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Generalized Schmidt decomposability and its relation to projective norms in multipartite entanglement

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Abstract

Projective norms are capable of measuring entanglement of multipartite quantum states. However, typically, the explicit computation of these distance-based geometric entanglement monotones is very difficult even for finite dimensional systems. Motivated by the significance of Schmidt decompositions for our quantitative understanding of bipartite quantum entanglement, a generalization of this concept to multipartite scenarios is proposed, in the sense that generalized Schmidt decomposability of a multipartite pure state implies that its projective norm can be calculated in a simple way analogous to the bipartite case. Thus, this concept of generalized Schmidt decomposability of multipartite quantum states is linked in a natural way to projective norms as entanglement monotones. Therefore, it may not only be a convenient tool for calculations, but may also shed new light onto the intricate features of multipartite entanglement in an analogous way as the 'classical' Schmidt decomposition does for bipartite quantum systems.

Keywords: Schmidt decomposition, projective norm, greatest cross norm, entanglement measures, multipartite entanglement

1. Introduction

Quantum entanglement is one of the characteristic quantum phenomena that has received considerable attention, not only from the fundamental point of view, but also from the point of view, of practical applications of quantum physics. From the fundamental point of view, its relation to 'counterintuitive' violations of local realism have been of particular interest already since the early theoretical investigations by Schrödinger [1] and Einstein, Podolsky and Rosen [2], and by the seminal work of Bell [3]. For quantum technological purposes, the potential usefulness of entanglement and of its 'counterintuitive features' for applications in quantum information processing is particularly important. For these latter purposes, it is important to quantify the degree of entanglement by *entanglement monotones*, which do not increase under local operations and classical communication (LOCC).

Various entanglement monotones have already been proposed [4–6] for bipartite quantum systems. Some of them also have operational meanings. The *distillable entanglement* [4], for example, quantifies the maximal rate at which noisy mixed states can be transformed into singlet states by LOCC operations. Similarly, the *entanglement cost* of a quantum state equals the maximal rate at which blocks of two-qubit maximally entangled states can be transformed into many copies of this quantum state by LOCC operations. There are also entanglement monotones related to the amount of information being contained in the state of a quantum system or of its constituents. Examples of such measures are the *entropy of entanglement* [5], which turns out to be equal to both the entanglement cost and the distillable entanglement for pure states, and an entire family of monotones known as *relative entropies of entanglement*. A particular representative of this latter family, usually called *relative entropy of entanglement* [6], can also be formulated for multipartite quantum systems.

Another intuitive but geometrically motivated approach to quantify entanglement is to look at the distance of a given quantum state to the set of separable states where the measure of distance is left open at first. In fact, the relative entropy of entanglement may be considered a distance-based *geometric measure*, provided one is willing to accept that relative entropies of two states are not symmetric with respect to exchange of these states. The *projective norm* [7], which we are going to focus on in our subsequent discussion, also qualifies as a geometric entanglement monotone, since it can be shown to be equal to the *Minkowski functional* of the convex hull of rank-one operators induced by the set of unit product vectors of the underlying Hilbert space [8]. Although in general, the projective norm defines an entanglement monotone only, Rudolph [9] has shown how these projective norms can be turned into entanglement measures. On a more abstract level, projective norms can also be understood as special realizations of the so-called *convex roof construction* [10, 11] constituting a general principle for the construction of entanglement monotones given a suitable computational criterion for separability of pure states. For the latter, one may choose the (multipartite) *concurrence* [12–14], the *Schmidt rank* [15], or others.

For bipartite quantum systems, significant progress has been achieved as far as our understanding of these entanglement monotones and of the control and manipulation of entanglement is concerned. Especially for pure bipartite states, several interesting questions, such as the perfect transformability of two given states into each other by LOCC operations, have been answered completely. A key to the systematic understanding of entanglement of bipartite quantum systems is the *Schmidt decomposition* [16], which can be considered the canonical normal form of pure bipartite quantum states. The Schmidt decomposition yields certain invariants of pure bipartite quantum states, the so-called *Schmidt coefficients*, which play a crucial role in the computation of some entanglement monotones, such as the relative entropy of entanglement [17] or the projective norm [9]. Furthermore, the investigation of the Schmidt coefficients of two pure bipartite quantum states allows one to decide whether or not these pure states can be transformed into one another by LOCC operations. Unfortunately, such a systematic understanding of entanglement is not yet available for mixed quantum states, even if the quantum systems are bipartite.

Contrary to the bipartite case, our understanding of quantum entanglement for general multipartite quantum systems with more than two distinguishable subsystems is still far from being satisfactory, and the range of entanglement monotones that are applicable to such scenarios is rather limited. Generally, in such multipartite scenarios, *maximally entangled* *states* do not exist so that there is no pure state that can be transformed into any other state by means of LOCC operations. Therefore, it is difficult to find multipartite generalizations of entanglement monotones, such as the entanglement cost or the distillable entanglement. Some progress has been made, however, in tripartite quantum systems where different entanglement types, such as *GHZ-type* or *W-type* entanglement [18], have been investigated. In particular, recently, interesting results have been obtained for three-qubit states with GHZ-symmetry [19]. Nevertheless, despite these promising particular results, our current understanding of multipartite entanglement is still rather incomplete. Known entanglement monotones, which can also be defined for multipartite quantum systems, include the relative entropy of entanglement, several *robustness measures*, *tangles* [4], and the projective norms [9, 20–22]. The latter projective norms offer particularly interesting perspectives for a systematic understanding of multipartite entanglement, as these measures, can even be defined for infinite dimensional subsystems and their distance-based geometric origin allows the application of powerful functional analytic methods [22]. Therefore, it is a main intention of our subsequent discussion to explore these particular measures and to develop methods for evaluating these measures, at least for particular classes of multipartite quantum states.

Although in general, projective norms of arbitrary multipartite quantum states are difficult to evaluate, this task is fairly simple for bipartite pure quantum states. This is due to the fact that pure bipartite quantum states can always be represented by a Schmidt decomposition with positive Schmidt coefficients so that the evaluation of the corresponding projective norm reduces to a summation over all Schmidt coefficients [9]. Therefore, for bipartite quantum systems, projective norms are well understood mathematical objects and they are closely related to trace norms of linear operators. Unfortunately, projective norms in multipartite scenarios are usually hard to compute. Nevertheless, we shall demonstrate in the following that a reasonable generalization of the 'classical' (bipartite) Schmidt decomposition is possible, which allows us to compute projective norms of several classes of multipartite pure and mixed quantum states in a completely analogous and simple way. The resulting concept of *generalized Schmidt decomposability* of a multipartite quantum state, which is linked in a natural way to projective norms as entanglement monotones, may not only be a useful tool for calculations, but it may also shed new light on the intricate features of multipartite entanglement analogous to how the 'classical' Schmidt decomposition does for bipartite quantum systems. Recently, other types of 'generalized Schmidt decompositions' have been introduced by several authors [23–25] with the intention to construct a canonical representation for arbitrary multipartite state vectors of a given quantum system. Our approach is rather different. We do not aim at constructing such a normal form for general multipartite quantum states, but at determining criteria as to when the projective norm of a multipartite quantum state can be evaluated in an analogously simple way as is possible for bipartite pure quantum states with the help of 'classical' Schmidt decompositions. In this case, we call the state vector 'generalized Schmidt decomposable.'

