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2012 J. Phys. A: Math. Theor. 45 485301

(http://iopscience.iop.org/1751-8121/45/48/485301)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **45** (2012) 485301 (15pp) doi:10.1088/1751-8113/45/48/485301

Asymptotic properties of quantum Markov chains

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Received 30 April 2012, in final form 3 October 2012
Published 19 November 2012
Online at stacks.iop.org/JPhysA/45/485301

Abstract

The asymptotic dynamics of discrete quantum Markov chains generated by the most general physically relevant quantum operations is investigated. It is shown that it is confined to an attractor space in which the resulting quantum Markov chain is diagonalizable. A construction procedure of a basis of this attractor space and its associated dual basis of 1-forms is presented. It is applicable whenever a strictly positive quantum state exists which is contracted or left invariant by the generating quantum operation. Moreover, algebraic relations between the attractor space and Kraus operators involved in the definition of a quantum Markov chain are derived. This construction is not only expected to offer significant computational advantages in cases in which the dimension of the Hilbert space is large and the dimension of the attractor space is small, but it also sheds new light onto the relation between the asymptotic dynamics of discrete quantum Markov chains and fixed points of their generating quantum operations. Finally, we show that without any restriction our construction applies to all initial states whose support belongs to the so-called recurrent subspace.

PACS numbers: 03.67.−a, 03.65.Yz, 02.30.Tb, 03.65.Ta

1. Introduction

Quantum operations, i.e. completely positive and trace non-increasing linear transformations acting on a Hilbert space, play a central role in quantum theory. They describe the most general dynamics of an open quantum system which can be induced by unitary transformations and selective von Neumann measurements involving an additional initially uncorrelated ancillary quantum system [1]. Quantum channels, i.e. trace preserving quantum operations, can always be described by a unitary operation acting on both the system under consideration and an appropriately chosen ancilla system, whereas general non-trace preserving quantum operations cannot be represented in this way. Also, they always involve a selective quantum measurement performed on the ancilla system. Physical applications of quantum operations range from quantum control theory and dissipation engineering [2] to quantum error correction [3], quantum percolation [4] and quantum computation and quantum programming [5]. Furthermore, quantum operations are also an indispensable theoretical tool for exploring the ultimate limits of quantum theory [6].

Recent quantum technological advances [7] have stimulated significant interest in the dynamics of large open quantum systems formed by many indistinguishable or distinguishable elementary quantum systems, such as Bose–Einstein condensates [8] or qubit-based quantum networks [9]. These technological developments raise interesting theoretical questions concerning the dynamics of large quantum systems under the action of iterated quantum operations, the so-called quantum Markov chains [10]. This is due to the fact that in the area of statistical physics many discrete models involving quantum Markov chains are employed to explore fundamental physical phenomena such as the approach to thermal equilibrium or the asymptotic dynamics and decoherence of macroscopic quantum systems.

For a general classification of quantum Markov chains an investigation of their asymptotic dynamics resulting from large numbers of iterations of their generating quantum operations constitutes a natural starting point. In this context the natural questions arise: which asymptotic dynamics is possible for a quantum Markov chain and how it is related to spectral properties of its generating quantum operation? It is the main intention of this paper to address these questions for arbitrary quantum Markov chains.

Recently, some results addressing these questions have already been found for special classes of quantum Markov chains. In particular, the asymptotic dynamics of Markov chains resulting from iterated random unitary quantum operations has been investigated in detail. These operations are contracting and two major results have been established [11, 12]. Firstly, it has been demonstrated that the asymptotic dynamics of such a quantum Markov chain is confined to an attractor space. This attractor space is spanned by all orthogonal eigenspaces of the generating random unitary transformations which are associated with eigenvalues of unit modulus. Based on an analysis of these particular spectral properties, convenient representations for the asymptotic dynamics of this special class of quantum Markov chains can be derived. Secondly, it has been shown how an orthogonal basis of such an asymptotic attractor space is determined by a set of linear equations involving the Kraus operators specifying the generating random unitary quantum operation. Taking advantage of the fundamental contraction property of this special class of unital and trace preserving quantum operations recently parts of these investigations have been generalized also to quantum Markov chains which are generated either by trace preserving and unital [13] or by trace preserving and sub-unital quantum operations [14]. Although these generalizations demonstrate that these quantum operations can be diagonalized on their asymptotic attractor spaces, they still leave important questions open concerning, for example, the explicit construction of bases of the attractor spaces and of their associated dual bases.

Despite recent developments [13–15], it is still unclear to what extent similarly powerful results apply to the most general and physically relevant quantum Markov chains. This is due to the fact that general quantum operations are not contracting, so that the ideas underlying the proofs of these previous results do not apply. The main purpose of this paper is to close this gap and to generalize these previous results to quantum Markov chains which are generated by arbitrary quantum operations. Our generalization reveals that the contraction property is not essential for deriving analogous results applicable to general discrete quantum Markov chains. The desired structural properties can be established from the fact that general quantum operations are trace non-increasing and completely positive. These properties imply the validity of generalized Schwartz inequalities [16, 17]. Saturating these inequalities imposes important structural constraints on the asymptotic attractor space. We show that the saturation of these generalized Schwartz inequalities is possible whenever a strictly positive operator exists which is contracted or left invariant by the generating the quantum operation of a quantum Markov chain. Our results shed new light onto characteristic properties of fixed points of quantum operations, thereby generalizing recent results on the theory of fixed points of quantum operations and of noiseless subsystems in quantum systems with finite-dimensional Hilbert spaces [18–20]. The main intention behind our subsequent investigation is a systematic exploration of the general structure of the asymptotic dynamics and of the attractor space of general quantum Markov chains. Thereby, questions concerning the convergence toward this asymptotic dynamics are beyond the scope of this paper. Despite numerous recent efforts, a systematic rigorous understanding of these convergence properties is still lacking.

This paper is organized as follows. Known basic notions and properties of completely positive quantum maps and of their iterations are summarized in section 2. Based on these results, it is demonstrated that the asymptotic dynamics of a quantum Markov chain resulting from iterations of a general trace non-increasing quantum operation is confined to an attractor space. This attractor space is spanned by, in general, non-orthogonal (simple) eigenvectors of generating quantum operation. Thus, in this attractor space also these most general physically relevant quantum Markov chains are diagonalizable. The determination of the asymptotic dynamics requires the projection of an arbitrary initially prepared quantum state onto this attractor space. For this purpose one has to construct a dual basis of 1-forms. In general, this is a complicated task already for underlying Hilbert spaces of moderate dimension, even though the dimension of the attractor space is small. The construction of these dual bases of 1-forms is addressed in section 3. It is shown that whenever a strictly positive linear operator exists which is contracted or left invariant by the generating quantum operation, it is possible to construct this dual basis in a straightforward way from a knowledge of the basis of the attractor space. In section 4, the relation between the basis of the attractor space and Kraus operators defining the generating quantum operation of a quantum Markov chain is found. Finally, in section 5, it is demonstrated that to some extent the theoretical treatment of sections 3 and 4 can also be applied to cases in which the quantum state which is contracted or left invariant by a quantum operation is not strictly positive.

