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The Jamiołkowski Isomorphism and a Simplified Proof for the Correspondence Between Vectors Having Schmidt Number k and k-Positive Maps

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Abstract. Positive maps which are not completely positive are used in quantum information theory as witnesses for convex sets of states, in particular as entanglement witnesses and, more generally, as witnesses for states having Schmidt number not greater than k. Such maps and witnesses are related to k-positive maps, and their properties may be investigated by making use of the Jamiołkowski isomorphism. In this article we review the properties of this isomorphism, noting that there are actually two related mappings bearing that name. As a new result, we give a simplified proof for the correspondence between vectors having Schmidt number k and k-positive maps and thus for the Jamiołkowski criterion for complete positivity. Another consequence is a special case of a result by Choi, namely that k-positivity implies complete positivity, if k is the dimension of the smaller one of the Hilbert spaces on which the operators act.

1. Introduction

In quantum mechanics, the state of a physical system is described by a density operator ρ, which is a positive semi-definite operator (i. e. all eigenvalues of ρ are non-negative: $\rho \geq 0$) acting on some Hilbert space associated with the physical system. When a physical system undergoes a dynamical process, at the end of that process the new state of the same physical system should also be represented by a valid density operator ρ'; this means that the dynamical process is represented by a positive map T, such that $\rho' = T(\rho)$. It was observed that this condition alone cannot sufficiently describe physical processes [10]: For any state ρ_{AB} of a joint physical system AB, where the subsystems A and B are spatially separated and each subsystem is transformed according to positive maps T_A and T_B, respectively, the global state must still be represented by a valid density operator. This leads to the condition that the tensor product of these maps, $T = T_A \otimes T_B$, must be a positive map. Thus, the maps T_A and T_B have to be completely positive in the following sense: *A linear map T is completely positive, if and only if $\mathbb{1}_k \otimes T$ is positive for all $k \in \mathbb{N}$.* Here, $\mathbb{1}_k$ denotes the identity map acting on the set of $k \times k$-matrices.

The structure of completely positive maps and their applications in quantum theory were extensively studied in the 1960s and 70s. However, positive maps which are not completely positive are also known to have applications in quantum information theory. In particular, they can be used as so-called *entanglement witnesses*. The simplest example of such a map is the transposition T_{tr}: if $\mathbb{1}_A \otimes T_{\mathrm{tr},B}(\rho_{AB})$ turns out to be non-positive, ρ_{AB} is entangled [12]. This observation has started considerable efforts to understand the positive maps which are not completely positive [1]. Later it was observed that there is a direct relation between positive maps and entanglement witnesses through the so-called Jamiołkowski isomorphism; the Peres criterion [12] and the reduction criterion for separability [2, 7] are manifestations of this relation [6].

Terhal and Horodecki extended the notion of the Schmidt number for pure states to mixed states [17], where it turned out to be a legitimate measure of entanglement. If we denote by S_k the convex hull of all pure states having Schmidt number not greater than k, then the set of separable states is S_1, whereas all other states are entangled. The states in these sets can be characterised with the aid of *Schmidt witnesses* [16]: A hermitian operator W is called Schmidt witness of class k, if and only if $\mathrm{Tr}(W\sigma) \geq 0$ for all $\sigma \in S_{k-1}$ and there exists at least one $\rho \in S_k$ such that $\mathrm{Tr}(W\rho) < 0$. Clarisse [5] noted, that the theorem underlying Terhal and Horodecki's considerations can be stated as follows: *A map is k-positive, if and only if the corresponding operator [under the so-called Jamiołkowski isomorphism] is positive on states with Schmidt number k or less.* In this article, we will present a simple, direct and explicit proof of that fact; although the basic idea of the proof is quite simple, it does not seem to have been published in the literature. Working to this aim, we investigate the notion of the term "Jamiołkowski isomorphism" and mention the fact that there are actually two related, but different isomorphisms sharing the same name. We review the relevant results by de Pillis [13] and Jamiołkowski [8] and adapt them to our purposes, thereby providing full and detailed proofs.

This work is organised as follows: In Sect. 2 we introduce the necessary notation and shortly discuss the notion of k-positivity and the Schmidt decomposition. Sections 3 and 4 contain a self-contained in-detail review of the two Jamiołkowski isomorphisms. In Sect. 5 we prove the relation between k-positive maps and vectors having Schmidt number k; we conclude the section with a further discussion of the two Jamiołkowski isomorphisms and note some consequences of the main theorem. Finally, we summarise our results in Sect. 6.