This paper is organized as follows. In section 2.1, already known basic facts about norms on algebraic tensor products of normed spaces are summarized, with particular emphasis on *cross norms* and on the projective norm or *greatest cross norm*. Following the pioneering work of Rudolph [9] and Arveson [22] in section 2.2, the relation between cross norms and entanglement monotones is summarized briefly. In section 3 our concept of generalized Schmidt decomposability is introduced, and our main result is stated in theorem 3. Its intriguing consequence in a quantum mechanical context is the reduction of an N-partite projective norm to a finite sum of $(N-2)$-partite projective norms. Furthermore, it is shown how with the additional use of a natural isometric isomorphism between the space, of trace-class-operators on an N-partite Hilbert space and another suitable $2N$-partite space, this theorem can also be used to evaluate projective norms of those mixed states easily, which can be mapped by this isomorphism to generalized Schmidt decomposable vectors. A detailed proof of theorem 3 is finally presented in section 4.

2. Cross norms and entanglement monotones

In this section, basic results are summarized that concern the projective norm, its relation to cross norms, and its usefulness as an entanglement monotone.

We consider finitely many finite dimensional normed spaces $X_1, \ldots, X_N$ and their (algebraic) tensor product $X := X_1 \otimes \ldots \otimes X_N$. In general, there are many different ways of turning X into a normed space but most of these obtained norms are not very interesting. However, there is a certain class of norms, the so called *cross norms*, that turns out to be quite useful. The defining property of a cross norm $\| \cdot \|_X$ on X is that it has to factorize on every product vector $x_1 \otimes \ldots \otimes x_N \in X$ according to

$$\| x_1 \otimes \ldots \otimes x_N \|_X = \| x_1 \|_{X_1} \cdot \ldots \cdot \| x_N \|_{X_N}$$

where $\| \cdot \|_{X_i}$ denotes the given norm on X_i.

Example 1. Certainly, the most familiar realization of a cross norm to a physicist is the canonical Hilbert space norm on the tensor product of finitely many finite dimensional Hilbert spaces $\mathcal{H} := \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$. Such a Hilbert space can describe N distinguishable quantum systems. By definition, this norm is induced by the *Hilbert–Schmidt scalar product* given by the sesquilinear extension of

$$\left\langle \psi_1 \otimes \ldots \otimes \psi_N, \varphi_1 \otimes \ldots \otimes \varphi_N \right\rangle := \left\langle \psi_1, \varphi_1 \right\rangle \cdot \ldots \cdot \left\langle \psi_N, \varphi_N \right\rangle$$

for $\psi_i, \varphi_i \in \mathcal{H}_i$. Thereby, the scalar product is linear in the first argument and anti-linear in the second argument, as will be our standard assumption. Consequently, we obtain for a product vector $\psi_1 \otimes \ldots \otimes \psi_N \in \mathcal{H}$

$$\begin{aligned} \left\| \psi_1 \otimes \ldots \otimes \psi_N \right\| &= \sqrt{\left\langle \psi_1, \psi_1 \right\rangle \cdot \ldots \cdot \left\langle \psi_N, \psi_N \right\rangle} \\ &= \sqrt{\left\langle \psi_1, \psi_1 \right\rangle} \cdot \ldots \cdot \sqrt{\left\langle \psi_N, \psi_N \right\rangle} \\ &= \left\| \psi_1 \right\| \cdot \ldots \cdot \left\| \psi_N \right\| . \end{aligned}$$

2.1. The Projective Norm

The class of cross norms on X is still very large. However, there is a unique cross norm $\| \cdot \|_\pi$ that dominates any other cross norm on X; that is, if $\| \cdot \|_X$ is an arbitrary cross norm on X, we have the relation

$$\| x \|_X \leqslant \| x \|_\pi \quad \text{for all } x \in X.$$

This norm can be constructed in a very intuitive way. To this end, let $x \in X$ be an arbitrary vector and choose a representation of x, as a finite sum of product vectors such as

$$x = \sum_{i=1}^{k} x_{i,1} \otimes \dots \otimes x_{i,N}.$$

By the triangle inequality and the cross norm property of $\| \cdot \|_X$ we find

$$\| x \|_X \leqslant \sum_{i=1}^{N} \left\| x_{i,1} \otimes \dots \otimes x_{i,N} \right\|_X = \sum_{i=1}^{N} \left\| x_{i,1} \right\|_{X_1} \cdot \dots \cdot \left\| x_{i,N} \right\|_{X_N}.$$

Since this holds for every representation of x, we obtain

$$\| x \|_X \leqslant \inf \left\{ \sum_{i=1}^{k} \left\| x_{i,1} \right\|_{X_1} \cdot \dots \cdot \left\| x_{i,N} \right\|_{X_N} : x = \sum_{i=1}^{k} x_{i,1} \otimes \dots \otimes x_{i,N} \right\} =: \| x \|_\pi$$

where the infimum runs over all representations of x. On the other hand, one can show (see [7] for a detailed proof) that the right-hand side of this inequality indeed defines a cross norm, denoted by $\| \cdot \|_\pi$. This norm is called the *greatest cross norm* or *projective norm* with respect to the norms $\| \cdot \|_{X_1}$,..., $\| \cdot \|_{X_N}$. Taking a look at the definition of this norm, it is clear that its computation is usually very difficult, although several techniques can be developed for handling this task. For instance, it can be shown (see [9]) that for finite dimensional Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and a vector $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ with Schmidt decomposition

$$\psi = \sum_{i=1}^{n} \lambda_i e_i \otimes f_i, \qquad \lambda_i \geqslant 0$$

the relation

$$\| \psi \|_\pi = \sum_{i=1}^{n} \lambda_i$$

holds.

It is a main aim of this work to generalize this simple relation to a much more general class of quantum states.

2.2. Projective norms as entanglement monotones

Let $\mathcal{H}_1$,..., $\mathcal{H}_N$ be finite dimensional Hilbert spaces and $\mathcal{H} := \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$. We denote the set of all linear operators on the i-th space $\mathcal{H}_i$ by $\mathcal{B}(\mathcal{H}_i)$. By defining the *trace norm* of an operator $A_i \in \mathcal{B}(\mathcal{H}_i)$ according to

$$\| A_i \|_1 := \mathrm{tr} \left(\sqrt{A_i^* A_i} \right)$$

we can turn $\mathcal{B}(\mathcal{H}_i)$ into a normed space denoted by $\mathcal{T}(\mathcal{H}_i)$. The associated projective norm on $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1) \otimes \dots \otimes \mathcal{B}(\mathcal{H}_N)$ with respect to the trace norms of an operator $A \in \mathcal{B}(\mathcal{H})$ is given by

$$\| A \|_{\otimes_{\pi i=1}^{N} \mathcal{T}(\mathcal{H}_i)} := \inf \left\{ \sum_{j=1}^{k} \| A_{j,1} \|_1 \cdot \dots \cdot \| A_{j,N} \|_1 : A = \sum_{j=1}^{N} A_{j,1} \otimes \dots \otimes A_{j,N} \right\}.$$

The usefulness of this projective norm for quantum information theoretical purposes, in particular concerning quantitative measures of entanglement, arises from several results of Rudolph [9, 20] obtained in 2000 and 2001.

Theorem 1. *(Rudolph) Let $\mathcal{H}_1,\ldots,\ \mathcal{H}_N$ be finite dimensional Hilbert spaces and ρ a density operator on $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$. Then we have the inequality*

$$\| \rho \|_{\otimes_{\pi}{}_{j=1}^{N}\mathcal{T}(\mathcal{H}_i)} \geqslant 1$$

and equality holds if and only if ρ is fully separable.