2. Basic properties of quantum Markov chains and their asymptotic dynamics

In this section, previously known basic properties of completely positive trace non-increasing quantum operations are summarized in order to introduce our notation. With the help of these properties the asymptotic dynamics of quantum Markov chains is determined which are generated by iterations of such operations.

2.1. General trace non-increasing quantum operations

In the following we consider an N-dimensional Hilbert space $\mathscr{H}$ equipped with a scalar product $(.,.)$. Let $\mathsf{B}(\mathscr{H})$ be the associated Hilbert space of all linear operators acting on $\mathscr{H}$ with the Hilbert–Schmidt scalar product $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}\{A^{\dagger}B\}$. The corresponding Hilbert–Schmidt norm is given by $\parallel A \parallel := \sqrt{\langle A, A\rangle_{\mathrm{HS}}}$ for all $A \in \mathsf{B}(\mathscr{H})$. Consequently, the induced norm of a linear operator $\mathcal{S} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ acting on the space $\mathsf{B}(\mathscr{H})$ can be defined by

$$\parallel \mathcal{S} \parallel = \sup_{\parallel A \parallel = 1} \parallel \mathcal{S}(A) \parallel . \tag{1}$$

This latter norm fulfils the important relation (see e.g. [16])

$$\parallel \mathcal{S} \parallel = \parallel \mathcal{S}^{\dagger} \parallel \tag{2}$$

characterizing a Banach* algebra [23]. Thereby, $\mathcal{S}^{\dagger}$ denotes the adjoint map of $\mathcal{S}$ with respect to the Hilbert–Schmidt scalar product. If a linear map satisfies the relation $\parallel \mathcal{S} \parallel \leqslant 1$, then it is called a contraction.

An arbitrary completely positive linear map $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ admits a decomposition into Kraus operators $\{A_j\}_{i=1}^{k} \subseteq \mathsf{B}(\mathscr{H})$ [1, 24], i.e.

$$\mathcal{P}(.) = \sum_{j=1}^{k} A_j(.)A_j^{\dagger}. \tag{3}$$

Its adjoint map $\mathcal{P}^{\dagger}$ is also a completely positive map with the Kraus operators $\{A_j^{\dagger}\}_{i=1}^{k}$, i.e.

$$\mathcal{P}^{\dagger}(.) = \sum_{j=1}^{k} A_j^{\dagger}(.)A_j. \tag{4}$$

In our subsequent discussion we call a completely positive map $\mathcal{P}$ which is also trace non-increasing, i.e. $\mathcal{P}^{\dagger}(I) = \sum_j A_j^{\dagger}A_j \leqslant I$, a quantum operation. In the special case $\mathcal{P}^{\dagger}(I) = \sum_j A_j^{\dagger}A_j = I$ such a map is called a quantum channel or a trace preserving quantum operation. A quantum operation which leaves maximally mixed states undisturbed is called unital and satisfies the relation $\mathcal{P}(I) = \sum_j A_jA_j^{\dagger} = I$. Prominent examples of unital quantum channels are random unitary operations (or random external fields) [11, 12]. In less restrictive cases in which $\mathcal{P}(I) = \sum_j A_jA_j^{\dagger} \leqslant I$ quantum operations are called sub-unital.

In the subsequent sections the dynamics of a quantum system with Hilbert space $\mathscr{H}$ is discussed which is governed by the iterated application of a quantum operation $\mathcal{P}$ as described by (3). This dynamics constitutes a quantum Markov chain with a generator $\mathcal{P}$ in analogy to the corresponding classical case (compare e.g. with [10, 25]). Thus, starting with an initial quantum state $0 \leqslant \rho(0) \in \mathsf{B}(\mathscr{H})$, after n iterations this state is transformed into the quantum state $\rho(n) = \mathcal{P}^n(\rho(0))$. Our main purpose is to analyze characteristic features of the resulting asymptotic behavior of $\mathcal{P}^n$ and its relation to the spectral properties of its generating quantum operation $\mathcal{P}$. In general, the commutation relation $[\mathcal{P}, \mathcal{P}^{\dagger}] = 0$ need not hold so that in general a diagonalization of the generator $\mathcal{P}$ is not possible. Therefore, in such cases the determination of the resulting asymptotic dynamics of $\mathcal{P}^n$ is complicated as this n-fold iteration may involve high powers of non-trivial Jordan normal forms of the generator $\mathcal{P}$.

Despite these possible complications the following useful theorems hold. The first one captures some basic spectral properties of the special class of quantum channels, i.e. of trace preserving completely positive quantum operations [26, 25]. The second one restricts the eigenvalues and eigenspaces of any trace non-increasing general quantum operation.

Theorem 2.1. *If $\mathcal{P}$ is a trace preserving quantum operation of the form (3) and if σ denotes the set of all its eigenvalues, then the following statements are equivalent:*

- *If $\lambda \in \sigma$, then $|\lambda| \leqslant 1$.*
- $1 \in \sigma$.
- *For every quantum state $0 \leqslant \rho \in \mathsf{B}(\mathscr{H})$ the limit*

$$\overline{\rho} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}^k(\rho) \tag{5}$$

exists and the quantum state $0 \leqslant \overline{\rho} \in \mathsf{B}(\mathscr{H})$ is a fixed point and the eigenvector of $\mathcal{P}$ with eigenvalue $\lambda = 1$.

In our subsequent discussion two concepts for analyzing characteristic properties of quantum operations $\mathcal{P}$ play an important role, namely eigenspaces and their ranges. Suppose λ is an eigenvalue of the map $\mathcal{P}$. Its corresponding eigenspace is denoted by

$$\mathsf{Ker}(\mathcal{P} - \lambda I) = \{X \in \mathcal{B}(\mathscr{H}) | \mathcal{P}(X) = \lambda X\}, \tag{6}$$

and the associated range of the map $\mathcal{P} - \lambda I$ is denoted by

$$\mathsf{Ran}(\mathcal{P} - \lambda I) = \{X \in \mathcal{B}(\mathscr{H}) | \exists Y \in \mathcal{B}(\mathscr{H}),\ \ X = \mathcal{P}(Y) - \lambda Y\}. \tag{7}$$

The possible eigenvalues and the structure of the corresponding eigenspaces of general trace non-increasing quantum operations are restricted considerably by the following properties.

Theorem 2.2. *For any quantum operation $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ as defined by (3), i.e. for any trace non-increasing and completely positive map, the following relations hold.*

- *Any eigenvalue λ of $\mathcal{P}$ fulfils the relation $|\lambda| \leqslant 1$.*
- *The kernel $\mathsf{Ker}\,(\mathcal{P} - \lambda I)$ and the range $\mathsf{Ran}\,(\mathcal{P} - \lambda I)$ of any eigenvalue λ with $|\lambda| = 1$ fulfil the relation*

$$\mathsf{Ker}\,(\mathcal{P} - \lambda I) \cap \mathsf{Ran}\,(\mathcal{P} - \lambda I) = \{0\}. \tag{8}$$

Different versions of this theorem and of special cases thereof are known [21, 22, 5]. For the sake of completeness, we present a proof involving different methods in appendix A.

