

Cyclic mutually unbiased bases, Fibonacci polynomials and Wiedemann's conjecture

Ulrich Seyfarth and Kedar S. Ranade

Citation: *J. Math. Phys.* **53**, 062201 (2012); doi: 10.1063/1.4723825

View online: <http://dx.doi.org/10.1063/1.4723825>

View Table of Contents: <http://jmp.aip.org/resource/1/JMAPAQ/v53/i6>

Published by the [American Institute of Physics](http://www.aip.org).

Related Articles

Uncertainty inequalities among frequency moments and radial expectation values: Applications to atomic systems

J. Math. Phys. **53**, 043512 (2012)

Quantum spin holography with surface state electrons

Appl. Phys. Lett. **100**, 163112 (2012)

Macroscopic thermal entanglement in a spin chain caused by the magnetic field: Inhomogeneity effect

Low Temp. Phys. **38**, 210 (2012)

Continuity bounds on the quantum relative entropy — II

J. Math. Phys. **52**, 112201 (2011)

Some remarks on assignment maps

J. Math. Phys. **52**, 012101 (2011)

Additional information on *J. Math. Phys.*

Journal Homepage: <http://jmp.aip.org/>

Journal Information: http://jmp.aip.org/about/about_the_journal

Top downloads: http://jmp.aip.org/features/most_downloaded

Information for Authors: <http://jmp.aip.org/authors>

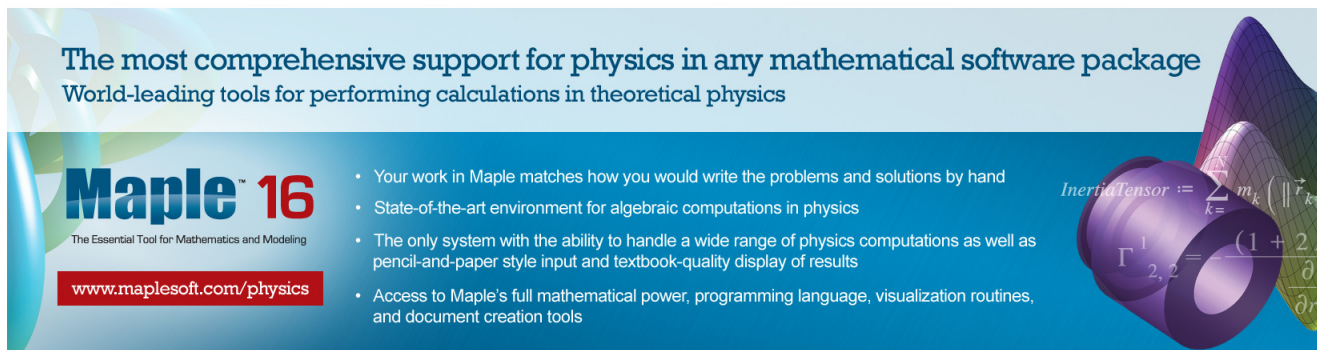
ADVERTISEMENT

The most comprehensive support for physics in any mathematical software package
World-leading tools for performing calculations in theoretical physics

Maple 16
The Essential Tool for Mathematics and Modeling
www.maplesoft.com/physics

- Your work in Maple matches how you would write the problems and solutions by hand
- State-of-the-art environment for algebraic computations in physics
- The only system with the ability to handle a wide range of physics computations as well as pencil-and-paper style input and textbook-quality display of results
- Access to Maple's full mathematical power, programming language, visualization routines, and document creation tools

Inertia Tensor := $\sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \mathbf{1} - \vec{r}_k \vec{r}_k^T \right)$



Cyclic mutually unbiased bases, Fibonacci polynomials and Wiedemann's conjecture

Ulrich Seyfarth^{1,a)} and Kedar S. Ranade^{2,b)}

¹*Institut für Angewandte Physik, Technische Universität Darmstadt, Hochschulstraße 4a, 64289 Darmstadt, Germany*

²*Institut für Quantenphysik, Universität Ulm, Albert-Einstein-Allee 11, 89081 Ulm, Germany*

(Received 10 May 2012; accepted 14 May 2012; published online 5 June 2012)

We relate the construction of a complete set of cyclic mutually unbiased bases, i.e., mutually unbiased bases generated by a single unitary operator, in power-of-two dimensions to the problem of finding a symmetric matrix over \mathbb{F}_2 with an irreducible characteristic polynomial that has a given Fibonacci index. For dimensions of the form 2^k , we present a solution that shows an analogy to an open conjecture of Wiedemann in finite field theory. Finally, we discuss the equivalence of mutually unbiased bases.

© 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4723825>]

I. INTRODUCTION

Mutually unbiased bases (MUBs) are used in various contexts in quantum information theory. The construction of MUBs is connected to the theory of finite fields^{1,2} and to construction problems of Latin squares³ or of maximally commuting bases of orthogonal unitary matrices.⁴ In this paper, we aim to construct complete sets of *cyclic* mutually unbiased bases,^{5,6} which exist only for even prime-power dimensions.⁷ We start with the definition of MUBs.

Definition 1.1 (Mutually unbiased bases):

Let $\mathcal{H} = \mathbb{C}^d$ be a d -dimensional complex Hilbert space. Two orthonormal bases $\mathcal{B}_1 = \{|\psi_1\rangle, \dots, |\psi_d\rangle\}$ and $\mathcal{B}_2 = \{|\phi_1\rangle, \dots, |\phi_d\rangle\}$ of \mathcal{H} are said to be *mutually unbiased*, if there holds $|\langle\psi_i|\phi_j\rangle|^2 = d^{-1}$ for all $i, j \in \{1, \dots, d\}$.

The standard examples for MUBs are the z -, x - and y -basis for $d = 2$ or any basis and its Fourier transform for any dimension d . It is well known that the maximum number of pairwise mutually unbiased bases of \mathbb{C}^d , denoted $N(d)$, is upper bounded by $d + 1$. If d is a prime power, then $N(d) = d + 1$ bases can be constructed by using methods from finite field theory. In any other dimension, the precise number $N(d)$ is unknown.