In this context, full separability of a state ρ means that it belongs to the convex hull of all product states on $\mathcal{H}$. Therefore, the projective norm of a density operator on $\mathcal{H}$ with respect to the trace norms is capable of quantifying entanglement in general. In 2009, Rudolph's results were generalized by Arveson [22] to arbitrary multipartite quantum systems, including infinite dimensional ones. Arveson uses an elegant geometric approach in order to quantify general entanglement by an extended real valued function E: $\mathcal{B}(\mathcal{H}) \to [0, \infty]$, which has several, but not necessarily all, properties of a norm. If at least two of the involved Hilbert spaces are infinite dimensional, this real valued function may attain the value '∞' leading to the interesting subject of infinitely entangled states. This function, whose explicit definition can be found in [22], satisfies the following theorem.

Theorem 2. *(Arveson) Let $\mathcal{H}_1,\ldots,\ \mathcal{H}_N$ be Hilbert spaces and ρ a density operator on $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$. Then the inequality*

$$E(\rho) \geqslant 1$$

holds with equality if and only if ρ is fully separable.

As proven in detail in [22], the function E turns out to be the projective norm $\| \cdot \|_{\otimes_{\pi}{}_{j=1}^{N}\mathcal{T}(\mathcal{H}_j)}$ as defined earlier if at most one of the involved Hilbert spaces is infinite dimensional. Thus, Rudolph's result for purely finite dimensional systems is recovered.

For a vector $\psi \in \mathcal{H}$ let

$$\| \psi \|_{\otimes_{\pi}{}_{j=1}^{N}\mathcal{H}_j} := \inf \left\{ \sum_{j=1}^{k} \| \psi_{j,1} \| \cdot \ldots \cdot \| \psi_{j,N} \| \colon \psi = \sum_{j=1}^{k} \psi_{j,1} \otimes \ldots \otimes \psi_{j,N} \right\}$$

denote the projective norm of ψ with respect to the Hilbert space norms on $\mathcal{H}_i$. Furthermore, let

$$\begin{aligned} t_{\psi,\psi} \colon \mathcal{H} &\mapsto \mathcal{H} \\ \xi &\mapsto \psi \langle \xi, \psi \rangle \end{aligned}$$

denote the rank-one-operator induced by ψ. It is proven in [22] that the projective norm of ψ is related to the projective norm of $t_{\psi,\psi}$ by the equation

$$\left\| t_{\psi,\psi} \right\|_{\otimes_{\pi}{}_{j=1}^{N}\mathcal{T}\left(\mathcal{H}_j\right)} = \| \psi \|^2_{\otimes_{\pi}{}_{j=1}^{N}\mathcal{H}_j}.$$

It is worth mentioning that the projective norm for density operators may also be obtained by the classical convex roof construction [10, 11] given that the square of the projective norm for state vectors is used as entanglement monotone on the set of pure states in accordance with the previous equation. Thus, we directly get a necessary and sufficient separability criterion for pure states.

Corollary 1. *Let $\mathcal{H}_1,\ldots,\mathcal{H}_N$ be finite dimensional Hilbert spaces and $\psi \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$ be a unit vector. Then, the relation*

$$\| \psi \|_{\otimes_{\pi}{}_{j=1}^{N}\mathcal{H}_j} \geqslant 1$$

is fulfilled with equality if and only if ψ is a product vector.

Example 2. As pointed out in section 2.1, the projective norm of a unit vector of a bipartite, finite dimensional quantum system is given by the sum of its Schmidt coefficients. For arbitrary complex numbers a_i the inequality

$$\sqrt{\sum_{i=1}^{n} |a_i|^2} \leqslant \sum_{i=1}^{n} |a_i|$$

holds with equality if and only if $a_i \neq 0$ for at most one i. This implies

$$\| \psi \|_{\pi} = \sum_{i=1}^{n} \lambda_i \geqslant \sqrt{\sum_{i=1}^{n} \lambda_i^2} = 1$$

and equality is given if and only if all Schmidt coefficients but one vanish; that is, if and only if ψ is a product vector. Furthermore, from the relation

$$\max \left\{ \sum_{i=1}^{n} |a_i| : a_1,\ldots,a_n \in \mathbb{C},\ \sum_{i=1}^{n} |a_i|^2 = 1 \right\} = \sqrt{n}$$

and from the fact that this maximum is attained precisely if $|a_1| = \ldots = |a_n| = \frac{1}{\sqrt{n}}$ we see that the maximally entangled vectors with respect to this measure are those of the form

$$\psi = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} e_i \otimes f_i$$

for orthonormal systems $(e_i)_{1 \leqslant i \leqslant k} \subset \mathcal{H}_1$, $(f_i)_{1 \leqslant i \leqslant k} \subset \mathcal{H}_2$ where

$$k = \min \left\{ \dim \mathcal{H}_1, \dim \mathcal{H}_2 \right\}.$$

In this case, one obtains $\| \psi \|_{\pi} = \sqrt{k}$. Specializing to $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$ we see, for instance, that Bell states are maximally entangled.

3. Generalized Schmidt Decomposability

As we have seen so far, the problem of characterizing entanglement in terms of projective norms is completely solvable for pure bipartite states in finite dimensional quantum systems by computing Schmidt coefficients. However, since there is no such result for multipartite quantum systems in the first place, we are left with the quite challenging task of computing projective norms by their definition. In the following, we demonstrate that there is a reasonable generalization of the 'classical' Schmidt decomposition, which allows us to compute projective norms in a completely analogous way for several classes of multipartite pure and mixed states. Our main result formulated in theorem 3 applies to pure multipartite quantum states. However, by using a natural isometric isomorphism, this theorem can also be used for the evaluation of projective norms of large classes of mixed multipartite quantum states.

In order to prepare grounds for our main result of theorem 3, we first of all reformulate the classical Schmidt decomposition. For this purpose, we start from the elementary fact that the set of complex numbers $\mathbb{C}$ can be regarded as a 'one-dimensional Hilbert space' over the field $\mathbb{C}$

with 'norm' $\|\cdot\|_{\mathbb{C}} :=|\cdot|$ where $|\cdot|$ is the usual modulus on $\mathbb{C}$. Thus, for given Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we can consider the 'product space' $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}$. Using $\dim \mathbb{C} = 1$ and the fact that

$$\dim \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C} = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2 \cdot \dim \mathbb{C} = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2$$

we see that the spaces $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ are isomorphic. Moreover, there is a natural isomorphism given by the linear extension of the assignment

$$\iota\colon \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C} \to \mathcal{H}_1 \otimes \mathcal{H}_2,\ \psi_1 \otimes \psi_2 \otimes \lambda \mapsto \lambda\psi_1 \otimes \psi_2 .$$

Therefore, we can think of these two spaces as 'being the same in principal.' It makes no difference whether a quantum system is modeled by the former one or the latter one. Furthermore, it is simple to see that this map is isometric with respect to the Hilbert space norm **and** the projective norm, i.e.,

$$\| \psi \|_{\mathcal{H}_1\otimes_\pi\mathcal{H}_2} = \|\iota^{-1}(\psi)\|_{\mathcal{H}_1\otimes_\pi\mathcal{H}_2\otimes_\pi\mathbb{C}} \quad \text{for all } \psi \in \mathcal{H}_1 \otimes \mathcal{H}_2 .$$