2.2. Asymptotic dynamics of iterated trace non-increasing quantum operations

Recently, the asymptotic behavior of quantum Markov chains which are generated by random unitary transformations has been investigated in detail [11, 12]. As a major result, it has been shown that all Jordan blocks corresponding to eigenvalues λ with $|\lambda| = 1$ are one-dimensional and that the asymptotic dynamics of $\mathcal{P}^n$ can be diagonalized on the associated attractor space

$$\mathrm{Attr}(\mathcal{P}) := \bigoplus_{|\lambda|=1} \mathsf{Ker}(\mathcal{P} - \lambda I) \tag{9}$$

formed by the direct sum of all eigenspaces of $\mathcal{P}$ with eigenvalues $|\lambda| = 1$. The original idea of the proof of this asymptotic structure as presented in [12] relies on the fact that random unitary transformations are contractions. Recently, this result has been generalized to Markov chains generated by unital quantum channels [13] or by sub-unital quantum operations [14]. Although these investigations address a large class of iterated quantum operations, the extension of these results to quantum Markov chains generated by arbitrary trace non-increasing quantum operations, which are generally non-contracting, is unclear as the previously applied arguments do no longer apply. However, based on theorem 2.2 it can be shown that for quantum operations, i.e. trace non-increasing completely positive quantum maps $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ (compare with equation (3)) with $\mathcal{P}^\dagger(I) \leqslant I$, in the limit $n \to \infty$ the dynamics of the iterated map $\mathcal{P}^n$ is confined to an attractor space $\mathrm{Attr}(\mathcal{P})$. This latter space is spanned by all eigenvectors of $\mathcal{P}$ with eigenvalues λ of unit modulus, i.e. $|\lambda| = 1$. According to theorem 2.2, these eigenvectors are simple but non-orthogonal, in general. Thus, on this asymptotically relevant subspace $\mathrm{Attr}(\mathcal{P})$ the iterated quantum operation $\mathcal{P}^n$ can be diagonalized. This fact is summarized by the following theorem.

Theorem 2.3. *Asymptotically the iterative dynamics of any quantum operation $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ is given by*

$$X_\infty(n) = \sum_{\lambda \in \sigma_1, i=1}^{d_\lambda} \lambda^n X_{\lambda,i} \mathrm{Tr}(X^{\lambda,i\dagger} X(0)) \tag{10}$$

with

$$\lim_{n\to\infty} \parallel X_\infty(n) - \mathcal{P}^n(X(0)) \parallel = 0. \tag{11}$$

Thereby, the eigenvectors $X_{\lambda,i}$ *are determined by the relation* $P(X_{\lambda,i}) = \lambda X_{\lambda,i} (i = 1 \ldots d_\lambda)$ *with* $\lambda \in \sigma_1 := \{\lambda||\lambda| = 1\}$*. Their dual vectors* $X^{\lambda,i} \in \mathsf{B}(\mathscr{H})(i = 1 \ldots d_\lambda)$ *with respect to the Hilbert–Schmidt scalar product are defined by the property* $\mathrm{Tr}(X^{\lambda,i\dagger} X_{\lambda',i'}) = \delta_{\lambda\lambda'}\delta_{ii'}$ *for all* $\lambda, \lambda' \in \sigma_1$ *and by the property that for* $\lambda \in \sigma_1$ *each* $X^{\lambda,i}$ *is orthogonal to all eigenspaces* $\mathsf{Ker}(\mathcal{P} - \lambda' I)$ *with* $|\lambda'| < 1$.

The proof of this theorem involves a straightforward application of theorem 2.2 and of an upper bound restricting the influence of non-trivial Jordan blocks [12]. For the sake of completeness, a proof of this theorem is presented in appendix B.

Theorem 2.3 generalizes previous results on the asymptotic dynamics of quantum Markov chains which apply only to subunital channels and unital quantum operations or even to more restrictive cases, such as unital channels, i.e. $\mathcal{P}(I) = I, \mathcal{P}^\dagger(I) \leqslant I$ or random unitary operations [11, 12]. It implies considerable simplifications as far as the determination of the asymptotic dynamics is concerned because only eigenspaces corresponding to eigenvalues of unit modulus contribute and these eigenspaces are associated with trivial one-dimensional Jordan blocks. However, in general these eigenspaces spanning the asymptotic attractor space $\mathrm{Attr}(\mathcal{P})$ may still be non-orthogonal. This complicates the construction of the relevant dual vectors which project onto the attractor space because for this purpose typically the knowledge of the complete Jordan basis of generalized eigenvectors of the quantum operation $\mathcal{P}$ is required. This fact is summarized in the following corollary.

Corollary 2.4. *Let* $\{X_{\lambda,i}\}$ *with* $|\lambda| \leqslant 1$ *and* $i = 1 \ldots, d_\lambda$ *be a complete Jordan basis of the quantum operation* $\mathcal{P}$*. The corresponding non-singular Hermitian matrix*

$$g_{\lambda i,\lambda' i'} = \mathrm{Tr}\big(X_{\lambda,i}^\dagger X_{\lambda',i'}\big) \tag{12}$$

with $|\lambda|, |\lambda'| \leqslant 1$*,* $i = 1 \ldots, d_\lambda$*,* $i' = 1 \ldots, d_{\lambda'}$ *contains all relevant information about the non-orthogonality of this Jordan basis. The corresponding dual basis* $\{X^{\lambda,i}\}$ *with* $|\lambda| \leqslant 1$ *and* $i = 1 \ldots, d_\lambda$ *is then given by*

$$X^{\lambda,i} = \sum_{|\lambda'|\leqslant 1, i'=1}^{d_{\lambda'}} (g^{-1})_{\lambda i,\lambda' i'} X_{\lambda' i'}. \tag{13}$$

In terms of this Jordan basis and its dual, the projection operator Π *onto the attractor space* $\mathrm{Attr}(\mathcal{P})$ *is given by*

$$\Pi. = \sum_{\lambda\in\sigma_1, i=1}^{d_\lambda} X_{\lambda,i} \mathrm{Tr}(X^{\lambda,i\dagger}.). \tag{14}$$

Thus, despite the simplifications resulting from theorem 2.3, in general the determination of the required dual vectors of the asymptotic attractor space $\mathrm{Attr}(\mathcal{P})$ still constitutes a formidable task in particular in cases in which the dimension of the Hilbert space is large. Nevertheless, in the subsequent sections it is demonstrated that under additional restrictions on the generating quantum operations $\mathcal{P}$ of a quantum Markov chain both tasks, namely the construction of a basis for the asymptotic attractor space $\mathrm{Attr}(\mathcal{P})$ and the construction of its associated dual basis, can be simplified considerably.

3. Construction of the dual asymptotic basis

A major open problem which has not been addressed in the previous section is whether there exist convenient methods which simplify the construction of the dual vectors $X^{\lambda,i}(i = 1\ldots, d_\lambda)$ for all possible eigenvalues $\lambda \in \sigma_1$. The knowledge of these dual vectors is crucial for projecting any linear operator $X(0)$ or any initially prepared quantum state $\rho(0)$ onto the attractor space $\mathrm{Attr}(\mathcal{P})$ according to (10).