A basis of $\mathcal{H} = \mathbb{C}^d$ can always be identified with a unitary matrix U on this space, since the columns of such matrix define an orthonormal basis and vice versa. Since U^2, U^3 , etc., are unitary, they also define bases of \mathcal{H} .

In the following, let \mathbb{N} be the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a finite field of order q , we write \mathbb{F}_q , for the complex field \mathbb{C} , and for the $n \times n$ -matrices over an arbitrary field (or ring) K , we write $M_n(K)$.

Definition 1.2 (Cyclic mutually unbiased bases):

A set $\{\mathcal{B}_0, \dots, \mathcal{B}_{n-1}\}$ of mutually unbiased bases is called *cyclic*, if there exists a unitary matrix $U \in M_d(\mathbb{C})$ of finite order $n \in \mathbb{N}$, such that the columns of the matrices $U, U^2, U^3, \dots, U^n = U^0 = \mathbb{1}_d$ coincide with the bases $\{\mathcal{B}_0, \dots, \mathcal{B}_{n-1}\}$.

^{a)}Electronic mail: ulrich.seyfarth@physik.tu-darmstadt.de.

^{b)}Electronic mail: kedar.ranade@uni-ulm.de.

We are interested in *complete sets of cyclic MUBs*, i.e., sets of $N(d)$ cyclic MUBs, and want to construct a unitary generator U . Since $N(d)$ is known for prime powers only and it was shown that such operators cannot exist for odd dimensions,⁷ we shall only consider $d = 2^m$ for some $m \in \mathbb{N}$. (We will keep this notion of d and m throughout this paper.)

II. METHODS AND FORMALISM

In this section, we introduce the methods which are relevant for our treatment of cyclic MUBs. We will start with a subsection on Fibonacci polynomials and then relate the construction of cyclic MUBs to properties of a particular matrix.

A. Fibonacci polynomials

Our analysis of complete sets of cyclic MUBs is closely related to the properties of the so-called *Fibonacci polynomials*. Note that these polynomials differ by one from the f_n used previously, i.e., $F_n = f_{n-1}$.⁸

Definition 2.1 (Fibonacci polynomials):

Over an arbitrary field K , we define the *Fibonacci polynomials* $(F_n)_{n \in \mathbb{N}_0}$, by $F_0(x) := 0$, $F_1(x) := 1$, and the recursion $F_{n+1}(x) := x \cdot F_n(x) + F_{n-1}(x)$.

If $K = \mathbb{C}$, the sequence $(F_n(1))_{n \in \mathbb{N}_0}$ is the usual Fibonacci sequence. In the context of cyclic MUBs, we exclusively deal with the ground field $K = \mathbb{F}_2$ and possibly its extensions and make use of a block matrix $C = \begin{pmatrix} B & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in M_{2m}(\mathbb{F}_2)$ with $\mathbb{1}$, $0 \in M_m(\mathbb{F}_2)$ and some $B \in M_m(\mathbb{F}_2)$. The powers of C are easily seen to be of the form

$$C^k = \begin{pmatrix} F_{k+1}(B) & F_k(B) \\ F_k(B) & F_{k-1}(B) \end{pmatrix} \in M_{2m}(\mathbb{F}_2), \quad (1)$$

where the Fibonacci polynomials naturally appear. The Fibonacci numbers satisfy several well-known divisibility relations (pp. 67–69 in Ref. 9) and we will need their counterparts for Fibonacci polynomials over $K = \mathbb{F}_2$. We start with a definition.

Definition 2.2 (Fibonacci index):

The *Fibonacci index* (sometimes *depth*) of an irreducible polynomial $p \in K[x]$ is defined as the minimum number $d \in \mathbb{N}$, such that p divides F_d .

Several properties of these polynomials are discussed by Goldwasser *et al.*;^{10,11} for example, F_n divides F_m , if and only if n divides m for $n, m \in \mathbb{N}$. In the proof of the following lemma (Lemmas 5 and 7(a) of Ref. 11) we use the fact that there holds $F_{n-t}(x) + F_{n+t}(x) = xF_n(x)F_t(x)$ over \mathbb{F}_2 for $n, t \in \mathbb{N}$ and $t \leq n$ (Lemma 4(2) of Ref. 10).

Lemma 2.1 (Fibonacci polynomials and Fibonacci index):

For every $m \in \mathbb{N}$, the product $F_{2^m+1}(x) \cdot F_{2^m-1}(x)$ is equal to the square of the product of all irreducible polynomials—except x —whose degree divides m . Therefore, the Fibonacci index of an irreducible polynomial $p \in \mathbb{F}_2[x]$ of degree m is well defined and, for $p(x) \neq x$, is odd and divides either $2^m + 1$ or $2^m - 1$.

Proof: We set $n = 2^m + 1$ and $t = n - 2$ and use that (i) over any finite field \mathbb{F}_q the product of all monic irreducible polynomials whose degree divide some $m \in \mathbb{N}$ is given by $x^{q^m} - x$ (Theorem 3.20 of Ref. 12), and (ii) $F_{2^m}(x) = x^{2^m-1}$ (Lemma A.3 of Ref. 8). Note further that F_{2^m-1} and F_{2^m+1} are coprime (Proposition 5 of Ref. 10). \square

The following lemma was proven by Sutner (Theorem 3 of Ref. 13).

Lemma 2.2 (Irreducible factors of $F_{2^m \pm 1}$):

Let $p = \sum_{i=0}^m a_i x^i \in \mathbb{F}_2[x]$ be an irreducible polynomial of order $m \in \mathbb{N}$. Then, p divides F_{2^m+1} , if $a_1 = 1$; otherwise it divides F_{2^m-1} .