Now suppose we are given a vector $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ of the form

$$\psi = \sum_{i=1}^{k} \lambda_i e_i \otimes f_i$$

with orthonormal systems $(e_i)_{1\leqslant i\leqslant k} \subset \mathcal{H}_1$, $(f_i)_{1\leqslant i\leqslant k} \subset \mathcal{H}_2$ and arbitrary complex numbers $(\lambda_i)_{1\leqslant i\leqslant k}$. The Schmidt coefficients of this particular vector are obviously given by $|\lambda_1|,\ldots,|\lambda_k|$ since the phases of $\lambda_i = |\lambda_i| \cdot e^{i\varphi_i}$ can be absorbed by the system $(e_i)_{1\leqslant i\leqslant k}$, for instance, to obtain a new orthonormal system $(\tilde{e}_i)_{1\leqslant i\leqslant k} := (e^{i\varphi_i} \cdot e_i)_{1\leqslant i\leqslant k}$ such that

$$\psi = \sum_{i=1}^{k} |\lambda_i|\ \tilde{e}_i \otimes f_i .$$

Therefore, the projective norm of ψ equals $\sum_{i=1}^{k} |\lambda_i|$. The 'counterpart' $\psi' := \iota^{-1}(\psi) \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}$ of the vector ψ is given by

$$\psi' = \sum_{i=1}^{k} e_i \otimes f_i \otimes \lambda_i$$

and using the 'norm' $\|\cdot\|_{\mathbb{C}}$ on $\mathbb{C}$, the projective norm of ψ' reads

$$\| \psi' \|_{\mathcal{H}_1\otimes_\pi\mathcal{H}_2\otimes_\pi\mathbb{C}} = \| \psi \|_{\mathcal{H}_1\otimes\mathcal{H}_2} = \sum_{i=1}^{k} |\lambda_i| = \sum_{i=1}^{k} \| \lambda_i \|_{\mathbb{C}} .$$

Our goal is now to prove the following main result, which shows that the previous elementary observations are a special case of a much more general behavior that provides powerful tools for computing projective norms.

Theorem 3. *Let $\mathcal{H}_1, \mathcal{H}_2$ be finite dimensional Hilbert spaces and X be an arbitrary finite dimensional normed space. Moreover, suppose we are given a vector $\xi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes X$ such that there are orthonormal systems $(e_i)_{1\leqslant i\leqslant k} \subset \mathcal{H}_1$, $(f_i)_{1\leqslant i\leqslant k} \subset \mathcal{H}_2$ and a family of vectors $x_1,\ldots,x_k \subset X$ satisfying*

$$\xi = \sum_{i=1}^{k} e_i \otimes f_i \otimes x_i .$$

Then, the relation

$$\| \xi \|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2 \otimes_\pi X} = \sum_{i=1}^{k} \left\| x_i \right\|_X$$

is fulfilled where $\| \cdot \|_X$ *is the given norm on X.*

Note that if the space X happens to be a projective tensor product of Hilbert spaces; that is, $X = \mathcal{H}_3 \otimes_\pi \ldots \otimes_\pi \mathcal{H}_N$, the previous theorem allows the reduction of an N-partite projective norm to a finite sum of $(N-2)$-partite projective norms whenever its assumptions are met. Hence, we have a reduction of an N-partite entanglement monotone to a finite sum of $(N-2)$-partite entanglement monotones. We will give a detailed proof of this theorem in section 4. In order to demonstrate its usefulness for the evaluation of projective norms, let us first of all take a look at how this result can be applied to some physically relevant cases.

Example 3. Let $\{e_1, e_2\}$ be an orthonormal basis of $\mathbb{C}^2$. We consider a tripartite qubit system $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and an arbitrary unit vector of the form

$$\psi = a \cdot e_1 \otimes e_1 \otimes e_1 + b \cdot e_1 \otimes e_1 \otimes e_2 + c \cdot e_2 \otimes e_2 \otimes e_1 + d \cdot e_2 \otimes e_2 \otimes e_2$$

with $a, b, c, d \in \mathbb{C}$. Using theorem 3, we find

$$\begin{aligned} \| \psi \|_{\mathbb{C}^2 \otimes_\pi \mathbb{C}^2 \otimes_\pi \mathbb{C}^2} &= \big\| a \cdot e_1 \otimes e_1 \otimes e_1 + b \cdot e_1 \otimes e_1 \otimes e_2 \\ &\quad + c \cdot e_2 \otimes e_2 \otimes e_1 + d \cdot e_2 \otimes e_2 \otimes e_2 \big\|_{\mathbb{C}^2 \otimes_\pi \mathbb{C}^2 \otimes_\pi \mathbb{C}^2} \\ &= \big\| e_1 \otimes e_1 \otimes \left(ae_1 + be_2 \right) + e_2 \otimes e_2 \otimes \left(ce_1 + de_2 \right) \big\|_{\mathbb{C}^2 \otimes_\pi \mathbb{C}^2 \otimes_\pi \mathbb{C}^2} \\ &= \left\| ae_1 + be_2 \right\| + \left\| ce_1 + de_2 \right\| \\ &= \sqrt{|a|^2 + |b|^2} + \sqrt{|c|^2 + |d|^2} \end{aligned}$$

yielding an explicit and simple-to-evaluate formula for the projective norm. Using

$$1 = \| \psi \|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2$$

we get

$$\| \psi \|_{\mathbb{C}^2 \otimes_\pi \mathbb{C}^2 \otimes_\pi \mathbb{C}^2} = \sqrt{\left| a \right|^2 + \left| b \right|^2} + \sqrt{1 - \left| a \right|^2 - \left| b \right|^2}.$$

The function $f: [0, 1] \to \mathbb{R},\ x \mapsto \sqrt{x} + \sqrt{1-x}$ attains its maximum at $x = \frac{1}{2}$. Hence, the maximally entangled states within this class of tripartite pure qubit states are those that satisfy $|a|^2 + |b|^2 = |c|^2 + |d|^2 = \frac{1}{2}$. In particular, this is the case for the GHZ-state [26]

$$\frac{1}{\sqrt{2}} \left(e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \right).$$

The results and observations we considered so far motivate the following concept and its nomenclature.

Definition 1. Let $\mathcal{H}_1, \mathcal{H}_2$ be finite dimensional Hilbert spaces and X a finite dimensional normed space. We call a vector $\xi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes X$ a *generalized Schmidt decomposable vector (gsd-vector)* if it is of the form as in theorem 3.

The following table shows a comparison between the classical and the generalized Schmidt decomposition.

	classical Schmidt decomposition	generalized Schmidt decomposition
space	$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}$	$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes X$
vectors	$\sum_{i=1}^{k} e_i \otimes f_i \otimes \lambda_i$	$\sum_{i=1}^{k} e_i \otimes f_i \otimes x_i$
projective norm	$\sum_{i=1}^{k} \lVert \lambda_i \rVert_{\mathbb{C}}$	$\sum_{i=1}^{k} \lVert x_i \rVert_X$

Given a randomly chosen vector $\xi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes X$, it is certainly very improbable that ξ is of the type as in definition 3. In fact, one can show that the *W-state* [18]

$$\xi_W := \frac{1}{\sqrt{3}} \left(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

where $\{e_1, e_2\}$ is an orthonormal basis of $\mathbb{C}^2$ satisfies

$$\lVert \xi_W \rVert_{\mathbb{C}^2 \otimes_\pi \mathbb{C}^2 \otimes_\pi \mathbb{C}^2} \geqslant \frac{3}{2} .$$

Furthermore, the maximum of the projective norm over the set of all unit gsd-vectors of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ equals $\sqrt{2}$. Therefore, ξ_W cannot be generalized Schmidt decomposable. Nevertheless, whenever the gsd-structure is present, we have a powerful computational tool for evaluating projective norms.