In this section, it is shown that under the additional assumption, that the generating quantum operation $\mathcal{P}$ of a quantum Markov chain supports the existence of a strictly positive quantum state $0 < \rho \in \mathsf{B}(\mathscr{H})$ which is contracted or left invariant, i.e. $\mathcal{P}(\rho) \leqslant \rho$, a straightforward construction of these dual basis vectors is possible from the knowledge of all eigenvectors with eigenvalues $\lambda \in \sigma_1$. For a quantum channel, i.e. a trace preserving quantum operation, a quantum state ρ with $\mathcal{P}(\rho) < \rho$ cannot exist. Therefore, for quantum channels this property reduces to the existence of a not necessarily uniquely determined full rank stationary state. For the construction of the required dual basis vectors, we exploit the basic property that the generating quantum operation $\mathcal{P}$ of a quantum Markov chain is trace non-increasing and thus fulfils characteristic generalized Schwartz inequalities [16, 17]. Saturating these inequalities we arrive at the following theorem.

Theorem 3.1. *If $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ is a quantum operation with the additional property that there exists a quantum state $0 < \rho \in \mathsf{B}(\mathscr{H})$ fulfilling the inequality $\mathcal{P}(\rho) \leqslant \rho$, then for all kernels of eigenvalues $\lambda \in \sigma_1$ the following equivalences hold:*

- $X \in \mathsf{Ker}(\mathcal{P} - \lambda I) \Leftrightarrow X\rho^{-1} \in \mathsf{Ker}(\mathcal{P}^\dagger - (1/\lambda)I)$
- $X \in \mathsf{Ker}(\mathcal{P} - \lambda I) \Leftrightarrow \rho^{-1}X \in \mathsf{Ker}(\mathcal{P}^\dagger - (1/\lambda)I)$
- $X \in \mathsf{Ker}(\mathcal{P} - \lambda I) \Leftrightarrow \rho^{-1}X\rho \in \mathsf{Ker}(\mathcal{P} - \lambda I)$.

Note that $\lambda \in \sigma_1$ implies the relation $\overline{\lambda} = 1/\lambda$ with $\overline{\lambda}$ denoting the complex conjugate of λ.

Proof. For the proof of the first statement *(1)* we investigate the following linear map:

$$\mathcal{V}(X) = \mathcal{P}^\dagger(X\rho^{-\frac{1}{2}})\rho^{\frac{1}{2}} \tag{15}$$

with its adjoint map

$$\mathcal{V}^\dagger(X) = \mathcal{P}(X\rho^{\frac{1}{2}})\rho^{-\frac{1}{2}}. \tag{16}$$

First of all we demonstrate that both maps are contractions. Because $\mathcal{P}^\dagger$ is subunital, i.e. $\mathcal{P}^\dagger(I) \leqslant I$, the Schwartz operator inequality $\mathcal{P}^\dagger(X)\mathcal{P}^\dagger(X^\dagger) \leqslant \mathcal{P}^\dagger(XX^\dagger)$ applies [16, 17]. Thus, for a density operator $0 < \rho \in \mathsf{B}(\mathscr{H})$ which fulfils the relation $\mathcal{P}(\rho) \leqslant \rho$, we obtain the inequality

$$\begin{aligned}\| \mathcal{V}(X) \|^2 &= \mathrm{Tr}\{(\mathcal{V}(X))^\dagger \mathcal{V}(X)\} = \mathrm{Tr}\{\mathcal{P}^\dagger(\rho^{-\frac{1}{2}}X^\dagger)\mathcal{P}^\dagger(X\rho^{-\frac{1}{2}})\rho\} \\ &\leqslant \mathrm{Tr}\{\mathcal{P}^\dagger(\rho^{-\frac{1}{2}}X^\dagger X\rho^{-\frac{1}{2}})\rho\} = \mathrm{Tr}\{\rho^{-\frac{1}{2}}X^\dagger X\rho^{-\frac{1}{2}}\mathcal{P}(\rho)\} \\ &\leqslant \mathrm{Tr}\{\rho^{-\frac{1}{2}}X^\dagger X\rho^{-\frac{1}{2}}\rho\} = \| X \|^2.\end{aligned} \tag{17}$$

Consequently, $\| \mathcal{V} \| = \| \mathcal{V}^\dagger \| \leqslant 1$, and both $\mathcal{V}$ and $\mathcal{V}^\dagger$ are contracting linear maps.

If $X \in \mathsf{Ker}(\mathcal{P} - \lambda I)$, then a simple calculation reveals that

$$\| \mathcal{V}^\dagger(X\rho^{-\frac{1}{2}}) \| \;=\; \| \mathcal{P}(X)\rho^{-\frac{1}{2}} \| = |\lambda| \| X\rho^{-\frac{1}{2}} \|. \tag{18}$$

Furthermore, Schwartz's inequality and the contracting properties of $\mathcal{V}$ and $\mathcal{V}^\dagger$ imply the inequalities

$$\begin{aligned}\| \mathcal{V}^\dagger(X\rho^{-1/2}) \|^2 &= \mathrm{Tr}((\mathcal{V}^\dagger(X\rho^{-1/2}))^\dagger \mathcal{V}^\dagger(X\rho^{-1/2})) = \mathrm{Tr}((X\rho^{-1/2})^\dagger \mathcal{V}\mathcal{V}^\dagger(X\rho^{-1/2})) \\ &\leqslant \| \mathcal{V}\mathcal{V}^\dagger(X\rho^{-1/2}) \| \| X\rho^{-1/2} \| \leqslant \| \mathcal{V} \| \| \mathcal{V}^\dagger \| \| (X\rho^{-1/2}) \|^2 \leqslant \| (X\rho^{-1/2}) \|^2 .\end{aligned} \tag{19}$$

For $\lambda \in \sigma_1$, the very left hand side of these inequalities equals the very right hand side, so that we can conclude

$$\mathcal{V}\mathcal{V}^\dagger(X\rho^{-\frac{1}{2}}) = \mathcal{P}^\dagger(\mathcal{P}(X)\rho^{-1})\sqrt{\rho} = X\rho^{-\frac{1}{2}} \tag{20}$$

for $\lambda \in \sigma_1$. Thus, using $\mathcal{P}(X) = \lambda X$ we finally arrive at the relation

$$\mathcal{P}^\dagger(X\rho^{-1}) = \frac{1}{\lambda}X\rho^{-1} = \overline{\lambda}X\rho^{-1} \tag{21}$$

in view of $|\lambda| = 1$. Analogously, it can be demonstrated that

$$\mathcal{V}^\dagger\mathcal{V}(X\rho^{-\frac{1}{2}}) = \mathcal{P}(\mathcal{P}^\dagger(X\rho^{-1})\rho)\rho^{-1/2} = X\rho^{-\frac{1}{2}} \tag{22}$$

for $\lambda \in \sigma_1$, so that we can conclude $\mathcal{P}(X) = \lambda X$ provided $\mathcal{P}^\dagger(X\rho^{-1}) = \lambda^{-1}X\rho^{-1}$.

Statement (2) can be proven by applying the same reasoning to the linear map $\mathcal{W}(X) = \rho^{1/2}\mathcal{P}^\dagger(\rho^{-1/2}X)$ and to its adjoint map $\mathcal{W}^\dagger(X) = \rho^{-1/2}\mathcal{P}(\rho^{1/2}X)$.

