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Equiangular tight frames from Paley tournaments

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Abstract

We prove the existence of equiangular tight frames having $n = 2d - 1$ elements drawn from either $\mathbb{C}^d$ or $\mathbb{C}^{d-1}$ whenever n is either $2^k - 1$ for $k \in \mathbb{N}$, or a power of a prime such that $n \equiv 3 \bmod 4$. We also find a simple explicit expression for the prime power case by establishing a connection to a $2d$-element equiangular tight frame based on quadratic residues.

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1. Introduction

Define an (n, d) equiangular tight frame to be a set of n normalized vectors $\phi_k \in \mathbb{C}^d$ such that

$$|\langle \phi_j, \phi_k \rangle|^2 = \frac{n-d}{d(n-1)} \qquad \forall j \neq k. \tag{1}$$

This is merely the equiangular criterion, but any frame meeting this condition is automatically tight [1,2]. Many constructions of equiangular tight frames are known [2], from the ordinary regular simplex, to frames based on perfect difference sets [3], quadratic residues [4], both symmetric [5,6] and anti-symmetric [7,8] conference matrices, Hadamard matrices [9], adjacency matrices of strongly regular graphs or regular two-graphs [9,10], and (d^2, d) ETFs based on the

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Heisenberg group [4,11–13]. To this list we add a new construction using adjacency matrices of Paley tournaments. In Section 2, we demonstrate the existence of $(2d-1, d)$ and $(2d-1, d-1)$ equiangular tight frames whenever d is a power of two or $2d-1 \equiv 3 \bmod 4$ and is a power of a prime. In Section 3, we find an explicit description of the $(2d-1, d)$ ETFs for odd prime powers by connecting them to the construction of $(2d, d)$ ETFs using quadratic residues due to Zauner [4].

2. Gram matrix construction

To construct an equiangular tight frame it is sufficient to construct an appropriate Gram matrix for the set, as shown by the following lemma, adapted from Corollary 2.4 in [2].

Lemma 1. *Let G be an $n \times n$ Hermitian matrix with the following properties*:

(i) $|G_{jk}|^2 = (n-d)/d(n-1)$, *and*
(ii) *G has only one nonzero eigenvalue n/d of degeneracy d.*

Then an (n, d) ETF may be explicitly constructed.

Proof. The frame vectors are given by the columns of the $d \times n$ rank d matrix T satisfying $G = T^*T$ [14]. More concretely, the frame vectors can be found by factorizing G into $V\Lambda V^*$, where Λ is the diagonal matrix of eigenvalues and V is the unitary matrix whose columns are the eigenvectors. Arranging Λ such that the nonzero eigenvalues of G are located in its first d entries, the ϕ_k are given by $\phi_k = \sqrt{n/d}\{V_{kl}\}_{l=1}^{d}$. □

We shall use adjacency matrices of Paley tournaments in the construction of the Gram matrix, a variation of the conference matrix construction [2,5–8]. Recall that the $n \times n$ adjacency matrix A of the Paley tournament is defined as $A_{jk} = \chi(j-k)$ for any prime power $n \equiv 3 \bmod 4$. Here $\chi(a)$ is the quadratic character of $\mathbb{F}_n$, equal to zero for $a = 0$ and ± 1 when a is or is not a nonzero quadratic residue. It is easily seen by direct calculation that $A^2 = J - nI$, where J is the matrix of all 1s, using the fact that $\sum_{a \in \mathbb{F}_n} \chi(a)\chi(a-b) = -1$ for all $b \neq 0$ [15].

Beyond this case, we also have the following construction for $n = 2^k - 1$ (cf. Corollary 2.6b in [2]). Consider the recursive construction of an $(n+1) \times (n+1)$ conference matrix C satisfying $CC^{\mathrm{T}} = nI$

$$C_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_{2m} = \begin{pmatrix} C_m & C_m - I_m \\ C_m + I_m & -C_m \end{pmatrix}. \tag{2}$$

Let e be the column matrix of n 1s. By induction, C_{2^k} has zero diagonal and the following block form:

$$C_{2^k} = \begin{pmatrix} 0 & -e^{\mathrm{T}} \\ e & A \end{pmatrix}. \tag{3}$$

By the conference matrix condition, $A^2 = J - nI$ and also has zero diagonal and entries ± 1. Now we can prove our first main result.

Theorem 1. *Let $n = 2d - 1$ for either $n = 2^k - 1$, $k \in \mathbb{N}$ or $n = p^k$ for prime p such that $p^k \equiv 3 \bmod 4$. Then there exist (n, d) and $(n, d-1)$ equiangular tight frames which can be explicitly constructed.*

Proof. First we construct the (n, d) ETF. The Gram matrix G may be written

$$G_{(n,d)} = \frac{1}{2d}(J + nI + i\sqrt{n}A). \tag{4}$$

Note that because G has complex entries, the resulting frame vectors will generally be elements of $\mathbb{C}^d$. Condition (i) of Lemma 1 is clear by inspection. To find the eigenvalues of G, note that $Je = ne$, while $Ae = 0$ since half the nonzero elements of a finite field are quadratic residues. As J has no other nonzero eigenvalues, A and $J + nI$ commute. The eigenvalues are then easily found to satisfy condition (ii) using the fact that $A^2 = J - nI$.

