/2$ or $r_2 > 1/2$. The function $F(r_1, r_2)$ increases its value after application of the map (15) when the strict inequality is met. In this case, as $F(r_1, r_2)$ is a monotonic function of r_1 and r_2, iteration of the map converges to the fixed point with $r_1 = 1$ which is the only fixed point with the condition $r_1 > 1/2$. The equality in (20) happens only in the case $r_3 = r_4 = 0$, but after each iteration of the map (15) with initial values $r_3 = r_4 = 0$, the values of r_3 and r_4 cannot be simultaneously zero and thus $F(r_1'', r_2'') > F(r_1', r_2')$ unless $r_2 = 0$ but that would be the case of the fixed point with $r_1 = 1$. This proves that the condition to purify to a Bell state $|\Psi^-\rangle$, i.e., $r_1 = 1$, can be written as

$$(2r_1 - 1)(1 - 2r_2) > 0. \tag{21}$$

In complete analogy due to the symmetry $r_1 \leftrightarrow r_4$ and $r_2 \leftrightarrow r_3$, one can find the condition to purify to Bell state $|\Psi^+\rangle$, i.e., $r_1 = 1$, can be written as

$$(2r_4 - 1)(1 - 2r_3) > 0. \tag{22}$$

These inequalities define four purifiable regions in the three-dimensional space of the diagonal parameters shown in Fig. 1. These findings are similar to Ref. [7], where purifications of the states $|\Phi^+\rangle$ and $|\Psi^+\rangle$ have been proven. Here, our proof is extended, taking into account off-diagonal elements of X states subject to our protocol. We have shown in Eqs. (17) and (18) that the absolute value of the off-diagonal parameters decreases in the regions subject to the conditions (21) and (22). Therefore each of the regions depicted in blue (gray) and green (light gray) in Fig. 1 have one attracting fixed point. In the appendix we evaluate numerically the stability of the fixed points.

IV. CONDITIONS FOR GENERAL TWO-QUBIT DENSITY MATRICES

In this section we extend our analysis to general two-qubit density matrices with the representation given in Eq. (8). In this case after one iteration one obtains the density matrix of an X state [see Eq. (12)]. Therefore, the most important question is to define the conditions for which after the first iteration of the map (11) any of the four r_i''s is larger than 1/2. First

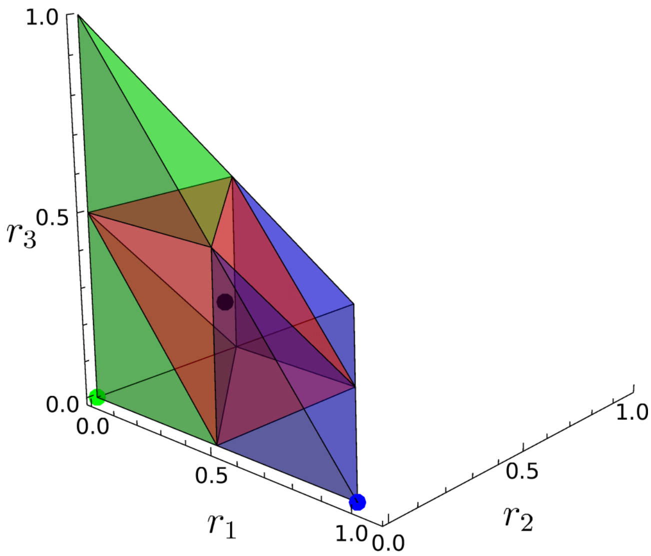

FIG. 1. Three-dimensional space of the independent diagonal elements (r_1,r_2,r_3). The other parameter r_4 is obtained from the trace condition Tr$\rho = 1$. In blue (gray): the region defined by condition (21) whose points lead to the fixed point $r_1 = 1$. In green (light gray): the region defined by condition (22) that leads to the fixed point $r_4 = 1$. In red (middle gray): all other conditions that do not lead to an entangled state. The blue (gray) and green (light gray) dots are the stable fixed point that corresponds to Bell states $|\Psi^-\rangle$ and $|\Psi^+\rangle$, whereas the black dot in the middle is the one representing the maximally mixed state.