B. Finding a stabilizer matrix

Bandyopadhyay *et al.* showed that the problem of finding MUBs on \mathbb{C}^d can be reduced to finding certain partitions of operator bases of $M_d(\mathbb{C})$ (cf. also the Appendix).⁴ The problem of constructing cyclic MUBs can accordingly be reduced to finding a suitable symplectic *stabilizer matrix* $C \in M_{2m}(\mathbb{F}_2)$, where $d = 2^m$.⁸ Numerically, it was shown that, at least for $m \leq 24$, this matrix may be chosen of the form

$$C = \begin{pmatrix} B & \mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix} \in M_{2m}(\mathbb{F}_2) \quad (2)$$

with $B \in M_m(\mathbb{F}_2)$, which we may call *reduced stabilizer matrix*, and we will see in this paper that this form can be maintained for all $m \in \mathbb{N}$. (We summarize the explicit construction of the generating unitary U from such B in the Appendix.) Using the Fibonacci polynomials, we demand that $C^{d+1} = \mathbb{1}_{2m}$, i.e.,

$$C^{d+1} = \begin{pmatrix} F_{d+2}(B) & F_{d+1}(B) \\ F_{d+1}(B) & F_d(B) \end{pmatrix} = \begin{pmatrix} \mathbb{1}_m & 0_m \\ 0_m & \mathbb{1}_m \end{pmatrix}. \quad (3)$$

We thus require $F_d(B) = \mathbb{1}_m$ and $F_{d+1}(B) = 0_m$. By the recursion relation, the condition $F_{d+2}(B) = \mathbb{1}_m$ is automatically fulfilled. The conditions on B are the following:⁸

- (i) B is symmetric, i.e. $B = B^t$,
- (ii) $F_k(B)$ is invertible for $k \in \{1, \dots, d\}$,
- (iii) $F_d(B) = B^{d-1} = \mathbb{1}_m$.

Since the minimal polynomial of a matrix B is the normalized polynomial of minimal degree that annihilates B , we see that it is a factor of F_{d+1} .

Lemma 2.3 (Characteristic polynomial of reduced stabilizer matrices):

A matrix $B \in M_m(\mathbb{F}_2)$, $m \in \mathbb{N}$, that is at first annihilated by F_{2^m-1} or F_{2^m+1} , has an irreducible characteristic polynomial. The characteristic polynomial thus coincides with the minimal polynomial of B .

Proof: If the first Fibonacci polynomial that annihilates B equals $F_{2^m \pm 1}$, B is annihilated by a polynomial p with Fibonacci index $2^m \pm 1$ that has degree m and is therefore irreducible. Since the degree of p coincides with the dimension of B , it is the irreducible characteristic polynomial of B . \square

Lemma 2.4 (Multiplicative order of B):

If the characteristic polynomial of B has Fibonacci index $d + 1$, then $F_d(B) = \mathbb{1}_m$.

Proof: By Lemma 2.3, χ_B is irreducible and its splitting field over \mathbb{F}_2 is isomorphic to \mathbb{F}_d with $d = 2^m$. By the Hamilton-Cayley theorem, B is a root of χ_B and thus, as an element of the multiplicative group of a finite field of order d , there holds $B^{d-1} = \mathbb{1}_m$. The assertion now follows from $F_d(B) = B^{d-1}$ (cf. Lemma A.3 of Ref. 8). \square

Proposition 2.5 (Representation of fields):

Let L/K be a field extension of order $n \in \mathbb{N}$ and let $A \in M_n(K)$ be a matrix, such that its characteristic polynomial $\chi_A \in K[x]$ is irreducible (and therefore minimal). Then, the set $\{f(A) | f \in K[x]\}$ is isomorphic to the extension field L .

Note that the set of polynomials in A effectively includes only polynomials of degree less than n ; it also includes the zero polynomial with degree $-\infty$.

Proof: By the Hamilton-Cayley theorem, there holds $\chi_A(A) = 0$ as matrix equality, i.e., A is a root of the irreducible polynomial χ . By adjoining A to the field K , we obtain the extension L (cf. p. 66 ff. of Ref. 12). \square

With the help of this proposition and Lemma 2.4 we can replace conditions (ii) and (iii) on B from above by

(ii') the Fibonacci index of the characteristic polynomial χ_B of B is $d + 1$.

The following theorem shows that for every dimension $d = 2^m$ we can find a reduced stabilizer matrix B and thus a unitary generator U of a complete set of cyclic MUBs.

Theorem 2.6 (Existence of reduced stabilizer matrices):

For any dimension $d = 2^m$ with $m \in \mathbb{N}$ there exists a reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$ that fulfills conditions (i) and (ii').

Proof: We have to find a symmetric matrix $B \in M_m(\mathbb{F}_2)$ with irreducible characteristic polynomial and Fibonacci index $d + 1$. It is known that the number of polynomials in $\mathbb{F}_2[x]$ with degree m and Fibonacci index $d + 1$ is given by $\frac{\phi(d+1)}{2^m}$, where ϕ is Euler's totient function (Theorem 8 of Ref. 11). Since this expression is non-zero, there exists at least one polynomial for any dimension $d = 2^m$ that has maximal Fibonacci index $d + 1$.

It is well known that any polynomial $p \in K[x]$ possesses a companion matrix, whose characteristic polynomial is p . For every monic polynomial over a finite field $K = \mathbb{F}_q$ there exists a symmetric matrix that is similar to the companion matrix and, therefore, has the same characteristic polynomial (Lemma 2 of Ref. 14). \square

III. EXPLICIT CONSTRUCTION OF THE REDUCED STABILIZER MATRIX FOR $m = 2^k$

In this section, we deal with a specific solution of B for $m = 2^k$ and $k \in \mathbb{N}$. We start by presenting an iterated construction of B , which is followed by a discussion on the characteristic polynomial and its Fibonacci index. We show that this construction is related to an open conjecture in finite field theory by Wiedemann¹⁵ (cf. Prob. 28 of Ref. 16) that is still of current interest.¹⁷

Let us recall some already known solutions:⁸ For $k = 1$, there exist two different forms of B (by permutation), one of them being $B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{F}_2)$. If we set $k = 2$ a possible choice of B is the block matrix $B_4 = \begin{pmatrix} B_2 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in M_4(\mathbb{F}_2)$. By iterating this construction, we get a recursion

$$B_{2^k} = \begin{pmatrix} B_{2^{k-1}} & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in M_{2^k}(\mathbb{F}_2), \quad (4)$$

and we shall show that it is suitable for our purposes.

A. Irreducible self-reciprocal polynomials and the characteristic polynomial of B_{2^k}

In this subsection, we want to evaluate the characteristic polynomials of the matrices B_{2^k} of (4). It will turn out that the notion of reciprocal and self-reciprocal polynomials are of interest in the context of determining χ_B .