Statement (3) is a simple consequence of statements (1) and (2). Assuming that X is an eigenvector of $\mathcal{P}$ with eigenvalue $\lambda \in \sigma_1$, i.e. $\mathcal{P}(X) = \lambda X$, statement (2) implies $\mathcal{P}^\dagger(\rho^{-1}X) = \frac{1}{\lambda}\rho^{-1}X$ and statement (1) implies $\mathcal{P}(\rho^{-1}X\rho) = \lambda\rho^{-1}X\rho$. □

On the basis of this theorem a new scalar product can be defined in the space $\mathsf{B}(\mathscr{H})$. This scalar product is determined via any strictly positive operator $0 < \rho \in \mathsf{B}(\mathscr{H})$ by $\langle A, B\rangle_\rho \equiv \langle A, B\rho^{-1}\rangle_{\mathrm{HS}}$. It allows us to define the concept of ρ-orthogonality by the requirement that two operators, say $A, B \in \mathsf{B}(\mathscr{H})$, are ρ-orthogonal, i.e. $A\perp_\rho B$, iff $\langle A, B\rangle_\rho = 0$. Based on this concept, the following important ρ-orthogonality relations can be proved.

Theorem 3.2. *Let $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ be a quantum operation and let there be a strictly positive operator $0 < \rho \in \mathsf{B}(\mathscr{H})$ such that $\mathcal{P}(\rho) \leqslant \rho$, then the following statements are fulfilled.*

- *For any eigenvalue λ of $\mathcal{P}$ with $|\lambda| = 1$ kernel and range are orthogonal, i.e.*

$$\mathsf{Ker}(\mathcal{P} - \lambda I) \ \perp_\rho \ \mathsf{Ran}(\mathcal{P} - \lambda I) \tag{23}$$

and

$$\mathsf{Ker}\,(\mathcal{P} - \lambda I) \cap \mathsf{Ran}\,(\mathcal{P} - \lambda I) = \{0\}\,. \tag{24}$$

- *For any two different eigenvalues λ_1 and λ_2 of $\mathcal{P}$ with $|\lambda_1| = |\lambda_2| = 1$ the associated eigenspaces are orthogonal, i.e.*

$$\mathsf{Ker}(\mathcal{P} - \lambda_1 I) \ \perp_\rho \ \mathsf{Ker}(\mathcal{P} - \lambda_2 I). \tag{25}$$

Proof. Let us consider $X \in \mathsf{B}(\mathscr{H})$ with $X \in \mathsf{Ker}(\mathcal{P} - \lambda I)$ and $\lambda \in \sigma_1$, so that we obtain the relations $(\mathcal{P}(X))^\dagger = \mathcal{P}(X^\dagger) = \overline{\lambda}X^\dagger$ and $\mathcal{P}^\dagger(X\rho^{-1}) = \frac{1}{\lambda}X\rho^{-1}$ from theorem 3.1. Furthermore, let us consider $Y \in \mathsf{B}(\mathscr{H})$ with $Y \in \mathsf{Ran}(\mathcal{P} - \lambda I)$, i.e. there exists a $0 \neq Z \in \mathsf{B}(\mathscr{H})$ with $\mathcal{P}(Z) - \lambda Z = Y$. This implies the relation

$$\begin{aligned}\langle X, Y\rangle_\rho &= \mathrm{Tr}(X^\dagger Y\rho^{-1}) = \mathrm{Tr}(X^\dagger\mathcal{P}(Z)\rho^{-1}) - \lambda\mathrm{Tr}(X^\dagger Z\rho^{-1}) \\ &= \mathrm{Tr}([P^\dagger(X\rho^{-1})]^\dagger Z) - \lambda\mathrm{Tr}(\rho^{-1}X^\dagger Z) = 0\end{aligned} \tag{26}$$

for $\lambda \in \sigma_1$.

In order to prove statement (2) let us consider $X_1 \in \mathsf{Ker}(\mathcal{P} - \lambda_1 I)$ and $X_2 \in \mathsf{Ker}(\mathcal{P} - \lambda_2 I)$ with $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \sigma_1$. This implies the relation

$$\begin{aligned}\langle X_1, X_2\rangle_\rho &= \mathrm{Tr}\big(X_1^\dagger X_2 \rho^{-1}\big) = \frac{1}{\overline{\lambda}_1}\mathrm{Tr}\big(\mathcal{P}\big(X_1^\dagger\big) X_2 \rho^{-1}\big) = \frac{1}{\overline{\lambda}_1}\mathrm{Tr}\big(X_1^\dagger \mathcal{P}^\dagger (X_2\rho^{-1})\big)\\ &= \frac{1}{\lambda_2\overline{\lambda}_1}\mathrm{Tr}\big(X_1^\dagger X_2\rho^{-1})\big) = \frac{1}{\lambda_2\overline{\lambda}_1}\langle X_1, X_2\rangle_\rho. \end{aligned} \tag{27}$$

Because $\lambda_1, \lambda_2 \in \sigma_1$, the relation $\lambda_2\overline{\lambda}_1 = \lambda_2/\lambda_1$ applies, so that $\langle X_1, X_2\rangle_\rho = 0$ for $\lambda_1 \neq \lambda_2$. □

Based on these characteristic properties the dual asymptotic basis can be constructed in a simple way from the knowledge of all eigenspaces $\mathsf{Ker}(\mathcal{P} - \lambda I)$ for all $\lambda \in \sigma_1$. This central result of this section is summarized in the following theorem.

Theorem 3.3. *Let $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ be a quantum operation with the additional property that there exists a strictly positive quantum state $0 < \rho \in \mathsf{B}(\mathscr{H})$ with $\mathcal{P}(\rho) \leqslant \rho$. Under these conditions the dual vectors $X^{\lambda,i}$ of the eigenvectors $X_{\lambda,i}$ with $\lambda \in \sigma_1$ and $i = 1\ldots, d_\lambda$ which span the asymptotic attractor space $\mathrm{Attr}(\mathcal{P})$ are given by*

$$X^{\lambda,i} = X_{\lambda,i}\rho^{-1}\big[\mathrm{Tr}\big(X_{\lambda,i}^\dagger X_{\lambda,i}\rho^{-1}\big)\big]^{-1}. \tag{28}$$

Proof. The dimensions of the kernel and the range of the eigenspace of an arbitrary eigenvalue λ fulfil the general relation

$$\mathrm{Dim}(\mathsf{Ker}(\mathcal{P} - \lambda I)) + \mathrm{Dim}(\mathsf{Ran}(\mathcal{P} - \lambda I)) \geqslant \mathrm{Dim}(\mathsf{B}(\mathscr{H})).$$