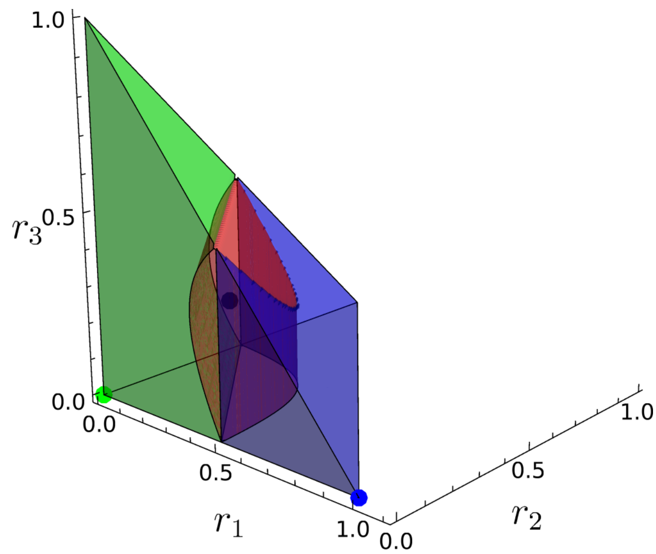

FIG. 2. Three-dimensional space of the independent diagonal elements (r_1,r_2,r_3). The other parameter r_4 is obtained from the trace condition Tr$\rho = 1$. In blue (gray): the region defined by condition (25) whose points lead to the fixed point $r_1 = 1$ and all others zero. In green (light gray): the region defined by condition (26) that leads to the fixed point $r_4 = 1$. In red (middle gray): all other conditions that do not lead to an entangled state. The initial coherences have the form in (27) with $\eta_a = 0.4$ and $\eta_b = 0.1$.

we note that one can never achieve $r_2' > 1/2$ or $r_3' > 1/2$ after one iteration. This due to the fact that $r_1' \geqslant r_3'$ and $r_4' \geqslant r_2'$, which is noted from (11) by comparing their numerators and considering that $x^2 + y^2 \geqslant 2xy$ for any $x,y \in \mathbb{R}$ and because $z^2 + z^{*2} \leqslant 2|z|^2$ for any $z \in \mathbb{C}$. This means that we only have to find those conditions that leave $r_1' > 1/2$ or $r_4' > 1/2$. For symmetry reasons it suffices to focus on only one; we choose r_1. Let us introduce the following abbreviations,

$$a = \left(r_1^2 + r_2^2 - r_{12}^2 - r_{21}^2\right),$$
$$b = (2r_1 - 1)(1 - 2r_2) - (r_{12} - r_{21})^2 + (r_{34} + r_{43})^2, \tag{23}$$

which relate to the normalization factor as $N = 2a - b$. This form of rewriting N allows us to identify the fidelity with respect to $|\Psi^-\rangle$ after one step of the map in Eq. (11), namely

$$r_1' = a/(2a - b). \tag{24}$$

This expression is larger than $1/2$ whenever $b > 0$ or equivalently when the following inequality is met:

$$(2r_1 - 1)(1 - 2r_2) > -(2\text{Im}[r_{12}])^2 - (2\text{Re}[r_{34}])^2. \tag{25}$$

This condition generalizes (21), as any state fulfilling it is transformed by the map into an X state of the form of (12), but with the coefficient $r_1' > 1/2$. In Sec. III we have proved that these type of states can be purified. Because the right-hand side of Eq. (25) is always zero or negative, the inequality is always fulfilled whenever r_1 or r_2 are larger than $1/2$ and actually allows them to be smaller than this threshold. This actually means that the purification protocol is able to convert the entanglement of all states fulfilling (25) into $|\Psi^-\rangle$. Symmetry arguments lead to an analog condition for purifying $|\Psi^+\rangle$ with r_3 and r_4, namely

$$(2r_4 - 1)(1 - 2r_3) > -(2\text{Im}[r_{34}])^2 - (2\text{Re}[r_{12}])^2. \tag{26}$$

Equations (25) and (26) are the main result of this work and define the conditions to purify entanglement in the presented protocol. These conditions compared with (21) and (22) relax the requirement for the diagonal parameters, because the right-hand side of the inequalities in (25) and (26) can be negative and thus r_1,r_2,r_3 and r_4 can be smaller than $1/2$. This clearly means that there exists states which do not have an overlap with a Bell state larger than $1/2$ but their ensemble can still be purified to $|\Psi^-\rangle$ or $|\Psi^+\rangle$. In the subsequent discussion, we investigate the inequalities of Eqs. (25) and (26).