Definition 3.1 (Reciprocal polynomial):

The *reciprocal* of a polynomial $f \in K[x]$ is defined by $f^*(x) := x^n f(x^{-1})$, where $n \in \mathbb{N}$ is the degree of f . If $f = f^*$, it is called *self-reciprocal*.

By applying the *reciprocal operator* Q on an arbitrary polynomial f of degree n , we get a self-reciprocal polynomial as

$$f^Q(x) := x^n f(x + x^{-1}). \quad (5)$$

Varshamov and Garakov¹⁸ have shown, that in the case of $f \in \mathbb{F}_2[x]$ and f being irreducible, f^Q again is irreducible if and only if the linear coefficient of f does not vanish; see also Meyn.¹⁹ We can now relate the notion of reciprocal polynomials to our construction from (4).

Lemma 3.1 (Characteristic polynomials):

Let K be a finite field of characteristic 2 and $S \in M_n(K)$ with characteristic polynomial χ_S . The characteristic polynomial of the matrix $S' := \begin{pmatrix} S & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in M_{2n}(K)$ is given by $\chi_{S'} = (\chi_S)^Q$.

In the following proof, we use the well-known fact that for block matrices there holds $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C)$, if D is invertible, which follows from the decomposition $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{1} & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & \mathbb{1} \end{pmatrix}$, because the determinant of block-triangular matrices is the product of the determinants of the blocks.

Proof: We calculate $\chi_{S'}(x) = \det(x\mathbb{1}_{2n} - S') = \det \begin{pmatrix} x\mathbb{1}_n - S & \mathbb{1}_n \\ \mathbb{1}_n & x\mathbb{1}_n \end{pmatrix}$ which yields $\chi_{S'}(x) = \det(x\mathbb{1}_n) \det(x\mathbb{1}_n - S - x^{-1}\mathbb{1}_n) = x^n \cdot \chi_S(x - x^{-1})$. Since $\text{char } K = 2$, this is the reciprocal of χ_S . \square

Given an arbitrary polynomial of degree n , $f(x) = \sum_{k=0}^n a_k x^k$, after some easy calculations we find $f^Q(x) = \sum_{k=0}^n \sum_{i=0}^k \binom{k}{i} a_k x^{n+k-2i}$. A contribution to the linear coefficient arises only from $k = i = n - 1$, so that it is given by a_{n-1} . A corollary thereof is the following.

Corollary 3.2 (Minimal polynomial of B_{2^k}):

The characteristic polynomial of $B_{2^k} \in M_{2^k}(\mathbb{F}_2)$ as in (4), $k \in \mathbb{N}$, is given by $f^Q(x)$ with $f(x) = 1 + x$. Therefore, it is irreducible and minimal.

Proof: By induction, the first part follows immediately from Lemma 3.1. According to Varshamov-Garakov, all polynomials are irreducible, provided that the linear coefficient does not vanish. This is guaranteed by the statements preceding this corollary. \square

This corollary implies that every $F_l(B_{2^k})$ is either invertible or zero, since by Proposition 2.5 it is a representative of an element of some finite field. However, we do not yet know anything about the Fibonacci index of the characteristic polynomial of B_{2^k} , but we shall relate it to an open conjecture in finite field theory in Subsection III B.

B. Fibonacci index of χ_B and Wiedemann's conjecture

Wiedemann¹⁵ considered iterated quadratic extensions of \mathbb{F}_2 using generators $x_{j+1} + x_{j+1}^{-1} = x_j$ for $j \in \mathbb{N}_0$ and $x_0 + x_0^{-1} = 1$; these extensions read $E_0 := \mathbb{F}_2(x_0)$, $E_1 := E_0(x_1)$, etc., where E_n is isomorphic to $\mathbb{F}_{2^{2^n+1}}$. He then showed that the order of x_n divides the n th Fermat number $\mathcal{F}_n = 2^{2^n} + 1$. Since the Fermat numbers are mutually coprime, the order of $\tilde{x} := x_0 x_1 \dots x_n \in E_n$ is the product of the orders of the x_i . If the order of each x_i is \mathcal{F}_i for $i \leq n$, then \tilde{x} is primitive in E_n , i.e., a generator of the multiplicative group of E_n . This is *Wiedemann's conjecture*, which is verified numerically for $n \leq 8$ (cf. p. 291/295 of Ref. 15). It turns out that our matrices C_{2^k} can be seen as a realization of the x_k .

Theorem 3.3 (Wiedemann analogy):

Wiedemann's conjecture is true, if and only if the characteristic polynomial of every B_{2^k} as in (4) has Fibonacci index $2^{2^k} + 1$.

Proof: Let C_{2^k} be the stabilizer matrix as in (2) using the construction of B_{2^k} from (4). Obviously, there holds $C_{2^0} + C_{2^0}^{-1} = \mathbb{1}$. Now, if C_{2^k} is of order $2^{2^k} + 1$, the characteristic polynomial of B_{2^k} has Fibonacci index $2^{2^k} + 1$. The characteristic polynomial of C_{2^k} over the field generated by $B_{2^k} = C_{2^{k-1}}$ is then given by $x^2 + C_{2^{k-1}}x + 1$. Identifying C_{2^k} with Wiedemann's x_k , the C_{2^k} are the roots of these polynomials, which are equivalent to those in Wiedemann's construction. \square

This theorem shows that our problem is an instance of Wiedemann's conjecture; however, a proof of his conjecture is still missing. We have thus shown that we are able to construct a reduced stabilizer matrix B , out of which we can construct a unitary operator U which generates a complete set of cyclic MUBs. In the Appendix, we show that this operator can be written in the form $U = e^{i\psi} H^{\otimes m} P$ with some global phase $e^{i\psi}$ and explicitly construct the diagonal phase system P from B .⁸

IV. EQUIVALENCE OF SETS OF MUTUALLY UNBIASED BASES

In this section, we shall discuss the equivalence of sets of mutually unbiased bases. At first, we will introduce methods which are related to those of Calderbank *et al.*,²⁰ Theorem 5.6 and Proposition 5.11; see also Kantor,²¹ Theorem 2.3. Our presentation is self-contained and stated in the language of operators, etc., rather than in terms of discrete geometry. We start with the usual definition of equivalence of MUBs.