According to theorem 3.2, the linear map $\mathcal{P}$ fulfils the relation $\mathsf{Ker}(\mathcal{P} - \lambda I) \perp_\rho \mathsf{Ran}(\mathcal{P} - \lambda I)$ so that we can conclude

$$\mathsf{Ker}(\mathcal{P} - \lambda I)) \oplus \mathsf{Ran}(\mathcal{P} - \lambda I) = \mathsf{B}(\mathscr{H}) \tag{29}$$

for all $\lambda \in \sigma_1$. In addition, theorem 3.2 also implies that $\mathsf{Ker}(\mathcal{P} - \lambda_1 I) \perp_\rho \mathsf{Ker}(\mathcal{P} - \lambda_2 I)$ for all $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \sigma_1$ which implies the relation

$$\bigoplus_{\lambda\in\sigma_1} \mathsf{Ker}(\mathcal{P} - \lambda I) \oplus \bigcap_{\lambda\in\sigma_1} \mathsf{Ran}(\mathcal{P} - \lambda I) = B(\mathscr{H}),$$

with $\bigcap_{\lambda\in\sigma_1} \mathsf{Ran}(\mathcal{P} - \lambda I)$ being orthogonal to the asymptotic attractor space $\mathrm{Attr}(\mathcal{P})$ and simultaneously containing all contributions of eigenspaces $\mathsf{Ker}(\mathcal{P} - \lambda I)$ with $|\lambda| < 1$. According to theorem 2.3, this latter subspace does not contribute to the dynamics of the iterated quantum operation $\mathcal{P}^n$ in the limit $n \to \infty$. Therefore, $\mathrm{Attr}(\mathcal{P})$ is spanned by the ρ-orthogonal eigenspaces $\mathsf{Ker}(\mathcal{P} - \lambda I)$ with $\lambda \in \sigma_1$, i.e.

$$\mathrm{Attr}(\mathcal{P}) = \bigoplus_{\lambda\in\sigma_1} \mathsf{Ker}(\mathcal{P} - \lambda I) \tag{30}$$

and the ρ-orthogonal projection onto this attractor space is achieved by the dual vectors of (28) and by the projection operator

$$\Pi\,(.) = \sum_{\lambda\in\sigma_1} X_{\lambda,i}\mathrm{Tr}(X^{\lambda,i\dagger}.). \tag{31}$$

Thereby, it is assumed that in the case of degeneracy of eigenspaces with $\lambda \in \sigma_1$, the corresponding eigenstates $X_{\lambda,i}$ are ρ-orthogonalized. □

4. Construction of the asymptotic attractor space

In this section, we address the final question how the relevant eigenvectors $X_{\lambda,i}$ $i = 1\ldots, d_\lambda$ with $\lambda \in \sigma_1$ which define the asymptotic attractor space $\mathrm{Attr}(\mathcal{P})$ are related to the Kraus operators which define the generating quantum operation of a quantum Markov chain.

For this purpose, we use the following property which holds for arbitrary completely positive maps (3) even if they not trace non-increasing.

Theorem 4.1. *Let $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ be a completely positive map defined by (3) and let there be a strictly positive operator $0 < \rho_1 \in \mathsf{B}(\mathscr{H})$ satisfying $\mathcal{P}(\rho_1) \leqslant \rho_1$ and a positive operator $\rho_2 \geqslant 0$ satisfying $P^\dagger(\rho_2) \leqslant \rho_2$, then any $X \in \mathsf{Ker}(\mathcal{P} - \lambda I)$ with $\lambda \in \sigma_1$ fulfils the set of equations*

$$\rho_2 A_j X \rho_1^{-1} = \lambda \rho_2 X \rho_1^{-1} A_j \tag{32}$$

for all $j \in \{1, \ldots, k\}$.

Proof. In order to prove this theorem we investigate the linear map

$$V_j(X) = \lambda \sqrt{\rho_2} X \rho_1^{-1} A_j \sqrt{\rho_1} - \sqrt{\rho_2} A_j X \rho_1^{-1/2} \tag{33}$$

with $X \in \mathsf{Ker}(\mathcal{P} - \lambda I)$ and evaluate the following sum of non-negative terms:

$$\begin{aligned}\sum_{j=1}^{k} \mathrm{Tr}\{V_j(X)[V_j(X)]^\dagger\} &= \mathrm{Tr}\left\{X\rho_1^{-1}X^\dagger \mathcal{P}^\dagger(\rho_2)\right\} + |\lambda|^2 \mathrm{Tr}\left\{X\rho_1^{-1}\mathcal{P}(\rho_1)\rho_1^{-1}X^\dagger \rho_2\right\} \\ &\quad - \lambda \mathrm{Tr}\left\{X\rho_1^{-1}\mathcal{P}(X^\dagger)\rho_2\right\} - \overline{\lambda}\mathrm{Tr}\left\{\mathcal{P}(X)\rho_1^{-1}X^\dagger\rho_2\right\} \\ &\leqslant (1 - |\lambda|^2)\mathrm{Tr}\left\{X\rho_1^{-1}X^\dagger\rho_2\right\}.\end{aligned} \tag{34}$$

Thereby, the relations $\mathcal{P}(X) = \lambda X$ and $\mathcal{P}(X^\dagger) = \overline{\lambda} X^\dagger$ together with the assumptions $\mathcal{P}(\rho_1) \leqslant \rho_1$ and $P^\dagger(\rho_2) \leqslant \rho_2$ have been taken into account. For $\lambda \in \sigma_1$, this leads to the conclusion $V_j(X) = 0$ for $X \in \mathsf{Ker}(\mathcal{P} - \lambda I)$ and for each $j \in \{1, \ldots, k\}$. □

Applying this theorem to a trace non-increasing quantum operation $\mathcal{P}$ and to its adjoint $\mathcal{P}^\dagger$ (which is generally not trace non-increasing) we obtain the following theorem.

Theorem 4.2. *Let $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ be a quantum operation and let there be a strictly positive operator $0 < \rho \in \mathsf{B}(\mathscr{H})$ satisfying $\mathcal{P}(\rho) \leqslant \rho$, then any $X \in \mathsf{Ker}(\mathcal{P} - \lambda I)$ with $\lambda \in \sigma_1$ fulfils the set of equations*

$$\begin{aligned} A_j X \rho^{-1} &= \lambda X \rho^{-1} A_j, \qquad & A_j^\dagger X \rho^{-1} &= (1/\lambda) X \rho^{-1} A_j^\dagger, \\ A_j \rho^{-1} X &= \lambda \rho^{-1} X A_j, \qquad & A_j^\dagger \rho^{-1} X &= (1/\lambda) \rho^{-1} X A_j^\dagger \end{aligned} \tag{35}$$

for all $j \in \{1, \ldots, k\}$.

Proof. The first equation is a simple application of theorem 4.1 with $\rho_1 = \rho$ and $\rho_2 = I$. The second equation is a consequence of theorem 3.1, namely $\mathcal{P}^\dagger(X\rho^{-1}) = (1/\lambda) X \rho^{-1}$, and of theorem 4.1 applied to $\mathcal{P}^\dagger$ with $\rho_1 = I$ and $\rho_2 = \rho$. The third and fourth equations are also consequences of these two theorems. □

This theorem states that for each eigenvalue $\lambda \in \sigma_1$ the set

$$\begin{aligned}\mathsf{D}_{\lambda,\rho} \equiv \big\{X \mid A_j X \rho^{-1} = \lambda X \rho^{-1} A_j, \quad & A_j^\dagger X \rho^{-1} = \overline{\lambda} X \rho^{-1} A_j^\dagger, \\ A_j \rho^{-1} X = \lambda \rho^{-1} X A_j, \quad & A_j^\dagger \rho^{-1} X = \overline{\lambda} \rho^{-1} X A_j^\dagger \qquad \text{for } j \in \{1, \ldots, k\}\big\}\end{aligned} \tag{36}$$

includes the eigenspace $\mathrm{Ker}(\mathcal{P}-\lambda I)$, i.e. $\mathrm{Ker}(\mathcal{P}-\lambda I) \subseteq \mathrm{D}_{\lambda,\rho}$. In order to address the question under which conditions we can achieve equality, i.e. $\mathrm{Ker}(\mathcal{P}-\lambda I) = \mathrm{D}_{\lambda,\rho}$, we add the following two corollaries of theorem 4.2 which state that either the conditions $\mathcal{P}^\dagger(I) \leqslant I$ and $\mathcal{P}(\rho) = \rho$ or the conditions $\mathcal{P}^\dagger(I) = I$ and $\mathcal{P}(\rho) = \rho$ are sufficient for this purpose.