Although the principle of the generalized Schmidt decomposition is merely a concept involving Hilbert spaces and is, therefore, dealing with state vectors, i.e., pure states, it is nevertheless possible to apply it to mixed states as well. This circumstance is based on the fact that there is an isometric isomorphism

$$\mathcal{T}(\mathcal{H}) \cong \mathcal{H} \otimes_\pi \mathcal{H}'$$

where $\mathcal{H}$ is an arbitrary Hilbert space and $\mathcal{H}'$ denotes its topological dual, which is a Hilbert space as well. For vectors $\psi, \varphi \in \mathcal{H}$ let $t_{\psi,\varphi}: \mathcal{H} \to \mathcal{H},\ \zeta \mapsto \psi \langle \zeta, \varphi \rangle$ denote the corresponding rank-one-operator. The previously mentioned isomorphism is then given by the continuous linear extension of the assignment

$$t_{\psi,\varphi} \mapsto \psi \otimes \langle \cdot, \varphi \rangle .$$

In fact, this isomorphism is certainly not too surprising for a physicist since Dirac's notation is based on it. Furthermore, in the case of $\mathcal{H} = \otimes_{i=1}^{N} \mathcal{H}_i$ the corresponding isomorphisms $\mathcal{I}_i: \mathcal{T}(\mathcal{H}_i) \to \mathcal{H}_i \otimes \mathcal{H}_i'$ induce an isomorphism

$$\mathcal{I} := \mathcal{I}_1 \otimes \dots \otimes \mathcal{I}_N: \otimes_{i=1}^{N} \mathcal{T}(\mathcal{H}_i) \to \otimes_{i=1}^{N} \left(\mathcal{H}_i \otimes \mathcal{H}_i \right)'.$$

Using the fact that forming projective norms is an associative procedure, we can reduce the projective norm with respect to trace norms of a given N-partite state (compare with theorem 1) to the projective norm with respect to Hilbert space norms of a $2N$-partite vector according to

$$\begin{aligned} \lVert \rho \rVert_{\mathcal{T}(\mathcal{H}_1) \otimes_\pi \dots \otimes_\pi \mathcal{T}(\mathcal{H}_N)} &= \lVert \mathcal{I}(\rho) \rVert_{(\mathcal{H}_1 \otimes_\pi \mathcal{H}_1') \otimes_\pi \dots \otimes_\pi (\mathcal{H}_N \otimes_\pi \mathcal{H}_N')} \\ &= \lVert \mathcal{I}(\rho) \rVert_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_1' \otimes_\pi \dots \otimes_\pi \mathcal{H}_N \otimes_\pi \mathcal{H}_N'} . \end{aligned}$$

Example 4. Let $\{e_1,...,e_n\}$ and $\{f_1,...,f_n\}$ be orthonormal bases of $\mathbb{C}^n$. We consider the tripartite quantum system $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m$ and the $n \cdot m$ - dimensional linear subspace

$$\mathcal{K} := \text{lin}\left\{e_i \otimes f_i : \ 1 \leqslant i \leqslant n\right\} \otimes \mathbb{C}^m.$$

We compute the projective norm of all states having support on this subspace. It is a simple text book exercise to show that for every linear operator Φ having support on $\mathcal{K}$, there is a family of linear operators Φ_{ij} on $\mathbb{C}^m$ such that

$$\Phi = \sum_{i,j=1}^{n} t_{e_i,e_j} \otimes t_{f_i,f_j} \otimes \Phi_{ij}.$$

Therefore, we may identify Φ with the operator-valued $n \times n$-matrix

$$\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix}.$$

Applying the isometric isomorphism $\mathcal{T}(\mathbb{C}^n) \cong \mathbb{C}^n \otimes_\pi \mathbb{C}^{n\prime}$ to the first two tensor factors, we can perform a transformation according to

$$\mathcal{T}\left(\mathbb{C}^n\right) \otimes_\pi \mathcal{T}\left(\mathbb{C}^n\right) \otimes_\pi \mathcal{T}\left(\mathbb{C}^m\right) \to \mathbb{C}^n \otimes_\pi \mathbb{C}^{n\prime} \otimes_\pi \mathbb{C}^n \otimes_\pi \mathbb{C}^{n\prime} \otimes_\pi \mathcal{T}\left(\mathbb{C}^m\right).$$

For the projective norm of Φ we obtain

$$\begin{aligned} &\| \Phi \|_{\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^m)} \\ &= \left\| \sum_{i,j=1}^{n} t_{e_i,e_j} \otimes t_{f_i,f_j} \otimes \Phi_{ij} \right\|_{\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^m)} \\ &= \left\| \sum_{i,j=1}^{n} e_i \otimes \left\langle \ \cdot\ , e_j \right\rangle \otimes f_i \otimes \left\langle \ \cdot\ , f_j \right\rangle \otimes \Phi_{ij} \right\|_{\mathbb{C}^n\otimes_\pi\mathbb{C}^{n\prime}\otimes_\pi\mathbb{C}^n\otimes_\pi\mathbb{C}^{n\prime}\otimes_\pi\mathcal{T}(\mathbb{C}^m)}. \end{aligned}$$

Taking into account that projective norms are invariant under performing flips of the involved spaces, flipping the second and third tensor factor yields

$$\begin{aligned} &\| \Phi \|_{\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^m)} \\ &= \left\| \sum_{i,j=1}^{n} e_i \otimes f_i \otimes \left\langle \ \cdot\ , e_j \right\rangle \otimes \left\langle \ \cdot\ , f_j \right\rangle \otimes \Phi_{ij} \right\|_{\mathbb{C}^n\otimes_\pi\mathbb{C}^n\otimes_\pi\mathbb{C}^{n\prime}\otimes_\pi\mathbb{C}^{n\prime}\otimes_\pi\mathcal{T}(\mathbb{C}^m)} \\ &= \left\| \sum_{i=1}^{n} e_i \otimes f_i \otimes \sum_{j=1}^{n} \left\langle \ \cdot\ , e_j \right\rangle \otimes \left\langle \ \cdot\ , f_j \right\rangle \otimes \Phi_{ij} \right\|_{\mathbb{C}^n\otimes_\pi\mathbb{C}^n\otimes_\pi\mathbb{C}^{n\prime}\otimes_\pi\mathbb{C}^{n\prime}\otimes_\pi\mathcal{T}(\mathbb{C}^m)}. \end{aligned}$$

A first application of theorem 3 implies

$$\begin{aligned} &\| \Phi \|_{\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^n)\otimes_\pi\mathcal{T}(\mathbb{C}^m)} \\ &= \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} \left\langle \ \cdot\ , e_j \right\rangle \otimes \left\langle \ \cdot\ , f_j \right\rangle \otimes \Phi_{ij} \right\|_{\mathbb{C}^{n\prime}\otimes_\pi\mathbb{C}^{n\prime}\otimes_\pi\mathcal{T}(\mathbb{C}^m)}. \end{aligned}$$

Noting that $\{\langle \cdot , e_1\rangle,...,\langle \cdot , e_n\rangle\}$ and $\{\langle \cdot , f_1\rangle,...,\langle \cdot , f_n\rangle\}$ are orthonormal bases of $\mathbb{C}^{n\prime}$, a second application gives the final result

$$\| \Phi \|_{\mathcal{T}(\mathbb{C}^n)\otimes_\pi \mathcal{T}(\mathbb{C}^n)\otimes_\pi \mathcal{T}(\mathbb{C}^m)} = \sum_{i,j=1}^{n} \left\| \Phi_{ij} \right\|_1 .$$

If Φ happens to be a density operator ρ with associated operators ρ_{ij} on $\mathbb{C}^m$ we have the additional conditions

$$\rho_{ij} = \rho_{ji}^* \quad \text{and} \quad \rho_{ii} \geqslant 0$$

yielding

$$1 = \mathrm{tr}\,(\rho) = \sum_{i=1}^{n} \mathrm{tr}\,(\rho_{ii}) = \sum_{i=1}^{n} \left\| \rho_{ii} \right\|_1$$

and

$$\left\| \rho_{ji} \right\|_1 = \left\| \rho_{ij}^* \right\|_1 = \left\| \rho_{ij} \right\|_1 .$$