We shall identify a basis of the Hilbert space $\mathcal{H} = \mathbb{C}^d$ with a unitary matrix \mathcal{B} consisting of the basis vectors as columns and consider two sets of mutually unbiased bases, $\mathfrak{S} = \{B_1, \dots, B_r\}$ and $\mathfrak{S}' = \{B'_1, \dots, B'_r\}$; in our case, $r = d + 1$.

Definition 4.1 (Equivalence of mutually unbiased bases):

We say, the sets \mathfrak{S} and \mathfrak{S}' are equivalent, if there exists a unitary operator U on \mathcal{H} , a permutation π on $\{1, \dots, r\}$ and monomial matrices²² $W_k, k \in \{1, \dots, r\}$, such that there holds $B'_k = UB_{\pi(k)}W_k$ for all $k \in \{1, \dots, r\}$.

The essential transformation is the unitary U ; the matrices W_k take care for the fact that for MUBs we are more interested in a set of one-dimensional subspaces than in bases as ordered sets and π represents the fact that we consider unordered sets of bases.

As outlined in the Appendix (for details cf. Refs. 4 and 8), our MUBs appear as eigenbases of operators, i.e., $\mathfrak{C} = \{C_1, \dots, C_{d+1}\}$, where each class C_k consists of $d - 1$ commuting Pauli operators. Therefore, $C_k = \{XZ(\vec{v}_1), \dots, XZ(\vec{v}_{d+1})\}$. We can now reformulate equivalence in a way that the order of the basis vectors and their total phase—as in the W_k —do not appear.

Lemma 4.1 (Equivalence of mutually unbiased bases):

The MUBs characterized by $\mathfrak{C} = \{C_1, \dots, C_{d+1}\}$ and $\mathfrak{C}' = \{C'_1, \dots, C'_{d+1}\}$ are equivalent, if and only if there exists a unitary U on \mathbb{C}^d and a permutation π on $\{1, \dots, d + 1\}$, such that $C'_k = UC_{\pi(k)}U^\dagger$, where U acts simultaneously on all operators within a class C_k .

Proof: Assume that the eigensystems of \mathfrak{C} and \mathfrak{C}' are equivalent MUBs. As the ordering within a basis does not appear here, the W_k are irrelevant, and the unitary U is the same as in the definition of equivalence, since Uv is an eigenvector of UAU^\dagger , if and only if v is one of A . \square

As we assume \mathfrak{C} and \mathfrak{C}' to consist of Pauli operators only, the conjugation by U must map Pauli operators onto Pauli operators, i.e., U belongs to the Clifford group. We can view the Pauli operators—essentially the Pauli group factorized by its center $\{\pm 1, \pm i\} \mathbb{1}$ —as a projective representation of the symplectic space $V_S := \mathbb{F}_2^m \times \mathbb{F}_2^m$, and the unitary U thus corresponds to an invertible linear mapping f on V_S , i.e., $f \in M_{2m}(\mathbb{F}_2)$. In order to preserve the commutation relations, we require f to be symplectic.

In general, the classes C_k with $d - 1 = 2^m - 1$ elements are generated by m elements. In terms of V_S we have m vectors which generate a subspace of V_S with $d - 1$ non-zero elements. Viewing

a class \mathcal{C}_k as a $2m \times (d-1)$ -matrix, where the columns represent the Pauli operators (as in the Appendix), we find a (non-unique) generator G_k , which is a $2m \times m$ -matrix, and write $\langle G_k \rangle = \mathcal{C}_k$. As we can permute the columns within the \mathcal{C}_k by a permutation Q_k , we find that \mathcal{C} and \mathcal{C}' are equivalent, if and only if there exists a mapping f and a permutation π on $\{0, \dots, d\}$, such that $\langle G'_k \rangle = f \langle G_{\pi(k)} \rangle Q_k$ for all k .

We can assume that the matrix representation of f is of the block-matrix form $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$; symplecticity is equivalent to the three conditions, that $s^t u$ and $t^t v$ are symmetric and $v^t s - t^t u = \mathbb{1}_m$. Since in cyclic MUBs, we can rotate through the bases, we can choose $\pi(0) = 0$, such that $\langle G_0 \rangle = \langle G'_0 \rangle$ consists of all Z -type Pauli operators with the standard basis as their eigenbasis. This implies $u = 0$ and, to keep symplecticity, that $v = (s^t)^{-1}$.

The elements within a class \mathcal{C}_k are given by⁸, i.e.,

$$\mathcal{C}_k = \{C^k \cdot (\mathbb{1}_m, 0_m)^t \cdot c : c \in \mathbb{F}_2^m\}. \quad (6)$$

The generators of the classes \mathcal{C}_k can be grouped into the generator of the Z -type Pauli operators $G_0 = (\mathbb{1}_m, 0_m)^t$ and the remaining generators $G_k = C^k \cdot (\mathbb{1}_m, 0_m)^t$ with $k \in \{1, \dots, d\}$, which can be written as $G_k = (F_{k+1}(B), F_k(B))^t$ with the reduced stabilizer matrix B . For a complete set of MUBs, the generators G_k are constructed in a way, that for $k \in \{1, \dots, d\}$ the polynomial $F_k(B)$ is non-zero and invertible (cf. Proposition 2.5). Since any invertible operator that acts in (6) on the elements c is given by a permutation of those elements, we will produce the same classes \mathcal{C}_k , if we replace G_k for $k \in \{1, \dots, d\}$ by

$$\tilde{G}_k = \begin{pmatrix} p_k(B) \\ \mathbb{1}_m \end{pmatrix} \quad (7)$$

with $p_k(B) = F_{k+1}(B) \cdot F_k^{-1}(B)$. We call this the standard form, which fits into the construction of Bandyopadhyay *et al.*⁴ As B is a root of its minimal polynomial with degree m , the 2^m different elements $p_k(B)$ represent the extension field \mathbb{F}_2^m , i.e., the p_k are all 2^m polynomials over \mathbb{F}_2 with degree less than m .