2. Preliminaries

We denote by $\mathbb{C}$ the field of complex numbers, the complex conjugate of $z \in \mathbb{C}$ by $\overline{z}$ and the transpose and adjoint of an operator A by A^t and A^*, respectively. Scalar products are denoted by $\langle \cdot\,, \cdot \rangle$ and are taken to be linear in the left and antilinear in the right argument; occasionally we indicate the space on which they are taken by a subscript.

We consider two finite-dimensional Hilbert spaces $\mathcal{H}_A := \mathbb{C}^n$ and $\mathcal{H}_B := \mathbb{C}^m$, to which there are assigned their respective algebras $\mathfrak{A} := \mathbb{C}^{n\times n}$ and $\mathfrak{B} := \mathbb{C}^{m\times m}$ of matrices acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, which themselves form Hilbert spaces using the Hilbert-Schmidt scalar product, i. e. $\langle A_1, A_2 \rangle = \mathrm{Tr}\, A_2^* A_1$ for A_1, $A_2 \in \mathfrak{A}$ and similarly for $\mathfrak{B}$.

The space of linear maps from $\mathfrak{A}$ to $\mathfrak{B}$ is denoted by $L(\mathfrak{A}, \mathfrak{B})$. A map $T \in$

$L(\mathfrak{A},\mathfrak{B})$ is called *hermiticity-preserving*, if it maps hermitian $A \in \mathfrak{A}$ to hermitian $T(A) \in \mathfrak{B}$; it is called *positive*, if it maps positive A to positive $T(A)$. For any $k \in \mathbb{N}$ a map $T \in L(\mathfrak{A},\mathfrak{B})$ gives rise to a map $T_k := \mathbb{1}_k \otimes T \in L(M_k(\mathfrak{A}), M_k(\mathfrak{B}))$, where $M_k(\mathfrak{A})$ and $M_k(\mathfrak{B})$ are the sets of matrices having elements of $\mathfrak{A}$ and $\mathfrak{B}$ as their entries; for a matrix $A = (a_{ij})_{i,j=1}^k$, where $a_{ij} \in \mathfrak{A}$, it is defined by

$$T_k \left[\begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} \right] := \begin{pmatrix} T(a_{11}) & \dots & T(a_{1k}) \\ \vdots & \ddots & \vdots \\ T(a_{k1}) & \dots & T(a_{kk}) \end{pmatrix}. \tag{1}$$

The map T is called *k-positive*, if T_k is positive; it is called *completely positive*, if it is k-positive for all $k \in \mathbb{N}$. A detailed mathematical discussion of these properties may be found in an article by Choi [3].

Physically, T_k may be understood as coupling an auxiliary system (an “ancilla”) of dimension k to a quantum system without performing any action on that ancilla. Thus, if the use of such ancilla is allowed and if arbitrary quantum states of the principal system can be prepared, an operation consistent with quantum mechanics must be k-positive, and in the general case of arbitrary dimension of the ancilla, it must be completely positive.

We will prove a theorem relating the k-positivity of a map $T \in L(\mathfrak{A},\mathfrak{B})$ to vectors on $\mathcal{H}_A \otimes \mathcal{H}_B$ having Schmidt number k or less. For this purpose, we will now state a well-known result known as the *Schmidt decomposition* of a vector in $\mathcal{H}_A \otimes \mathcal{H}_B$; a proof thereof may be found in the book by Nielsen and Chuang [11].