Therefore, the projective norm of ρ simplifies according to

$$\| \rho \|_{\mathcal{T}(\mathbb{C}^n)\otimes_\pi \mathcal{T}(\mathbb{C}^n)\otimes_\pi \mathcal{T}(\mathbb{C}^m)} = 1 + 2 \cdot \sum_{1 \leqslant i < j \leqslant n} \left\| \rho_{ij} \right\|_1 .$$

Comparing this to theorem 1, we see that ρ is separable if and only if $\rho_{ij} = 0$ for all $1 \leqslant i < j \leqslant n$. If $\{g_1, \dots, g_k\}$ is an arbitrary orthonormal basis of $\mathbb{C}^m$ the relations

$$\left\langle \rho\left(e_j \otimes f_j \otimes g_k \right), e_i \otimes f_i \otimes g_l \right\rangle = \left\langle \rho_{ij} g_k, g_l \right\rangle$$

imply that ρ is separable if and only if the equations

$$\left\langle \rho\left(e_j \otimes f_j \otimes g_k \right), e_i \otimes f_i \otimes g_l \right\rangle = 0 \quad \text{for all } 1 \leqslant i < j \leqslant n,\ 1 \leqslant k, l \leqslant m$$

are satisfied, which constitutes a fairly simple separability criterion.

It is worth taking a look at the geometric implications of this criterion. The previous equations mean that ρ belongs to the kernel of the linear functionals

$$\varphi_{ijkl}\colon \mathcal{B}\left(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m \right) \to \mathbb{C}, \quad \Phi \mapsto \mathrm{tr}\left[\Phi\left(t_{e_j, e_i} \otimes t_{f_j, f_i} \otimes t_{g_k, g_l} \right) \right].$$

In particular, the separable states on $\mathcal{K}$ are given by the intersection of the set of all states on $\mathcal{K}$ with the linear subspace of operators

$$\mathcal{A} := \bigcap_{1 \leqslant i < j \leqslant n,\ 1 \leqslant k, l \leqslant m} \ker \varphi_{ijkl} \subset \mathcal{B}\left(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m \right).$$

4. The proof of theorem 3

4.1. Preparatory steps

As preparatory steps for the proof of our main theorem 3, let us first of all discuss three important properties of projective norms.

First of all, it is important to emphasize that projective norms do not respect subspaces in the following sense. Suppose we are given two finite dimensional normed spaces E, F and subspaces $X \subset E$, $Y \subset F$. The norms on E and F induce norms on X and Y, respectively, turning them into normed spaces as well. We denote the projective norm on $X \otimes Y$ with respect to these inherited norms by $\| \cdot \|_{X \otimes_\pi Y}$. For a given vector $u \in X \otimes Y \subset E \otimes F$, we can compute the norms $\| u \|_{X\otimes_\pi Y}$ and $\| u \|_{E\otimes_\pi F}$. The crucial point is that, in general, the inequality

$$\| u \|_{X\otimes_\pi Y} \geqslant \| u \|_{E\otimes_\pi F}$$

holds so that the equality sign can be achieved in special cases only. The reason for this circumstance is that in $E \otimes F$ there are usually more possible representations of u as a finite sum of elementary tensors than in $X \otimes Y$, leading to a smaller value of the infimum according to the definition of the projective norm. However, under additional assumptions on the subspaces X and Y, we may achieve equality. In order to access these cases, we recall that a subspace X of a normed space E is called *complemented* if there is a bounded projection $P: E \to E$ such that $P(E) = X$. In this context, the word 'projection' means that $P^2 = P$; that is, P is not necessarily an orthogonal projection if E happens to be a Hilbert space. With this definition, the following proposition holds.

Proposition 1. *Let E, F be finite dimensional normed spaces and $X \subset E$, $Y \subset F$ linear subspaces that are complemented by projections of norm 1. Under this condition, the following statements are valid:*

(i) For every $u \in X \otimes Y$ we have

$$\| u \|_{X\otimes_\pi Y} = \| u \|_{E\otimes_\pi F} \ .$$

(ii) The subspace $X \otimes Y \subset E \otimes F$ is complemented by a projection of norm 1.

A proof of this result can be found in [7], for example. As a second important property for the proof of theorem 3, a statement about the behavior of trace norms under discarding all non diagonal entries of a matrix is needed. This required property can be formulated in the following lemma.

Lemma 1. *Let us denote the set of all complex valued $n\times n$-diagonal matrices by*

$$D_n(\mathbb{C}) := \left\{ \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \colon \alpha_i \in \mathbb{C} \right\} .$$

The linear operator

$$\begin{aligned} T\colon\ & M_n(\mathbb{C}) \to D_n(\mathbb{C}) \\ & A = (\alpha_{ij}) \mapsto \mathrm{diag}(\alpha_{11}, \ldots, \alpha_{nn}) \end{aligned}$$

fulfills the relation

$$\| T(A) \|_1 \leqslant \| A \|_1 \ .$$

Thus, the trace norm of a given $n\times n$ matrix does not increase if all its non diagonal entries are replaced by zeroes.

Proof. We consider the canonical embedding

$$\iota\colon D_n(\mathbb{C}) \hookrightarrow M_n(\mathbb{C}),\ \ \Delta \mapsto \Delta \ .$$

If we regard the matrix spaces $D_n(\mathbb{C})$ and $M_n(\mathbb{C})$ as being normed by the standard operator norm, the map ι is (as an embedding) a linear map of norm 1. Thus, the dual map ι^* of ι is a linear operator of norm 1 as well. A fundamental statement of operator theory [27] tells us that it is possible to describe the dual spaces $D_n(\mathbb{C})^*$, $M_n(\mathbb{C})^*$ by the spaces $D_n(\mathbb{C})$, $M_n(\mathbb{C})$ themselves via the isomorphisms

$$\Phi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})^*,\ A \mapsto \mathrm{tr}\,(A\ \cdot)$$

$$\Psi\colon D_n(\mathbb{C}) \to D_n(\mathbb{C})^*,\ \Delta \mapsto \mathrm{tr}\,(\Delta\ \cdot)$$

which can be proven to be $\|\cdot\|_1 - \|\cdot\|_{\mathrm{Op.}}$-isometric. We will now show that the map T factorizes according to

$$T = \Psi^{-1} \circ \iota^* \circ \Phi$$

i.e., the following diagram commutes

$$\begin{tikzcd} D_n(\mathbb{C})^* \arrow[d, "\Psi^{-1}"] & M_n(\mathbb{C})^* \arrow[l, "\iota^*"'] \\ D_n(\mathbb{C}) & M_n(\mathbb{C})\,. \arrow[l, "T"] \arrow[u, "\Phi"'] \end{tikzcd}$$

To this end, let $A \in M_n(\mathbb{C})$ be arbitrary. For the diagonal matrix

$$\mathrm{diag}\left(\lambda_1,\ldots,\lambda_n\right) := \Psi^{-1} \circ \iota^* \circ \Phi(A)$$

holds

$$\begin{aligned} \lambda_1 &= \mathrm{tr}\left[\mathrm{diag}(1,\,0,\ldots,0)\mathrm{diag}\left(\lambda_1,\ldots,\lambda_n\right)\right] \\ &= \Psi\left(\mathrm{diag}\left(\lambda_1,\ldots,\lambda_n\right)\right)(\mathrm{diag}(1,\,0,\ldots,0)) \\ &= \left(\Psi \circ \Psi^{-1} \circ \iota^* \circ \Phi(A)\right)(\mathrm{diag}(1,\,0,\ldots,0)) \\ &= \left(\iota^* \circ \Phi(A)\right)(\mathrm{diag}(1,\,0,\ldots,0)) \\ &= (\Phi(A) \circ \iota)(\mathrm{diag}(1,\,0,\ldots,0)) \\ &= \mathrm{tr}\,[A \cdot \mathrm{diag}(1,\,0,\ldots,0)] = \alpha_{11}. \end{aligned}$$

Analogously, one can show that $\lambda_i = \alpha_{ii}$ for $2 \leqslant i \leqslant n$. This means that $T(A) = \Psi^{-1} \circ \iota^* \circ \Phi(A)$ and consequently $T = \Psi^{-1} \circ \iota^* \circ \Phi$.