By expressing the generators of a set of MUBs in the standard form we avoid the use of permutations Q_k . Using the standard form, we can show to be free in the explicit choice of the symmetric matrix B for a given irreducible polynomial.

Lemma 4.2 (Equivalent sets of MUBs in certain cases):

All symmetric matrices $B \in M_m(\mathbb{F}_2)$ with the same irreducible characteristic polynomial p generate equivalent sets of MUBs.

Proof: Consider two symmetric matrices B and B' with characteristic polynomial p . We can find some orthogonal matrix $s \in M_m(\mathbb{F}_2)$ (i.e., $s^t = s^{-1}$), such that $B' = sBs^t$ (orthogonal similarity). The matrix $f = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ applied on the standard form $(p_k(B), \mathbb{1})^t$ of Eq. (7) results in $(s p_k(B), s)^t$ which in the standard form reads $(s p_k(B) s^t, \mathbb{1})^t = (p_k(B'), \mathbb{1})^t$ showing equivalence in this case. \square

Using this lemma, we can relate even more sets, i.e., all sets which can be written in the standard form of Eq. (7) with polynomials p_k and some symmetric matrix B .

Theorem 4.3 (Equivalent sets of MUBs):

All complete sets of MUBs which are eigenbases of classes $\mathcal{C} = \{\mathcal{C}_0, \dots, \mathcal{C}_d\}$ generated by $G_0 = (\mathbb{1}_m, 0_m)^t$ and $\tilde{G}_k = (p_k(B), \mathbb{1}_m)^t$ for $k \in \{1, \dots, d\}$, some symmetric matrix $B \in M_m(\mathbb{F}_2)$ and p_k being all $d = 2^m$ polynomials over \mathbb{F}_2 with degree less than m are equivalent.

Proof: Let p and p' be two different irreducible polynomials of degree m over the ground field \mathbb{F}_2 with roots β and β' , respectively. By adjunction of β to \mathbb{F}_2 we generate the field $\mathbb{F}_2(\beta) \cong \mathbb{F}_2[x]/p\mathbb{F}_2[x] \cong \mathbb{F}_{2^m}$. The same holds true for the adjunction of β' in a similar fashion. Since both extensions are Galois extensions (cf. Chap. 4.1 of Ref. 23), the elements in $\mathbb{F}_2(\beta)$ and $\mathbb{F}_2(\beta')$ are

equivalent up to permutation. If we represent β as a symmetric matrix $B \in M_m(\mathbb{F}_2)$, such that the minimal polynomial of B equals the minimal polynomial of β , we are free in the explicit choice of B as seen in Lemma 4.2. \square

As mentioned in Chap. 4.1 of Ref. 4, the choice of a basis to produce the elements $p_k(B)$ implies a field structure and generates the MUBs of Wootters and Fields.¹ By Theorem 4.3, they are equivalent to the construction given here, and they are also equivalent to the bases which were generated by Klappenecker and Rötteler^{2,24} and by Bandyopadhyay *et al.*⁴ Thus, many of the known MUBs can be constructed with cyclic structure, which yields to a short description and allows a simple implementation in experiments by quantum circuits.²⁵

V. CONCLUSIONS

In this paper, we have shown how to relate the problem of constructing unitary generators of cyclic mutually unbiased bases to Fibonacci polynomials and their properties. In particular, we have shown that the problem for $d = 2^m$ can be reduced to finding a symmetric matrix with irreducible characteristic polynomial of degree m with non-zero linear coefficient, which is known to exist. If m itself is of the form $m = 2^k$, we give an explicit construction of such matrix, provided Wiedemann's conjecture is true. We have proven that all complete sets of cyclic mutually unbiased bases our scheme provides are equivalent and belong to the same class as the mutually unbiased bases by Wootters and Fields. The methods of this work are used in another article,²⁵ where we give a simple direct construction of the unitary generator U , which can be applied without reference to finite field theory and show how this can be translated into a quantum circuit, which may be useful in experiments.

ACKNOWLEDGMENTS

The authors acknowledge funding by CASED and by BMBF/QuOReP.

APPENDIX: CONSTRUCTION OF THE MUB-GENERATING UNITARY OPERATOR

In this Appendix, we will review and slightly reformulate the construction of the unitary matrix $U \in M_d(\mathbb{C})$ from $C \in M_{2m}(\mathbb{F}_2)$ or $B \in M_m(\mathbb{F}_2)$, respectively, up to a global phase⁸ $e^{i\psi}$.

On the Hilbert space $\mathcal{H} = \mathbb{C}^p$, $p \in \mathbb{N}$ prime, with canonical orthonormal basis $\{|i\rangle | i \in \mathbb{F}_p\}$, we define

$$X|i\rangle := |i+1\rangle \quad \text{and} \quad Z|i\rangle := \omega_p^i |i\rangle \quad (\text{A1})$$

with $\omega_p := \exp(2\pi i/p)$. For $\vec{a} := (\vec{a}_z | \vec{a}_x)^t \in \mathbb{F}_p^{2m}$ with $\vec{a}_v = (a_1^{(v)}, \dots, a_m^{(v)})^t \in \mathbb{F}_p^m$ and $v \in \{z, x\}$, we define the set of *Pauli operators* (which is not the Pauli group!) by $P_p^m := \{XZ(\vec{a}) | \vec{a} \in \mathbb{F}_p^{2m}\}$ with

$$XZ(\vec{a}) := f(\vec{a}) \cdot X^{a_1^{(x)}} Z^{a_1^{(z)}} \otimes \dots \otimes X^{a_m^{(x)}} Z^{a_m^{(z)}}, \quad (\text{A2})$$

where $f(\vec{a}) := i^{a_1^{(x)} a_1^{(z)} + \dots + a_m^{(x)} a_m^{(z)}}$, if $p = 2$, and $f(\vec{a}) = 1$ otherwise. This implies $f(\vec{a})^{-1} f(\vec{b})^{-1} XZ(\vec{a}) XZ(\vec{b}) = \omega_p^{a_1^{(z)} b_1^{(x)} + \dots + a_m^{(z)} b_m^{(x)}} f(\vec{a} + \vec{b})^{-1} XZ(\vec{a} + \vec{b})$, and it follows $XZ(\vec{a}) XZ(\vec{b}) = \omega_p^{(\vec{a}, \vec{b})_{\text{sp}}} XZ(\vec{b}) XZ(\vec{a})$, where $\langle \cdot, \cdot \rangle_{\text{sp}} : \mathbb{F}_p^{2m} \times \mathbb{F}_p^{2m} \rightarrow \mathbb{F}_p$ with $(\vec{a}, \vec{b})_{\text{sp}} := \sum_{k=1}^m a_k^{(z)} b_k^{(x)} - a_k^{(x)} b_k^{(z)}$ denotes the *symplectic product*.