In order to illustrate the difference with the previous section we consider the following dependence of the coherences

$$r_{12} = \eta_a\sqrt{r_1 r_2}, \quad r_{34} = \eta_b\sqrt{r_3 r_4}, \quad \eta_a,\eta_b \in [0,1]. \tag{27}$$

This case can be visualized in Fig. 2 for the specific values of $\eta_a = 0.4$ and $\eta_b = 0.1$. In the limit when $\eta_a,\eta_b \to 1$, the volume of the red (middle gray) region tends to zero and is covered equally by the green (light gray) and blue (gray) regions. In fact the red (middle gray) region becomes a two-dimensional shape and defines the border between the other two regions.

Another possibility is the following behavior of the coherences:

$$r_{12} = i\eta_c\sqrt{r_1 r_2}, \quad r_{34} = \eta_d\sqrt{r_3 r_3}, \quad \eta_c,\eta_d \in [0,1]. \tag{28}$$

Figure 3 shows the regions defined in this case for $\eta_c = 0.3$ and $\eta_d = 0.5$. In this case, the limit $\eta_c,\eta_d \to 1$ corresponds to a vanishing volume of the red (middle gray) region (nonempty

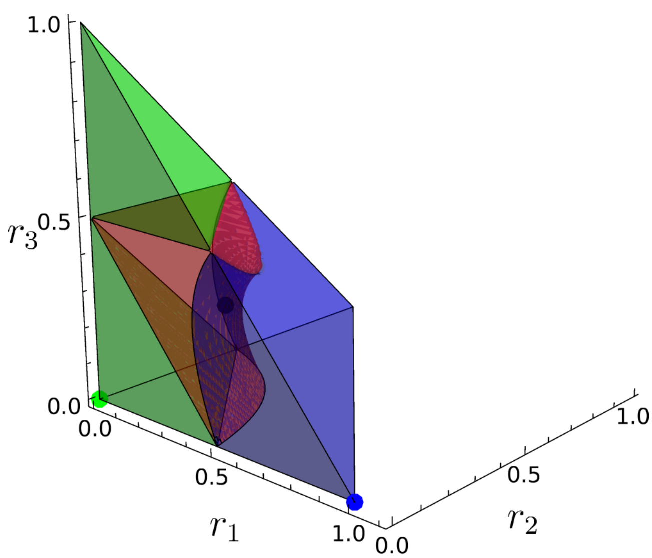

FIG. 3. Three-dimensional space of the independent diagonal elements (r_1,r_2,r_3). The other parameter r_4 is obtained from the trace condition Tr$\rho = 1$. In blue (gray): the region defined by condition (25) whose points lead to the fixed point $r_1 = 1$ and all others zero. In green (light gray): the region defined by condition (26) that leads to the fixed point $r_4 = 1$. In red (middle gray): all other conditions that do not lead to an entangled state. The initial coherences have the form in (28) with $\eta_c = 0.3$ and $\eta_d = 0.5$.

set), which is covered entirely by the blue (gray) region. The green (light gray) region remains intact.

These results demonstrate that the conditions on the diagonal parameters can indeed be relaxed. Figures 2 and 3 show that, depending on the type of initial states, the purifiable parameter space can be enlarged. Smaller values for the diagonal parameters require higher absolute values of the coherences, meaning that these states will still be entangled. This is of course expected, as one cannot create entanglement using only local operations.

V. EXAMPLES

In this section we provide examples of states that fulfill the condition in Eq. (25). The examples show how our protocol is able to purify initial entangled states with low fidelity with respect to Bell states.

Example 1. Consider the parameter $x \in (0.5,1]$ determining the state

$$\rho_1 = x|\Upsilon_{\text{ent}}\rangle\langle\Upsilon_{\text{ent}}| + (1-x)|\Phi^+\rangle\langle\Phi^+|$$
$$|\Upsilon_{\text{ent}}\rangle = \tfrac{1}{\sqrt{2}}(|\Psi^-\rangle + i|\Phi^-\rangle). \tag{29}$$

The state ρ_1 has no fidelity larger than one half with respect to any of the four Bell states. However, it has fidelity $x > 1/2$ with respect to the maximally entangled state $|\Upsilon_{\text{ent}}\rangle$. Taking into account that in this example $r_1 = r_2 = ir_{12} = x/2$ and $r_{34} = 0$ one can find that after one iteration of the map

$$r_1' = \frac{x^2}{x^2 + (1-x)^2},$$

which is always larger than x for $x \in (0.5,1)$. This is an example of an initial state where the fidelities with respect to any of the four Bell states are smaller than one half and still the state $|\Psi^-\rangle$ can be purified. Of course, here the fidelity with respect to the maximally entangled state $|\Upsilon_{\text{ent}}\rangle$ is being exploited by the protocol. In contrast, this feature would be lost with other known protocols [1,2] that use random unitary gates to bring a state into a Bell diagonal form. This procedure would destroy the entanglement of ρ_1.