The maps Φ and Ψ (as well as their inverse maps) are, as isometries, operators of norm 1. But this implies

$$\|\,T(A)\,\|_1 \leqslant \underbrace{\left\|\Psi^{-1}\right\|}_{=1} \cdot \underbrace{\left\|\iota^*\right\|}_{=1} \cdot \underbrace{\|\,\Phi\,\|}_{=1} \cdot \|\,A\,\|_1 = \|\,A\,\|_1$$

which proves our lemma. □

The third property that is important for the proof of theorem 3 is the connection between the projective norm and ℓ_1-spaces (see [7] for details). Let $\ell_1(\mathbb{C}^n)$ denote the space $\mathbb{C}^n$ endowed with the usual 1-norm $\|\cdot\|_1$. For a normed space X, we can also define an X-valued version of this space consisting of n-tuples of elements of X called $\ell_1(X^n)$. The norm of an element $(x_1,\ldots,x_n)^T \in X^n$ is then defined by

$$\left\|(x_1,\ldots,x_n)^T\right\|_{1,X} := \sum_{i=1}^{n} \|\,x_i\,\|_X\,.$$

We may embed elements of $\mathbb{C}^n \otimes X$ into X^n by the map

$$\begin{gathered} j\colon \mathbb{C}^n \otimes X \to X^n \\ \sum_{i=1}^{k} (z_{i,1},\ldots,z_{i,n}) \otimes x_i \mapsto \sum_{i=1}^{k} (z_{i,1}x_i,\ldots,z_{i,n}x_i). \end{gathered}$$

As can be shown (compare [7]), this map defines an isometric isomorphism of the spaces $\ell_1(\mathbb{C}^n) \otimes_\pi X$ and $\ell_1(X^n)$. This shows that ℓ_1 spaces behave well with respect to the construction of projective norms.

After these preparatory steps, let us now proceed to the proof of theorem 3, which will be performed in two steps.

4.2. Step 1. *In the first step, we concentrate on the case* $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$.

Without loss of generality, let us assume that the orthonormal systems $(e_i)_{1\leqslant i\leqslant k}$, $(f_i)_{1\leqslant i\leqslant k}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We denote by $\mathcal{K}$ the linear subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ generated by the tensor diagonal $\{e_1 \otimes f_1,\ldots,e_k \otimes f_k\}$. Thus, by definition, all vectors of this subspace are of the form

$$\psi = \sum_{i=1}^{k} \alpha_i e_i \otimes f_i \ .$$

As already pointed out in section 3, the projective norm of such a vector is given by

$$\| \psi \|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2} = \sum_{i=1}^{k} \left| \alpha_i \right| \ .$$

This implies that we have a $\| \cdot \|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2} - \| \cdot \|_1$ isometric isomorphism

$$\begin{gathered} \Lambda\colon \mathcal{K} \to \mathbb{C}^k \\ \sum_{i=1}^{k} \alpha_i e_i \otimes f_i \mapsto (\alpha_1,\ldots,\alpha_k)^T. \end{gathered}$$

It is now easy to see that this induces a $\| \cdot \|_{\mathcal{K}\otimes_\pi X} - \| \cdot \|_{\ell_1(\mathbb{C}^k)\otimes_\pi X}$-isometric isomorphism

$$\begin{gathered} \widetilde{\Lambda}\colon \mathcal{K} \otimes X \to \mathbb{C}^k \otimes X \\ \sum_{i=1}^{l} \eta_i \otimes x_i \mapsto \sum_{i=1}^{l} \Lambda(\eta_i) \otimes x_i. \end{gathered}$$

Thereby, $\| \cdot \|_{\mathcal{K}\otimes_\pi X}$ denotes the projective norm with respect to the norm on $\mathcal{K}$ inherited by the projective norm on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the norm on X.

Let us now demonstrate that $\mathcal{K}$ is complemented by a projection of norm 1. The canonical candidate is, of course, the orthogonal projection $P_\mathcal{K}\colon \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{K}$ and we have to show that $\| P_\mathcal{K} \| = 1$. For $e_1 \otimes f_1 \in \mathcal{K}$, for example, we have $\| P_\mathcal{K} e_1 \otimes f_1 \|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2} = \| e_1 \otimes f_1 \|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2} = 1$ so that $\| P_\mathcal{K} \| \geqslant 1$. Furthermore, the projective norm of an arbitrary unit vector $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ of the form

$$\psi = \sum_{i,j=1}^{k} \alpha_{ij} e_i \otimes f_j$$

is given by the sum of its Schmidt coefficients (compare with our discussion in section 2.1), which equals the trace norm of the matrix $A := (\alpha_{ij})$. Since

$$P_{\mathcal{K}}\psi = \sum_{i=1}^{k} \alpha_{ij} \underbrace{P_{\mathcal{K}}\left(e_i \otimes f_j\right)}_{\delta_{ij}} = \sum_{i=1}^{k} \alpha_{ii} e_i \otimes f_i$$

the representing matrix of $P_{\mathcal{K}}\psi$ is given by discarding all nondiagonal elements of A. Therefore, from lemma 1, we obtain the relation

$$\left|\left| P_{\mathcal{K}}\psi \right|\right|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2} = \left|\left| \mathrm{diag}\left(\alpha_{11}, \ldots, \alpha_{kk}\right) \right|\right|_1 \leqslant \| A \|_1 = \| \psi \|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2}$$

which proves that $\| P_{\mathcal{K}} \| \leqslant 1$ and consequently $\| P_{\mathcal{K}} \| = 1$. Let $s_1, \ldots, s_k$ be the canonical basis of $\mathbb{C}^k$. By proposition 1, we can conclude that

$$\begin{aligned} \left|\left| \xi \right|\right|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2 \otimes_\pi X} &= \left|\left| \xi \right|\right|_{\mathcal{K} \otimes_\pi X} = \left|\left| \widetilde{\Lambda}(\xi) \right|\right|_{\ell_1\left(\mathbb{C}^k\right) \otimes_\pi X} \\ &= \left|\left| \sum_{i=1}^{k} \Lambda\left(e_i \otimes f_i\right) \otimes x_i \right|\right|_{\ell_1\left(\mathbb{C}^k\right) \otimes_\pi X} = \left|\left| \sum_{i=1}^{k} s_i \otimes x_i \right|\right|_{\ell_1\left(\mathbb{C}^k\right) \otimes_\pi X} \end{aligned}$$

and using $\ell_1\left(\mathbb{C}^k\right) \otimes_\pi X \cong \ell_1\left(X^k\right)$, we finally obtain

$$\begin{aligned} \| \xi \|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2 \otimes_\pi X} &= \left|\left| \sum_{i=1}^{k} \left(\delta_{i1} x_i, \ldots, \delta_{ik} x_i\right)^T \right|\right|_{\ell_1\left(X^k\right)} \\ &= \left|\left| \left(x_1, \ldots, x_k\right)^T \right|\right|_{\ell_1\left(X^k\right)} = \sum_{i=1}^{k} \left|\left| x_i \right|\right|_X . \end{aligned}$$

This proves theorem 3 for the special case $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$.