THEOREM 1 (Schmidt decomposition) *For each non-zero vector $v \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exists a number $s \in \{1, \dots, \min\{n, m\}\}$, positive numbers $\lambda_1, \dots, \lambda_s$ and orthonormal systems $(e_i^A)_{i=1}^s$ and $(e_i^B)_{i=1}^s$ in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that $v = \sum_{i=1}^s \lambda_i e_i^A \otimes e_i^B$ holds.*

The number s is the rank of either reduced density matrix, thus well-defined; it is called the *Schmidt number* of the vector v. For later use in our proof, we note the following:

LEMMA 1 (Schmidt number of certain vectors) *Let $(v_i^A)_{i=1}^s$ and $(v_i^B)_{i=1}^s$ be systems of not necessarily orthogonal vectors in Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Then, the Schmidt number of the vector $v := \sum_{i=1}^s v_i^A \otimes v_i^B$ is not greater than s.*

Proof. The Hilbert spaces spanned by the vector systems $(v_i^A)_{i=1}^s$ and $(v_i^B)_{i=1}^s$ have dimension not greater than s. Thus, by Theorem 1 the Schmidt number of v cannot be greater than s. □

3. The Jamiołkowski Isomorphism and its Properties

De Pillis [13] considered a mapping $\mathcal{J}_1 : L(\mathfrak{A},\mathfrak{B}) \to \mathfrak{A} \otimes \mathfrak{B}$ which has the defining property that $\langle \mathcal{J}_1(T), A^* \otimes B \rangle_{\mathfrak{A}\otimes\mathfrak{B}} = \langle T(A), B \rangle_{\mathfrak{B}}$ should hold for all $T \in L(\mathfrak{A},\mathfrak{B})$, $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. He proved the following properties of such a map:

LEMMA 2 (Basic properties of $\mathcal{J}_1$) *The mapping $\mathcal{J}_1$ is uniquely defined, and for any orthonormal basis $(E_i)_{i\in I}$ of $\mathfrak{A}$ and every operator $T \in \mathfrak{A}$ the equation $\mathcal{J}_1(T) = \sum_{i\in I} E_i^* \otimes T(E_i)$ holds. Furthermore, $\mathcal{J}_1$ is an isometric isomorphism of the Hilbert spaces $L(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{A} \otimes \mathfrak{B}$.*

The following Lemma characterises hermiticity-preserving maps; it was used by de Pillis [13].

LEMMA 3 (Hermiticity-preserving maps) *A linear map $T \in L(\mathfrak{A}, \mathfrak{B})$ preserves hermiticity, if and only if $T(A^*) = T(A)^*$ holds for all $A \in \mathfrak{A}$. If $(E_i)_{i\in I}$ is a basis of $\mathfrak{A}$, this is the case, if and only if $T(E_i^*) = T(E_i)^*$ for all $i \in I$.*

Proof. If T preserves hermiticity, we have $T(A)^* = T(A) = T(A^*)$ for hermitian A; in general, an arbitrary $A \in \mathfrak{A}$ may be decomposed into $A = A_1 + \mathrm{i}A_2$, where A_1, $A_2 \in \mathfrak{A}$ are hermitian, so that

$$\begin{aligned} T(A^*) &= T(A_1^* - \mathrm{i}A_2^*) \;=\; T(A_1^*) - \mathrm{i}T(A_2^*) \\ &= T(A_1)^* - \mathrm{i}T(A_2)^* \;=\; [T(A_1) + \mathrm{i}T(A_2)]^* \;=\; T(A)^* \end{aligned} \tag{2}$$

holds. On the converse, we calculate $T(A) = T(A^*) = T(A)^*$ for hermitian $A \in \mathfrak{A}$, which shows the first statement. For the second statement, we use the decomposition $A = \sum_i a_i E_i$ and calculate

$$T(A^*) \;=\; \sum\nolimits_{i\in I} \overline{a_i}\, T(E_i^*) \;=\; \sum\nolimits_{i\in I} \overline{a_i}\, T(E_i)^* \;=\; T(A)^*, \tag{3}$$

where the second equality holds by assumption. The inverse statement is obvious, and this concludes the proof. □

THEOREM 2 (Maps which preserve hermiticity and/or positivity I) *A linear map $T \in L(\mathfrak{A}, \mathfrak{B})$ is*

1) hermiticity-preserving, if and only if $\mathcal{J}_1(T)$ is hermitian,

2) positive, if and only if $\langle \mathcal{J}_1(T)x \otimes y, x \otimes y\rangle \geq 0$ holds for all $(x, y) \in \mathcal{H}_A \otimes \mathcal{H}_B$.

The first part was proved by de Pillis [13], the second by Jamiołkowski [8]; the latter criterion can be interpreted as follows: *A map T is positive, if and only if $\mathcal{J}_1(T)$ is positive on separable vectors.* We will use a modified version of Jamiołkowski's criterion, which is appropriate for our proof.