Example 2. Now take the parameter $c \in (0,0.5]$ that parametrizes the state

$$\rho_2 = c|\Psi^-\rangle\langle\Psi^-| + (1-c)|\Upsilon_{\text{sep}}\rangle\langle\Upsilon_{\text{sep}}|$$
$$|\Upsilon_{\text{sep}}\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Psi^+\rangle). \tag{30}$$

In this example $r_1 = c$ and $r_3 = r_4 = r_{34} = (1-c)/2$. After a single iteration of the protocol one finds that the fidelity with respect to $|\Psi^+\rangle$ takes the value

$$r_1' = \frac{c^2}{2c^2 + (1-2c) - (1-c)^2} = 1.$$

This means that no matter the value of c, as long it is not zero, the state can be purified into $|\Psi^-\rangle$ in just one iteration of the protocol. The success probability is $c^2/2$ and therefore as the value of c is smaller, the occurrence of an event that generates $|\Psi^-\rangle$ becomes less probable. We remind the reader that in typical situations the overall success probability of an entanglement purification protocol scales exponentially with the number of iterations and the convergence is formally achieved in an asymptotic way [4]. Therefore, the fact that perfect purification is achieved in just one step makes this protocol with this initial state an attractive candidate. Furthermore, state ρ_2 is missed by known protocols [1,2], because it has fidelity smaller than one half with any maximally entangled pure state.

In order to prove this statement, let us consider the general form of these type of states in the Bell basis. First, we recall that the concurrence [11] is a measure of entanglement which takes unit value for maximally entangled pure states. For a pure state $|\Psi\rangle$ it is defined as

$$C(|\Psi\rangle) = |\langle\Psi|\sigma_y \otimes \sigma_y|\Psi\rangle^*|, \tag{31}$$

with the Pauli matrix σ_y and where $|\Psi\rangle^*$ is the complex conjugated vector of $|\Psi\rangle$. The Bell states are eigenstates of the operator in Eq. (31), namely $\sigma_y \otimes \sigma_y|\Psi^\pm\rangle = \pm|\Psi^\pm\rangle$ and $\sigma_y \otimes \sigma_y|\Phi^\pm\rangle = \mp|\Phi^\pm\rangle$. With this one can note that, up to a global phase, any maximally entangled pure state has the form

$$|\Psi\rangle = a_-|\Phi^-\rangle + ia_+|\Phi^+\rangle + ib_-|\Psi^-\rangle + b_+|\Psi^+\rangle, \tag{32}$$

with coefficients $a_\pm, b_\pm \in \mathbb{R}$ and satisfying the normalization condition $\langle\Psi|\Psi\rangle = 1$. Then, by introducing $y = (a_+^2 + b_+^2)/2$, the fidelity of ρ_2 with respect to any maximally entangled pure state takes the form

$$F_{\rho_2} = \langle\Psi|\rho_2|\Psi\rangle = cb_-^2 + (1-c)y \tag{33}$$
$$\leqslant c(1-y) + (1-c)y \leqslant \tfrac{1}{2}.$$

The first inequality takes into account the normalization conditions of $|\Psi\rangle$. The last inequality is valid provided both c

and y are positive numbers less than a half, which is true by our choice of c and by the definition of y.

Finally, let us point out that the parameter c is actually the concurrence [11] of the state ρ_2. Indeed, it is not hard to realize that the matrix $\rho_2(\sigma_y \otimes \sigma_y)\rho_2^*(\sigma_y \otimes \sigma_y)$ has only one eigenvalue given by c^2 and therefore, according to Ref. [11], the concurrence is c.

VI. CONCLUSIONS

We have studied the convergence of arbitrary input two-qubit states, under the influence of the entanglement purification protocol presented in Sec. II and first introduced in Ref. [8]. Our protocol was introduced in the context of a multiphoton-assisted quantum repeater, where a two-qubit quantum operation can be more efficiently realized than a controlled-NOT gate.