4.3. Step 2. Let us now consider the generalization to $n = \dim \mathcal{H}_1 \neq m = \dim \mathcal{H}_2$.

Without loss of generality, we assume that $n < m$. Furthermore, we may supplement the orthonormal systems $(e_i)_{1 \leqslant i \leqslant k}$, $(f_i)_{1 \leqslant i \leqslant k}$ to orthonormal bases $\mathcal{B} := (e_i)_{1 \leqslant i \leqslant n}$ of $\mathcal{H}_1$ and $\mathcal{B}' := (f_i)_{1 \leqslant i \leqslant m}$ of $\mathcal{H}_2$, respectively. Defining the map

$$\begin{aligned} \iota_{\mathcal{B},\mathcal{B}'}&: \mathcal{H}_1 \to \mathcal{H}_2 \\ &\sum_{i=1}^{n} \alpha_i e_i \mapsto \sum_{i=1}^{n} \alpha_i f_i , \end{aligned}$$

which is an isometric embedding of $\mathcal{H}_1$ into $\mathcal{H}_2$, we can identify $\mathcal{H}_1$ with the subspace $\mathcal{K} := \iota_{\mathcal{B},\mathcal{B}'}(\mathcal{H}_1) \subset \mathcal{H}_2$. The space $\mathcal{K}$ is complemented by the orthogonal projection $P_{\mathcal{K}}$ on $\mathcal{K}$ and $\|P_{\mathcal{K}}\| = 1$. Therefore, proposition 1.2 implies that the subspace $\mathcal{K} \otimes \mathcal{H}_2 \subset \mathcal{H}_2 \otimes \mathcal{H}_2$ is complemented by a projection of norm 1 as well, and by proposition 1.1, we obtain the relation

$$\left\| \sum_{i=1}^{k} e_i \otimes f_i \otimes x_i \right\|_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2 \otimes_\pi X} = \left\| \sum_{i=1}^{k} \iota_{\mathcal{B},\mathcal{B}'}(e_i) \otimes f_i \otimes x_i \right\|_{\mathcal{K} \otimes_\pi \mathcal{H}_2 \otimes_\pi X}$$
$$= \left\| \sum_{i=1}^{k} f_i \otimes f_i \otimes x_i \right\|_{\mathcal{K} \otimes_\pi \mathcal{H}_2 \otimes_\pi X} = \left\| \sum_{i=1}^{k} f_i \otimes f_i \otimes x_i \right\|_{\mathcal{H}_2 \otimes_\pi \mathcal{H}_2 \otimes_\pi X}$$
$$\stackrel{\text{Step1}}{=} \sum_{i=1}^{k} \| x_i \|_X$$

which finally proves the general form of theorem 3.

5. Summary and Conclusions

We have introduced the notion of generalized Schmidt decomposability of multipartite quantum systems. It has been demonstrated that projective norms of multipartite pure quantum states, which are induced by generalized Schmidt decomposable vectors, can be evaluated in a simple way. Furthermore, we have shown that this technique can be extended to the calculation of projective norms of certain classes of mixed states with the help of an isometric isomorphism between the space of trace-class-operators on a Hilbert space and the projective tensor product of this space with its topological dual.

In the future, we intend to explore possibilities of using the property of generalized Schmidt decomposability for the computation of projective norms of N-partite states by iteratively reducing them to finitely many projective norms of $N - 2$-partite states. Thus, in optimal cases, it should be possible to reduce this problem to the evaluation of a finite set of bipartite projective norms, i.e., to the evaluation of Schmidt coefficients. Using the previously mentioned isometric isomorphism, it should also be possible to extend this procedure to mixed states.

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References

[1] Schrödinger E 1935 Die gegenwärtige Situation in der Quantenmechanik *Die Naturwissenschaften* **49** 823–8
[2] Einstein A, Podolsky B and Rosen N 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47** 777–80
[3] Bell J S 1964 On the Einstein–Podolsky–Rosen paradox *Physics* **1** 195
[4] Plenio M B and Virmani S 2007 An introduction to entanglement measures *Quantum Inf. Comput.* **7** 1–51
[5] Bennett C H, Bernstein H, Popescu S and Schumacher B 1996 Concentrating partial entanglement by local operations *Phys. Rev.* A **53** 2046–52
[6] Plenio M B, Virmani S and Papadopoulus P 2000 Operator monotones, the reduction criterion and the relative entropy *J. Phys.* A **33** L193
[7] Ryan R A 2002 *Introduction to Tensor Products of Banach Spaces* (London: Springer-Verlag)
[8] Conway J B 1990 *A Course in Functional Analysis* 2nd edn (New York: Springer-Verlag)

[9] Rudolph O 2001 A new class of entanglement measures *J. Math. Phys.* **42** 5306–14
[10] Grabowski J, Kuś M and Marmo G 2005 Geometry of quantum systems: density states and entanglement *J. Phys. A: Math. Gen.* **38** 10217–44
[11] Uhlmann A 1998 Entropy and optimal decompositions of states relative to a maximal commutative subalgebra *Open Syst. Inf. Dyn.* **5** 209–28
[12] Uhlmann A 2000 Fidelity and concurrence of conjugated states *Phys. Rev.* A **62**
[13] Grabowski J, Kuś M and Marmo G 2006 Symmetries, group actions, and entanglement *Open Syst. Inf. Dyn.* **13** 343–62
[14] Mintert F, Kuś M and Buchleitner A 2005 Concurrence of mixed multipartite quantum states *Phys. Rev. Lett.* **95**
[15] Grabowski J, Kuś M and Marmo G 2007 On the relation between states and maps in infinite dimensions *Open Syst. Inf. Dyn.* **14** 355–70
[16] Schmidt E 1907 Zur Theorie der linearen und nichtlinearen Integralgleichungen *Math. Ann.* **63** 433
[17] Vedral V and Plenio M B 1998 Entanglement measures and purification procedures *Phys. Rev.* A **57** 1619–33
[18] Dür W, Vidal G and Cirac J I 2000 Three qubits can be entangled in two inequivalent ways *Phys. Rev.* A **62** 062314
[19] Eltschka C and Siewert J 2012 Entanglement of three-qubit Greenberger-Horne-Zeilinger-Symmetric states *Phys. Rev. Lett.* **108** 020502
[20] Rudolph O 2005 Further results on the cross norm criterion for separability *Quantum Inf. Process.* **4** 219
[21] Rudolph O 2000 A separability criterion for density operators *J. Phys. A: Math. Gen.* **33** 3951
[22] Arveson W 2009 Maximal vectors in Hilbert spaces and quantum entanglement *J. Funct. Anal.* **256** 1476–510
[23] Acín A, Andrianov A, Costa L, Jané E, Latorre J I and Tarrach R 2000 Generalized schmidt decomposition and classification of three-quantum-bit states *Phys. Rev. Lett.* **85** 1560–3
[24] Carteret H A, Higuchi A and Sudbery A 2000 Multipartite generalization of the Schmidt decomposition *J. Math. Phys.* **41** 7932
[25] Tamaryan L, Park D and Tamaryan S 2009 Generalized Schmidt decomposition based on injective tensor norm arXiv:0809.1290v2
[26] Greenberger D, Horne M A, Shimony A and Zeilinger A 1990 Bell's theorem without inequalities *Am. J. Phys.* **58** 1131
[27] Takesaki M 2002 *Theory of Operator Algebra III* (New York: Springer-Verlag)