For the $(n, d - 1)$ equiangular tight frame, take the Gram matrix to be

$$G_{(n,d-1)} = \frac{1}{2(d-1)}(nI - J + i\sqrt{n}A). \tag{5}$$

Existence follows from a similar argument to that above. □

3. Connections to $(2d, d)$ ETFs

Several examples from the previous section were found numerically by starting with $(2d, d)$ ETFs, removing one element, and forming the canonical tight frame from the resulting set. Recall that for a generic frame consisting of elements f_k, the canonical tight frame formed from $\{f_k\}$ is given by $\{\phi_k = S^{-1/2} f_k\}$, where S is the *frame operator* defined by $Sf_k = \sum_j \langle f_j, f_k\rangle f_j$ [16]. Theorem 2 formalizes this relationship between the $(2d, d)$ and $(2d - 1, d)$ equiangular tight frames for $2d - 1 \equiv 3 \bmod 4$.

First we shall need the following lemma on character sums of quadratic residues, adapted from Chapter 5 of [17]. Let $q = p^m$ for prime p and ψ_1 be the canonical additive character of $\mathbb{F}_q$, namely $\psi_1(a) = \mathrm{e}^{2\pi \mathrm{i}\,\mathrm{Tr}(a)/p}$, where Tr is the absolute trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. We then have

Lemma 2. *Define* $\sigma(a) = \sum_{b\in\mathbb{F}_q} \psi_1(ab^2)$ *and let* $r = 0(1)$ *for* $q \equiv 1(3) \bmod 4$. *Then* $\sigma(a) = i^r \chi(a)(-1)^m \sqrt{q}$ *for all* $a \neq 0$.

Proof. Consider the Gaussian sum associated to any additive character ψ and multiplicative character ξ of $\mathbb{F}_q$, defined by $\Sigma(\psi, \xi) = \sum_{a\in\mathbb{F}_q} \psi(a)\xi(a)$. Every additive character is given by some ψ_c, where $\psi_c(a) = \psi_1(ca)$, so the Gaussian sum obeys $\Sigma(\psi_c, \xi) = \overline{\xi(c)}\Sigma(\psi_1, \xi)$. Using the quadratic character χ we find $\sigma(a) = \Sigma(\psi_a, \chi) = \chi(a)\Sigma(\psi_1, \chi)$, since $\sum_{b\in\mathbb{F}_q} \psi_a(b) = 0$. The lemma then follows from the fact that $\Sigma(\psi_1, \chi) = i^r(-1)^m\sqrt{q}$. □

Now we make a slight modification of Zauner's construction of $(2d, d)$ ETFs. For $q = 2d - 1$ an power of an odd prime, call the $(q - 1)/2$ nonzero quadratic residues $b_1, b_2, \ldots, b_{(q-1)/2}$ and the elements of the field itself $a_1, a_2, \ldots, a_q$. Then the columns of the following $\frac{q+1}{2} \times (q + 1)$ matrix T form an ETF

$$T = \frac{1}{\sqrt{q}} \begin{pmatrix} i^r\sqrt{q} & 1 & 1 & \cdots & 1 \\ 0 & \sqrt{2}\psi(b_1a_1) & \sqrt{2}\psi(b_1a_2) & \cdots & \sqrt{2}\psi(b_1a_q) \\ 0 & \sqrt{2}\psi(b_2a_1) & \sqrt{2}\psi(b_2a_2) & \cdots & \sqrt{2}\psi(b_2a_q) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sqrt{2}\psi\left(b_{\frac{q-1}{2}}a_1\right) & \sqrt{2}\psi\left(b_{\frac{q-1}{2}}a_2\right) & \cdots & \sqrt{2}\psi\left(b_{\frac{q-1}{2}}a_q\right) \end{pmatrix}. \tag{6}$$

The factor i^r in the first column differs from the original construction. Each column vector ϕ_k is manifestly normalized, and the first clearly has inner product $i^r/\sqrt{q} = i^r/\sqrt{2d-1}$ with any of the others. Applying Lemma 2 establishes the remaining inner products.

These frames are shown to exist in [2] by using symmetric and anti-symmetric conference matrices to construct the appropriate Gram matrix.[1] By including the extra factor of i in the case $q \equiv 3 \bmod 4$ this construction leads to identical Gram matrices, again using $\sum_{a\in\mathbb{F}_n} \chi(a)\chi(a-b) = -1$ for all $b \neq 0$. However, the ETFs themselves are generally different. In particular, in the case of symmetric conference matrices the frame may be found in $\mathbb{R}^d$.

Theorem 2. *For all $q = p^m = 2d-1 = 3 \bmod 4$ a power of a prime p, there exists a $(2d-1, d)$ ETF with Gram matrix $G = (J + qI + i\sqrt{q}A)/2d$ which is given by the canonical tight frame associated with the set of vectors formed by removing the first element from Zauner's construction.*

By removing the first column from the matrix T and forming the frame operator $S = TT^*$, we find $S_{00} = 1$ and

$$S_{j0} = S_{0k}^* = \frac{\sqrt{2}}{q}\sum_m \psi(b_l a_m) = 0, \quad S_{jk} = \frac{2}{q}\sum_m \psi((b_j - b_k)a_m) = 2\delta_{jk},$$

so $S = \text{diag}\{1, 2, \ldots, 2\}$. The columns of the new matrix $\widetilde{T}$ are the normalized vectors $\sqrt{q/d}\,S^{-1/2}\phi_k$, $k = 1, \ldots, q$. The Gram matrix is easily found using Lemma 2.

We expect that a similar relationship holds between the $(2d, d)$ and $(2d-1, d)$ ETFs constructed via the conference matrices above, though at present we have only numerical evidence to support this claim.

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[1] Note that [2] contains a slight error in claiming the Paley construction produces *symmetric* conference matrices for all $n = p^k + 1$ (p an odd prime). The case $p^k \equiv 3 \bmod 4$ produces *anti-symmetric* conference matrices.

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