Consider the vectors $\vec{z}_i, \vec{x}_i \in \mathbb{F}_p^{2m}$, $i \in \{0, \dots, m-1\}$, where the i th component of the first resp. second half of the vector is one and all other entries vanish. These vectors form a *symplectic* (or *hyperbolic*) *basis* of \mathbb{F}_p^{2m} , i.e., for all $i, j \in \{1, \dots, p-1\}$ there hold the equations $\langle \vec{x}_i, \vec{x}_j \rangle_{\text{sp}} = \langle \vec{z}_i, \vec{z}_j \rangle_{\text{sp}} = 0$ and $\langle \vec{x}_i, \vec{z}_j \rangle_{\text{sp}} = \delta_{ij}$. For a matrix $C \in M_{2m}(\mathbb{F}_p)$, the images $\vec{v}'_i := C \vec{v}_i$, $v \in \{z, x\}$, form a symplectic basis, if and only if C is a symplectic matrix, i.e., $C^t S C = S$ for $S = \begin{pmatrix} 0_m & \mathbb{1}_m \\ -\mathbb{1}_m & 0_m \end{pmatrix} \in M_{2m}(\mathbb{F}_p)$.

For any symplectic basis, we can define *logical operators* by $\bar{X}_i := XZ(C\bar{x}_i)$ and $\bar{Z}_i := XZ(C\bar{z}_i)$ for C as above. By definition, the *logical state* $|0\rangle_L$ is the joint eigenstate of all \bar{Z}_i with eigenvalue $+1$, and we get the other logical states by $|j\rangle_L = \bar{X}_1^{j_1} \otimes \cdots \otimes \bar{X}_m^{j_m} |0\rangle_L$ for $j = (j_1, \dots, j_m)^t \in \mathbb{F}_2^m$. If we choose the block matrix form $C = \begin{pmatrix} B & \mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix} \in M_{2m}(\mathbb{F}_2)$ with $B \in M_m(\mathbb{F}_2)$, the matrix C is symplectic, if and only if B is symmetric, and we simply have $\bar{Z}_i = X_i$ for all $i \in \{1, \dots, m\}$ and $|0\rangle_L = 2^{-m/2} \sum_{j \in \mathbb{F}_2^m} |j\rangle$ (unique up to a global phase).

Given a partition $P_p^m = \{\mathbb{1}\} \cup \mathcal{C}_0 \cup \cdots \cup \mathcal{C}_d$, where the sets \mathcal{C}_k are pairwise disjoint and contain $d-1$ commuting operators each, the eigenbases of the \mathcal{C}_k are mutually unbiased.⁴ Let $\mathcal{C}_0 := \{XZ(\vec{a}_z|0) | \vec{a}_z \in \mathbb{F}_2^m \setminus \{0\}\}$ be the non-identity Z -type operators and suppose $U \in M_d(\mathbb{C})$ is unitary such that $UC_k U^\dagger = C_{(k+1) \bmod (d+1)}$ for all $k \in \{0, \dots, d\}$; then, $U^{d+1} = U^0 = \mathbb{1}_d$, and the columns of U^k , $k \in \{0, \dots, d\}$, are the eigenbases of the \mathcal{C}_k , i.e., U is the generator of cyclic MUBs. (In particular, U belongs to the *Clifford group*, i.e., the group of unitary operators which leave the set of Pauli operators invariant, possibly up to a p th root of unity: $\mathcal{C}_p^m := \{U \in \mathcal{U}(p^m) | (\forall \vec{a} \in \mathbb{F}_p^{2m})(\exists \vec{b} \in \mathbb{F}_p^{2m}, k \in \mathbb{F}_p)(U XZ(\vec{a}) U^\dagger = \omega_p^k XZ(\vec{b}))\}$.) Thus, for all $j \in \mathbb{F}_2^m$ there holds $U|j\rangle = |j\rangle_L = \prod_{k=1}^m XZ(C\bar{x}_k)^{j_k} |0\rangle_L$ and

$$U = 2^{-m/2} \sum_{i, j \in \mathbb{F}_2^m} \prod_{k=1}^m XZ(C\bar{x}_k)^{j_k} |i\rangle \langle j|. \quad (\text{A3})$$

With $B = (b_{ij})_{i, j=1}^m$ as above, we have $XZ(C\bar{x}_k) = (\otimes_{i=1}^{k-1} X^{b_{ik}}) \otimes i^{b_{kk}} X^{b_{kk}} Z \otimes (\otimes_{i=k+1}^m X^{b_{ik}})$, and the k th tensor factor in (A3) is given by $X^{b_{1k}j_1 + \dots + b_{kk}j_k} \cdot Z^{j_k} \cdot X^{b_{k+1,k}j_{k+1} + \dots + b_{mk}j_m}$, where we can shift the factor Z to the beginning, which yields $(-1)^{b_{1k}j_1 j_k + \dots + b_{kk}j_k^2} \cdot Z^{j_k} \cdot X^{b_{1k}j_1 + \dots + b_{mk}j_m}$. By defining new abbreviations $p_j := i^{\sum_{k=1}^m b_{kk}j_k} (-1)^{\sum_{k=1}^m b_{1k}j_1 j_k + \dots + b_{kk}j_k^2}$ and $\tilde{X}_j := \otimes_{k=1}^m X^{b_{1k}j_1 + \dots + b_{mk}j_m}$, we find

$$U = 2^{-m/2} \sum_{i, j \in \mathbb{F}_2^m} p_j (Z^{j_1} \otimes \cdots \otimes Z^{j_m}) \tilde{X}_j |i\rangle \langle j|; \quad (\text{A4})$$