Corollary 4.3. *Let $\mathcal{P} : \mathrm{B}(\mathscr{H}) \to \mathrm{B}(\mathscr{H})$ be a quantum operation defined by (3) and let there be a strictly positive operator $0 < \rho \in \mathrm{B}(\mathscr{H})$ satisfying $\mathcal{P}(\rho) = \rho$, then the following relations hold.*

- $\mathrm{Ker}(\mathcal{P}-\lambda I) = \mathrm{D}_{\lambda,\rho}$ *for all* $\lambda \in \sigma_1$.
- *If* $X_1 \in \mathrm{Ker}(\mathcal{P}-\lambda_1 I)$, $X_2 \in \mathrm{Ker}(\mathcal{P}-\lambda_2 I)$ *then* $X_1X_2\rho^{-1} \in \mathrm{Ker}(\mathcal{P}-\lambda_1\lambda_2 I)$.

Proof. For the proof of (1) we have to show that $\mathrm{D}_{\lambda,\rho} \subseteq \mathrm{Ker}(\mathcal{P}-\lambda I)$. For this purpose, let us assume that $X \in \mathrm{D}_{\lambda,\rho}$ and therefore satisfies the relations $A_jX\rho^{-1} = \lambda X\rho^{-1}A_j$ for $j \in \{1\ldots,k\}$. Multiplying both sides of these equations by $\rho A_j^\dagger$ and summing over all values of $j = 1\ldots,k$ we obtain the result $\mathcal{P}(X) = \lambda X\rho^{-1}\mathcal{P}(\rho) = \lambda X$, i.e. $X \in \mathrm{Ker}(\mathcal{P}-\lambda I)$.

If $X_1 \in \mathrm{Ker}(\mathcal{P}-\lambda_1 I)$ and $X_2 \in \mathrm{Ker}(\mathcal{P}-\lambda_2 I)$, then statement (1) ensures that $X_1 \in \mathrm{D}_{\lambda_1,\rho}$ and $X_2 \in \mathrm{D}_{\lambda_2,\rho}$. Thus, from theorem 3.1 we can conclude that $\rho X_2\rho^{-1} \in \mathrm{D}_{\lambda_2,\rho}$. Consequently, the following two identities are fulfilled:

$$\begin{aligned} A_jX_1X_2(\rho^{-1})^2 &= \lambda_1X_1\rho^{-1}A_j\rho X_2\rho^{-1}\rho^{-1} = \lambda_1\lambda_2X_1X_2(\rho^{-1})^2A_j,\\ A_j\rho^{-1}X_1X_2\rho^{-1} &= \lambda_1\rho^{-1}X_1A_jX_2\rho^{-1} = \lambda_1\lambda_2\rho^{-1}X_1X_2\rho^{-1} \end{aligned} \tag{37}$$

and analogous equalities for the adjoint linear operators $A_j^\dagger$, so that we can conclude that $X_1X_2\rho^{-1} \in \mathrm{D}_{\lambda_1\lambda_2,\rho} = \mathrm{Ker}(\mathcal{P}-\lambda_1\lambda_2 I)$. □

Unital quantum operations are examples of quantum operations fulfilling the assumptions of this theorem.

This corollary generalizes previous work on the theory of fixed points and noiseless subsystems of unital channels in Hilbert spaces of finite dimensions [18–20]. Firstly, these results generalize previously developed procedures for evaluating asymptotically relevant eigenspaces of quantum Markov chains, which apply to the special eigenvalue $\lambda = 1$ only, to all asymptotically relevant eigenvalues $\lambda \in \sigma_1$. Secondly, this result applies not only to unital channels. Indeed, according to 4.3 only one restriction is required for obtaining not only a necessary but also a sufficient condition for the construction of $\mathrm{Ker}(\lambda - I)$ for all $\lambda \in \sigma_1$, namely $\mathcal{P}(\rho) = \rho$ for some strictly positive $0 < \rho \in \mathrm{B}(\mathscr{H})$.

5. A generalization

In our previous considerations, we required the existence of a strictly positive quantum state such that $\mathcal{P}(\rho) \leqslant \rho$. However, there are quantum operations which do not fulfil this condition. Therefore, the natural question arises whether our previous results also apply to such situations. In the following, we show that our theory applies to all initial states whose support belong to the so-called recurrent Hilbert subspace [29]. It is a subspace of all pure states orthogonal to decaying states [15].

According to theorem 2.1 a quantum channel $\mathcal{P}$, i.e. a trace preserving completely positive map, is always equipped with a state ρ such that $\mathcal{P}(\rho) = \rho$. Although this state satisfies the condition $\mathcal{P}(\rho) \leqslant \rho$, it need not be strictly positive. Let

$$\rho = \sum_i \alpha_i|\psi_i\rangle\langle\psi_i| \tag{38}$$

be its diagonal form with the orthonormal pure states $|\psi_i\rangle$ and $\alpha_i > 0$. The orthogonal projection $P_\rho = \sum_i |\psi_i\rangle\langle\psi_i|$ onto the support (or range) of this state ρ satisfies the inequality

$$\mathcal{P}(P_\rho) \leqslant \frac{1}{\alpha_{\min}}\mathcal{P}(\rho) \leqslant \frac{1}{\alpha_{\min}\alpha_{\max}}\mathcal{P}(P_\rho) \tag{39}$$

with $\alpha_{\min}$ (resp. $\alpha_{\max}$) denoting the minimal (resp. maximal) nonzero eigenvalue of the state ρ. According to work [29], the projection P_ρ always reduces a quantum operation $\mathcal{P}$, i.e. $\mathcal{P}(P_\rho X P_\rho) = P_\rho \mathcal{P}(P_\rho X P_\rho) P_\rho$. Hence the corresponding reduced map $\mathcal{P} : \mathsf{B}(P_\rho\mathscr{H}) \to \mathsf{B}(P_\rho\mathscr{H})$ constitutes a well-defined quantum operation. Indeed, complete positivity is apparent and if the original map $\mathcal{P}$ is trace non-increasing (resp. trace preserving) then also its reduction is trace non-increasing (resp. trace preserving). Moreover, the fixed state ρ is strictly positive on $P_\rho\mathscr{H}$. Hence, all requirements for the applicability of our theorems from sections 3 and 4 are met for the restriction of a quantum channel $\mathcal{P}$ onto the subalgebra $\mathsf{B}(P_\rho\mathscr{H})$.