4. The Modified Jamiołkowski Isomorphism

In this section, we focus on a particular basis of $\mathfrak{A}$, namely the basis $(E_{ij})_{i,j=1}^n$ which consists of matrices E_{ij} which have entry one in the j-th column of the i-th row, whereas all other entries are zero; this basis is sometimes called *Weyl basis*. It has the property that $E_{ij} = \overline{E_{ij}} = E_{ji}^* = E_{ji}^t$ holds for all i, $j \in \{1, \ldots, n\}$.

We now consider a variant of the Jamiołkowski isomorphism, which we will call $\mathcal{J}_2$ and which is defined by (cf. e.g. [4, 9, 14])

$$\mathcal{J}_2(T) \;:=\; \sum_{i,j=1}^{n} E_{ij} \otimes T(E_{ij})\,. \tag{4}$$

The difference between $\mathcal{J}_1$ and $\mathcal{J}_2$ is that in the first tensor factor there is no adjoint. Note that for T_1, $T_2 \in L(\mathfrak{A},\mathfrak{B})$, due to $\langle E_{ij}, E_{kl}\rangle = \delta_{ik}\delta_{jl}$ we have

$$\begin{aligned}\langle \mathcal{J}_1(T_1), \mathcal{J}_1(T_2)\rangle &= \Big\langle \sum_{ij} E_{ij}^* \otimes T_1(E_{ij}), \sum_{kl} E_{kl}^* \otimes T_2(E_{kl})\Big\rangle \\ &= \sum_{ijkl} \big\langle E_{ij}^*, E_{kl}^*\big\rangle \langle T_1(E_{ij}), T_2(E_{kl})\rangle \\ &= \sum_{ij} \langle T_1(E_{ij}), T_2(E_{ij})\rangle \ ; \end{aligned} \tag{5}$$

a similar calculation for $\mathcal{J}_2$ shows $\langle \mathcal{J}_1(T_1), \mathcal{J}_1(T_2)\rangle = \langle \mathcal{J}_2(T_1), \mathcal{J}_2(T_2)\rangle$, i.e. $\mathcal{J}_2$ indeed is an isomorphism, and we state the adaption of Theorem 2 to this modified isomorphism.

LEMMA 4 (Maps which preserve hermiticity and/or positivity II)
A linear map $T \in L(\mathfrak{A},\mathfrak{B})$ is
1) hermiticity-preserving, if and only if $\mathcal{J}_2(T)$ is hermitian,
2) positive, if and only if $\langle \mathcal{J}_2(T)x \otimes y, x \otimes y\rangle \geq 0$ holds for all $(x, y) \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Proof. For showing the first statement, one can calculate

$$\begin{aligned}\mathcal{J}_2(T)^* &= \Big(\sum_{ij} E_{ij} \otimes T(E_{ij})\Big)^* = \sum_{ij} E_{ij}^* \otimes T(E_{ij})^* \\ &\stackrel{?}{=} \sum_{ij} E_{ji} \otimes T(E_{ij}^*) = \sum_{ij} E_{ji} \otimes T(E_{ji}) = \mathcal{J}_2(T)\,, \end{aligned} \tag{6}$$

where the equality in question holds, if and only if T preserves hermiticity (according to Lemma 3). The proof of the second statement is nearly the same as Jamiołkowski's original proof [8]: As any positive operator T may be decomposed into a real linear combination of projection operators, we have to show that $T(P_x)$ is positive for any one-dimensional projection P_x projecting on the vector space spanned by some unit vector $x \in \mathcal{H}_A$. Using an orthonormal basis $(f_p)_{p=1}^n$ of $\mathcal{H}_A$, we therefore calculate

$$\begin{aligned}T(P_x) &= \sum_{ij} \langle P_x, E_{ij}\rangle_{\mathfrak{A}} T(E_{ij}) = \sum_{ij} \mathrm{Tr}(E_{ij}^* P_x) T(E_{ij}) \\ &= \sum_{ijp} \langle E_{ji} P_x f_p, f_p\rangle_{\mathcal{H}_A} T(E_{ij}) \\ &= \sum_{ijp} \langle E_{ji} x, f_p\rangle_{\mathcal{H}_A} \langle f_p, x\rangle_{\mathcal{H}_A} T(E_{ij}) = \sum_{ij} \langle E_{ji} x, x\rangle_{\mathcal{H}_A} T(E_{ij})\,. \end{aligned} \tag{7}$$