we can pull through the sum over i and see that $\sum_i |i\rangle$ is invariant under \tilde{X}_j , which thus can be absorbed. This results in $U = 2^{-m/2} \sum_{ij} p_j (-1)^{ij} |i\rangle \langle j|$ or, using the Hadamard matrix $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2} \in M_2(\mathbb{C})$, in $U = H^{\otimes m} P$, with diagonal *phase system* matrix $P = \text{diag}((p_j)_{j \in \mathbb{F}_2^m})$. Since B is symmetric, we find $(-1)^{\sum_{k=1}^m b_{1k}j_1 j_k + \dots + b_{kk}j_k^2} = i^{(j|B|j)} i^{\sum_{k=1}^m b_{kk}j_k^2}$ with $(j|B|j) = \sum_{k, l=1}^m b_{kl} j_k j_l$ being a quadratic form, which leads to $p_j := i^{(j|B|j)} (-1)^{\sum_{k=1}^m b_{kk}j_k^2}$.

¹ W. K. Wootters and B. D. Fields, "Optimal state-determination by mutually unbiased measurements," *Ann. Phys.* **191**, 363 (1989).

² A. Klappenecker and M. Rötteler, "Constructions of mutually unbiased bases," *Lect. Notes Comput. Sci.* **2948**, 262 (2004).

³ T. Paterek, B. Dakić, and Č. Brukner, "Mutually unbiased bases, orthogonal Latin squares, and hidden-variable models," *Phys. Rev. A* **79**, 012109 (2009).

⁴ S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, and F. Vatan, "A new proof for the existence of mutually unbiased bases," *Algorithmica* **45**, 512 (2002); e-print [arXiv:quant-ph/0103162v3](https://arxiv.org/abs/quant-ph/0103162v3).

⁵ H. F. Chau, "Unconditionally secure key distribution in higher dimensions by depolarization," *IEEE Trans. Inf. Theory* **51**, 1451 (2005); e-print [arXiv:quant-ph/0405016v2](https://arxiv.org/abs/quant-ph/0405016v2).

⁶ R. Gow, "Generation of mutually unbiased bases as powers of a unitary matrix in 2-power dimensions," e-print [arXiv:math/0703333v2](https://arxiv.org/abs/math/0703333v2).

⁷ D. M. Appleby, "Properties of the extended Clifford group with applications to SIC-POVMs and MUBs," e-print [arXiv:0909.5233v1](https://arxiv.org/abs/quant-ph/0909.5233v1) [quant-ph].

⁸ O. Kern, K. S. Ranade, and U. Seyfarth, "Complete sets of cyclic mutually unbiased bases in even prime-power dimensions," *J. Phys. A: Math. Theor.* **43**, 275305 (2010); e-print [arXiv:09124.661v1](https://arxiv.org/abs/09124.661v1) [quant-ph].

⁹ H. Scheid, *Zahlentheorie*, 2nd ed. (B.I.-Wissenschaftsverlag, 1994).

¹⁰ J. Goldwasser, W. Klostermeyer, and G. Trapp, "Characterizing switch-setting problems," *Linear Multilinear Algebra* **43**, 121 (1997).

¹¹ J. Goldwasser, W. Klostermeyer, and H. Ware, "Fibonacci polynomials and parity domination in grid graphs," *Graphs Combin.* **18**, 271 (2002).

¹² R. Lidl and H. Niederreiter, *Finite Fields (Encyclopedia of Mathematics and its Applications)*, 2nd ed. (Cambridge University Press, 2008).

¹³ K. Sutner, " σ -Automata and Chebyshev-polynomials," *Theor. Comput. Sci.* **230**, 49 (2000).

¹⁴ J. V. Brawley and T. C. Teitloff, "Similarity to symmetric matrices over finite fields," *Finite Fields Appl.* **4**, 261 (1998).

- ¹⁵D. Wiedemann, "An iterated quadratic extension of GF (2)," *Fibonacci Quart.* **26**, 290 (1988), <http://www.fq.math.ca/Scanned/26-4/wiedemann.pdf>.
- ¹⁶G. L. Mullen and I. Shparlinski, "Open problems and conjectures in finite fields," in *Finite Fields and Applications*, S. Cohen and H. Niederreiter (eds.), London Math. Society Lecture Note Series, 233 (Cambridge Univ. Press, 1996), pp. 243–268.
- ¹⁷J. F. Voloch, "Elements of high order on finite fields from elliptic curves," *Bull. Aust. Math. Soc.* **81**, 425 (2010).
- ¹⁸R. Varshamov and G. Garakov, "On the theory of selfdual polynomials over a Galois field (Russian)," *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **13**, 403 (1969).
- ¹⁹H. Meyn, "On the construction of irreducible self-reciprocal polynomials over finite fields," *Appl. Algebra Eng. Commun. Comput.* **1**, 43 (1990).
- ²⁰A. R. Calderbank, P. J. Cameron, W. K. Kantor, and J. J. Seidel, "Z₄-Kerdock codes, orthogonal spreads and extremal Euclidean line-sets," *Proc. London Math. Soc.* **75**, 436 (1997).
- ²¹W. K. Kantor, "MUBs inequivalence and affine planes," *J. Math. Phys.* **53**, 032204 (2012); e-print [arXiv:1104.3370v1](https://arxiv.org/abs/1104.3370v1) [math.CO].
- ²²A matrix W is called monomial, if $W = D\Pi$, where $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ is diagonal and Π a permutation; we assume $|\lambda_i| = 1$, i. e. W is unitary.
- ²³S. Bosch, *Algebra*, 6th ed. (Springer, 2005).
- ²⁴C. Godsil and A. Roy, "Equiangular lines, mutually unbiased bases, and spin models," *Eur. J. Comb.* **30**, 246 (2009); e-print [arXiv:quant-ph/0511004v3](https://arxiv.org/abs/quant-ph/0511004v3).
- ²⁵U. Seyfarth and K. S. Ranade, "Construction of mutually unbiased bases with cyclic symmetry for qubit systems," *Phys. Rev. A* **84**, 042327 (2011); e-print [arXiv:1106.3515v1](https://arxiv.org/abs/1106.3515v1) [quant-ph].