In a first approach, we have studied in detail the convergence in a subset of density matrices which is left invariant by the protocol. This subset is characterized by seven real parameters, three of which are diagonal elements of the density matrix in the Bell basis. These diagonal elements do not depend on the coherences after the iteration of the protocol. Therefore, we have based our proof on the work of Macchiavello [7]. In order to show that the off-diagonal elements monotonically decrease, we have introduced a quadratic form which decreases under each iteration of the map. Combining these results we were able to define conditions for the initial states that lead to the purification of two Bell states. We have shown that these conditions can be generalized to arbitrary density matrices with 15 real parameters. This contrasts with the two seminal entanglement purification protocols [1,2] which consider density matrices characterized by one or three real parameters. Random unitary rotations are required in those cases to bring any state into a Bell diagonal state. This step in the original protocols may destroy useful entanglement. The fact that our protocol does not require such step is the reason why we obtain less constrained conditions for purifiable initial states.

Exploiting these new findings, we have shown that our protocol allows purification of entangled states having an overlap less than one half with any Bell state. Some states might even have fidelity less than one half with respect to any maximally entangled pure state. We have shown a class of these type of states that can be converted into a Bell state in just one step of the purification protocol. These states are parametrized by their concurrence c and can be purified with success probability $c^2/2$. This feature is interesting, as the resources of entanglement purification scale exponentially with the number of states. Therefore, our result could initiate the study of other type of local two-qubit operations, which might lead to an optimal entanglement purification for specific type of states. Provided that the source of these entangled pairs is well under control, the engineering of an efficient local two-qubit operation might be more resourceful than the application of a controlled-NOT gate.

TABLE I. List of fixed points of the map (15) with their respective stability. Two fixed points are given with four-digit precision.

Fixed points $(r_1,r_2,r_3,r_4,r_{14},r_{23})$	Stability
(1,0,0,0,0,0)	Stable
(0,0,0,1,0,0)	Stable
(0.1409,0.2344,0.1245,0.5,0,0)	Unstable
(0.5,0.1245,0.2344,0.1409,0,0)	Unstable
(0.25,0.25,0.25,0.25,0,0)	Stable
(0.25,0.25,0.25,0.25,0.25,−0.25)	Unstable
(0.25,0.25,0.25,0.25,0.25,0.25)	Unstable
(0.5,0,0,0.5,0,0)	Unstable
(0.5,0,0,0.5,0.5,0)	Unstable

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APPENDIX: FIXED POINT ANALYSIS OF *X* STATES

In order to show that there are no other stable fixed points in regions defined by the conditions (21) and (22), we have numerically investigated the stability of the fixed points of the map (15). This is accomplished by studying the Jacobian matrix with entries J_{ij}. The entries can be defined defined with the help of the function $\boldsymbol{f} : \mathbb{R}^8 \to \mathbb{R}^8$ as $J_{i,j} = \frac{\partial f_i}{\partial x_j}$, where each x_i is a component of the eight-dimensional vector $\boldsymbol{v} = (r_1,r_2,r_3,r_4,\mathrm{Re}[r_{14}],\mathrm{Im}[r_{14}],\mathrm{Re}[r_{23}],\mathrm{Im}[r_{23}])$. The introduced function fulfills the relation $\boldsymbol{v}' = \boldsymbol{f}(\boldsymbol{v})$ where the primed variables are defined in the nonlinear map (15). Fixed points are obtained by solving the equation $\boldsymbol{v}_\star = \boldsymbol{f}(\boldsymbol{v}_\star)$ and their stability is specified with the eigenvalues of $J|_{\boldsymbol{v}=\boldsymbol{v}_\star}$. If all eigenvalues have absolute value smaller than one, the point is stable [12]. If the absolute value of any eigenvalue is larger than one, the fixed point is unstable. Table I shows a list of the fixed points that represent a valid density matrix (non-negative eigenvalues). There are three stable fixed points, of which two represent the cases when we purify the states $|\Psi^-\rangle$ and $|\Psi^+\rangle$, and the other when we converge toward the maximally mixed state.

We have also found that when considering initial density matrices with one of the parameters $r_i = 0$ and the rest $r_j,r_k,r_l \in [0,0.5]$ the iteration of the protocol can lead to fixed points of at least second order, meaning that they are fixed points of the second iteration of the function $\boldsymbol{f}$, namely $\boldsymbol{v}_{\star\star} = \boldsymbol{f}[\boldsymbol{f}(\boldsymbol{v}_{\star\star})]$. In this case, the map cycles between two of these second-order or period 2 fixed points.

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