Thus, $T(P_x)$ is positive, if and only if $\sum_{ij} \langle E_{ji}x, x\rangle_{\mathcal{H}_A} \langle T(E_{ij})y, y\rangle_{\mathcal{H}_B} \geq 0$ for all $x \in \mathcal{H}_A$ and all $y \in \mathcal{H}_B$. If we use $x = (x_1, \ldots, x_n) \in \mathcal{H}_A$ and its complex conjugate $\overline{x} = (\overline{x_1}, \ldots, \overline{x_n}) \in \mathcal{H}_A$, we get $\langle E_{ji}x, x\rangle = x_i \cdot \overline{x_j} = \langle E_{ij}\overline{x}, \overline{x}\rangle$, that is

$$\begin{aligned}\sum_{ij} \langle E_{ji}x, x\rangle_{\mathcal{H}_A} \langle T(E_{ij})y, y\rangle_{\mathcal{H}_B} &= \sum\nolimits_{ij} \langle E_{ij}\overline{x}, \overline{x}\rangle_{\mathcal{H}_A} \langle T(E_{ij})y, y\rangle_{\mathcal{H}_B} \\ &= \langle \mathcal{J}_2(T)\overline{x} \otimes y, \overline{x} \otimes y\rangle \ , \end{aligned} \tag{8}$$

which has to hold for all $x \in \mathcal{H}_A$ and all $y \in \mathcal{H}_B$. Since $x \in \mathcal{H}_A$ implies $\overline{x} \in \mathcal{H}_A$ (for $\mathcal{H}_A = \mathbb{C}^n$) and vice versa, the Lemma is proved. □

A notable difference between the isomorphisms $\mathcal{J}_1$ and $\mathcal{J}_2$ is that $\mathcal{J}_1$ does not depend upon a particular basis of $\mathfrak{A}$, whereas $\mathcal{J}_2$ does: consider another basis $(F_{ij})_{i,j=1}^n$ of $\mathfrak{A}$, which may be expressed as $F_{ij} = \sum_{kl} \langle F_{ij}, E_{kl}\rangle_{\mathfrak{A}} E_{kl}$ for $i, j \in \{1, \dots, n\}$. Using this basis to define $\mathcal{J}_2'$ we find

$$\mathcal{J}_2'(T) \;=\; \sum_{klpq} \Big[\sum_{ij} \langle F_{ij}, E_{kl}\rangle \, \langle F_{ij}, E_{pq}\rangle \Big] E_{kl} \otimes T(E_{pq})\,, \tag{9}$$

and this is equal to $\mathcal{J}_2(T)$ for all T, if and only if the inner bracket is $\delta_{kp}\delta_{lq}$. As an example, consider the canonical basis $(e_i)_{i=1}^n$ of $\mathcal{H}_A$ and a unitary operator U on $\mathcal{H}_A$; thus, the vectors $f_i := Ue_i$ also form an orthonormal basis of $\mathcal{H}_A$. Defining $F_{ij} := \langle \,\cdot\,, f_j\rangle \, f_i$, it may be shown that $\mathcal{J}_2 = \mathcal{J}_2'$, if and only if U is orthogonal in the sense, that $U^t U = \mathbb{1}$. Since there are unitary, but non-orthogonal transformations, $\mathcal{J}_2$ is basis-dependent.

5. The Main Theorem and its Consequences

We will now state the main theorem and prove it using the material from the previous section.

THEOREM 3 (Main theorem) *A linear mapping $T \in L(\mathfrak{A}, \mathfrak{B})$ is k-positive, if and only if the inequality $\langle \mathcal{J}_2(T)v, v\rangle \geq 0$ holds for all vectors $v \in \mathcal{H}_A \otimes \mathcal{H}_B$ having Schmidt number less or equal than k.*

The basic idea of the proof is quite simple: by definition, T is k-positive, if and only if the operator $T_k := \mathbb{1}_k \otimes T \in L(M_k(\mathfrak{A}), M_k(\mathfrak{B}))$ is positive. The latter property can be checked by using the modified Jamiołkowski criterion of Lemma 4. This will be formalised in the following proof.

Proof. Let $\mathcal{H}_{A;k} := \mathbb{C}^k \otimes \mathcal{H}_A \cong \oplus_{\alpha=1}^k \mathcal{H}_A$ and $\mathcal{H}_{B;k} := \mathbb{C}^k \otimes \mathcal{H}_B \cong \oplus_{\beta=1}^k \mathcal{H}_B$ be two Hilbert spaces with associated matrix algebras $\mathfrak{A}_k := M_k(\mathfrak{A})$ and $\mathfrak{B}_k := M_k(\mathfrak{B})$. The operator T_k is positive, if for all $x \in \mathcal{H}_{A;k}$ and $y \in \mathcal{H}_{B;k}$ the inequality $\langle \mathcal{J}_{2;k}(T_k)x \otimes y, x \otimes y\rangle \geq 0$ holds, where $\mathcal{J}_{2;k} : L(\mathfrak{A}_k, \mathfrak{B}_k) \to \mathfrak{A}_k \otimes \mathfrak{B}_k$ denotes the modified Jamiołkowski isomorphisms on the respective spaces.

Considering the orthonormal basis $(E_{ij})_{i,j=1}^n$ of $\mathfrak{A}$ and $(e_{\alpha\beta})_{\alpha,\beta=1}^k$ of $M_k(\mathbb{C})$, we have the orthonormal basis $(e_{\alpha\beta} \otimes E_{ij})_{i,j=1}^{n}{}_{\alpha,\beta=1}^{k}$ of $M_k(\mathfrak{A})$; we thus calculate

$$\begin{aligned} \mathcal{J}_{2;k}(\mathbb{1}_k \otimes T) &= \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{k} e_{\alpha\beta} \otimes E_{ij} \otimes [\mathbb{1}_k \otimes T](e_{\alpha\beta} \otimes E_{ij}) \\ &= \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{k} e_{\alpha\beta} \otimes E_{ij} \otimes e_{\alpha\beta} \otimes T(E_{ij})\,. \end{aligned} \tag{10}$$

If $(f_p)_{p=1}^k$ denotes the canonical basis of $\mathbb{C}^k$, any vector $x \in \mathcal{H}_{A;k}$ may be written as $x = \sum_{p=1}^k f_p \otimes x_p$ with elements $x_p \in \mathcal{H}_A$; similarly, we write $y = \sum_{q=1}^k f_q \otimes y_q \in$

$\mathcal{H}_{B;k}$ with elements $y_q \in \mathcal{H}_B$. By Lemma 4 we have to find conditions, such that

$$\langle \mathcal{J}_{2;k}(\mathbb{1}_k \otimes T)x \otimes y, x \otimes y\rangle \;\geq\; 0 \tag{11}$$

holds for all $x \in \mathcal{H}_{A;k}$ and all $y \in \mathcal{H}_{B;k}$. We write $x \otimes y = \sum_{pq} f_p \otimes x_p \otimes f_q \otimes y_q = \sum_{rs} f_r \otimes x_r \otimes f_s \otimes y_s$ and rewrite the left hand side of (11) as

$$\sum_{i,j,\alpha,\beta,p,q,r,s} \Big\langle e_{\alpha\beta} f_p \otimes E_{ij} x_p \otimes e_{\alpha\beta} f_q \otimes T(E_{ij}) y_q, f_r \otimes x_r \otimes f_s \otimes y_s \Big\rangle_{\mathbb{C}^k \otimes \mathcal{H}_A \otimes \mathbb{C}^k \otimes \mathcal{H}_B}$$

$$= \sum_{i,j,\alpha,\beta,p,q,r,s} \langle e_{\alpha\beta} f_p, f_r\rangle_{\mathbb{C}^k} \cdot \langle E_{ij} x_p, x_r\rangle_{\mathcal{H}_A} \cdot \langle e_{\alpha\beta} f_q, f_s\rangle_{\mathbb{C}^k} \cdot \langle T(E_{ij}) y_q, y_s\rangle_{\mathcal{H}_B}$$

$$= \sum_{p,q,r,s} \Big[\sum_{\alpha,\beta} \langle e_{\alpha\beta} f_p, f_r\rangle_{\mathbb{C}^k} \, \langle e_{\alpha\beta} f_q, f_s\rangle_{\mathbb{C}^k} \Big] \langle \mathcal{J}_2(T) x_p \otimes y_q, x_r \otimes y_s\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B} \,.$$

For the bracket due to $\langle e_{\alpha\beta} f_p, f_r\rangle_{\mathbb{C}^k} = \delta_{\alpha r}\delta_{\beta p}$ etc. we calculate

$$\sum_{\alpha,\beta} \langle e_{\alpha\beta} f_p, f_r\rangle_{\mathbb{C}^k} \, \langle e_{\alpha\beta} f_q, f_s\rangle_{\mathbb{C}^k} \;=\; \Big(\sum\nolimits_\beta \delta_{\beta p}\delta_{\beta q}\Big)\Big(\sum\nolimits_\alpha \delta_{\alpha r}\delta_{\alpha s}\Big) \;=\; \delta_{pq}\delta_{rs}\,, \tag{12}$$

which yields

$$\sum_{p,r=1}^{k} \langle \mathcal{J}_2(T) x_p \otimes y_p, x_r \otimes y_r\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B} \;=\; \langle \mathcal{J}_2(T) v, v\rangle_{\mathcal{H}_A \otimes \mathcal{H}_B}\,, \tag{13}$$

where $v = \sum_{p=1}^{k} x_p \otimes y_p$. By Lemma 1, this vector v has a Schmidt number not greater than k, and this completes the proof. □

Note that in this proof we had to use $\mathcal{J}_2$; if we used $\mathcal{J}_1$ instead, the first term in the bracket of (12) would read $\langle e^*_{\alpha\beta} f_p, f_r\rangle_{\mathbb{C}^k} = \delta_{\alpha p}\delta_{\beta q}$, which would lead to a different result. As an example, that the result does not hold for $\mathcal{J}_1$, consider the case $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and $\mathfrak{A} = \mathfrak{B} = \mathbb{C}^{2\times 2}$, where $T : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ is the identity map, which obviously is completely positive. However, $v := e_1 \otimes e_2 - e_2 \otimes e_1 \in \mathbb{C}^2$ is an eigenvector of $\mathcal{J}_1(T)$ having eigenvalue -1, so that $\langle \mathcal{J}_1(T)v, v\rangle$ is negative. It has Schmidt number 2, so the analogue of Theorem 3 with $\mathcal{J}_1$ instead of $\mathcal{J}_2$ is wrong.

We will conclude this section with two corollaries of the main theorem: the first corollary is a special case of a more general result by Choi [3], the latter is known as the Jamiołkowski criterion for complete positivity (cf. e.g. [4, 9, 14]).

COROLLARY 2 (Complete positivity I) *If a linear map* $T \in L(\mathfrak{A}, \mathfrak{B})$ *is* $\min\{n, m\}$*-positive, it is also completely positive.*

Proof. By Theorem 3, T is k-positive, if and only if it is positive on all vectors having Schmidt number not greater than k. Since any vector on $\mathcal{H}_A \otimes \mathcal{H}_B$ has Schmidt number not greater than $\min\{n, m\}$, any $\min\{n, m\}$-positive map is completely positive. □

COROLLARY 3 (Complete positivity II) *A linear mapping* $T \in L(\mathfrak{A}, \mathfrak{B})$ *is completely positive, if and only if* $\mathcal{J}_2(T)$ *is positive semidefinite.*

Proof. If T is completely positive, by Theorem 3, it must be positive on vectors having arbitrary Schmidt number. Since any vector has a well-defined Schmidt number, T must be positive semidefinite. The converse statement is obvious. □

6. Summary

We reviewed the properties of two maps, $\mathcal{J}_1$ and $\mathcal{J}_2$, both of them being called "Jamiołkowski isomorphism". We used Jamiołkowski's criterion for positivity to give a concise and simple proof of a relation between vectors having Schmidt number k and k-positive maps, namely that a map T is k-positive, if and only if $\mathcal{J}_2(T)$ is positive on vectors having Schmidt number not greater than k. Using this theorem we rederived the Jamiołkowski criterion for complete positivity and that $\min\{n, m\}$-positivity implies complete positivity.

The generalisation of the methods presented here to infinite-dimensional Hilbert spaces seems to be straightforward, and most of the results should stay valid in appropriate contexts (cf. e.g. [15]); however, this is beyond the scope of this work.

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