This construction is feasible for an arbitrary not necessarily strictly positive state ρ satisfying $\mathcal{P}(\rho) \leqslant \rho$. Furthermore, there is always a maximal state $\tilde{\rho}$ satisfying this property. 'Maximal' means that $P_\sigma \leqslant P_{\tilde{\rho}}$ for an arbitrary state σ satisfying $\mathcal{P}(\sigma) \leqslant \sigma$. In this sense, the orthonormal projection $P_{\tilde{\rho}}$ on the support of $\tilde{\rho}$ defines the so-called recurrent subspace $P_{\tilde{\rho}}\mathscr{H}$ [29], the maximal Hilbert space for which our theory applies.

6. Summary and conclusions

We have investigated the asymptotic dynamics of quantum Markov chains generated by general quantum operations, i.e. by completely positive and trace non-increasing linear maps acting in a Hilbert space. It has been shown that their resulting asymptotic dynamics is confined to attractor spaces spanned by typically non-orthogonal eigenspaces of the generating quantum operations associated with eigenvalues of unit modulus. These eigenvalues have trivial Jordan blocks, so that asymptotically also these most general and physically relevant quantum Markov chains are diagonalizable on their attractor spaces. Furthermore, provided a strictly positive operator can be found, which is contracted or left invariant by the generating quantum operation of such a quantum Markov chain, an explicit construction of a basis of the attractor space and of its associated dual basis has been presented. The basis vectors of the attractor space are determined by linear equations which depend in a simple way on the Kraus operators defining the generating quantum operation. Each of these basis vectors is related to its dual one by a linear transformation. This linear transformation is defined by the inverse of the positive operator which is contracted or left invariant by the generating quantum operation. This explicit construction of the asymptotic dynamics of an arbitrary iterated quantum operation is expected to offer significant advantages whenever the dimension of the Hilbert space is large and at the same time the dimension of the attractor space is small. Thus, the theoretical description of the asymptotic dynamics described here may be particularly useful for applications in large quantum systems.

Apart from possible practical advantages, our discussion also explicitly demonstrates the close connection between the existence of a quantum state which is contracted or even left invariant by the generating quantum operation of a quantum Markov chain and the resulting asymptotic dynamics. It generalizes recent results on the theory of fixed points of quantum operations and of noiseless subsystems in quantum systems with finite-dimensional Hilbert spaces [18–20].

Finally, we want to emphasize again that the results presented here are restricted to finite-dimensional Hilbert spaces. This is mainly due to arguments involved in the proof of theorem 2.2. They make use of a general theorem due to Choi [27] and Jamiołkowski [28] which applies to finite-dimensional Hilbert spaces only and which relates the complete positivity of a linear map to the positivity of an extended map. As most arguments involved in the proofs of our subsequent theorems also apply to infinite-dimensional Hilbert spaces, it is expected that at least parts of the construction methods for attractor spaces presented here can be extended to certain physically relevant situations described by infinite-dimensional Hilbert spaces.

Acknowledgments

JN and IJ acknowledge support by the grant MSM 6840770039 of the Czech Ministry of Education and the Doppler Institute, FNSPE CTU in Prague, GA acknowledges support by CASED AB1.

Appendix A. Proof of theorem 2.2

According to a general theorem [27, 28] a linear map $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ is completely positive iff the quantum state $\mathcal{I} \otimes \mathcal{P}(|\Phi\rangle\langle\Phi|)$ resulting from the extended map $\mathcal{I} \otimes \mathcal{P} : \mathsf{B}(\mathscr{H}) \otimes \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H}) \otimes \mathsf{B}(\mathscr{H})$ by acting on the entangled pure state $|\Phi\rangle = 1/\sqrt{N}\sum_{i=1}^{N}|i\rangle|i\rangle \in \mathscr{H} \otimes \mathscr{H}$ is positive, i.e. $\mathcal{I} \otimes \mathcal{P}(|\Phi\rangle\langle\Phi|) \geqslant 0$. The positivity of this quantum state implies

$$
\begin{aligned}
|\ \langle i|\langle l|\mathcal{I} \otimes \mathcal{P}(|\Phi\rangle\langle\Phi|)|j\rangle|k\rangle\ |^2 &= \left|\frac{1}{N}\langle l|\mathcal{P}(|i\rangle\langle j|)|k\rangle\right|^2 \\
&\leqslant \frac{1}{N}\langle l|\mathcal{P}(|i\rangle\langle i|)|l\rangle\frac{1}{N}\langle k|\mathcal{P}(|j\rangle\langle j|)|k\rangle
\end{aligned}
\tag{A.1}
$$

with $|i\rangle|l\rangle$, $|j\rangle|k\rangle$ denoting arbitrary elements of an orthonormal basis in the extended Hilbert space $\mathscr{H} \otimes \mathscr{H}$. As a consequence, the relations

$$
\sum_{l,k=1}^{N} |\ \langle l|\mathcal{P}(|i\rangle\langle j|)|k\rangle\ |^2\ \leqslant 1 \tag{A.2}
$$

are fulfilled for all possible values of i, j if, in addition, the linear map $\mathcal{P}$ is trace non-increasing.

Assuming that λ is an eigenvalue of a completely positive and trace non-increasing quantum operation $\mathcal{P}$ with its corresponding eigenvector X, we can conclude

$$
\begin{aligned}
\|\ \mathcal{P}(X)\ \|^2 = |\ \lambda\ |^2 \|\ X\ \|^2 &= \|\ \sum_{i,j=1}^{N}\sum_{l,k=1}^{N} |k\rangle\langle l|\langle k|\mathcal{P}(|i\rangle\langle j|)|l\rangle\langle i|X|j\rangle\ \|^2 \\
&= \sum_{l,k=1}^{N} |\ \sum_{i,j=1}^{N}\langle i|X|j\rangle\langle k|\mathcal{P}(|i\rangle\langle j|)|l\rangle\ |^2 \\
&\leqslant \sum_{i,j=1}^{N} |\ \langle i|X|j\rangle\ |^2 \underbrace{\sum_{l,k=1}^{N} |\ \langle k|\mathcal{P}(|i\rangle\langle j|)|l\rangle\ |^2}_{\leqslant 1} \leqslant \|\ X\ \|^2
\end{aligned}
\tag{A.3}
$$

with $\|\ A\ \| = \sqrt{\sum_{l,k=1}^{N} |\ \langle l|A|k\rangle\ |^2}$ denoting the Hilbert–Schmidt norm of a linear operator $A \in \mathsf{B}(\mathscr{H})$. Thus, we finally arrive at the first part (1) of this theorem, i.e. $|\lambda\ | \leqslant 1$.

For the second part of this theorem, let us assume that this statement is false. Thus, there is an operator $0 \neq A \in \mathsf{B}(\mathscr{H})$ with $A \in \mathsf{Ker}\,(\mathcal{P} - \lambda I) \cap \mathsf{Ran}\,(\mathcal{P} - \lambda I)$. This implies $\mathcal{P}(A) = \lambda A$ and there is an operator $0 \neq B \in \mathsf{B}(\mathscr{H})$ such that $\mathcal{P}(B) = \lambda B + A$. By induction it can be verified that

$$\lambda^{n-1}A = \frac{1}{n}\mathcal{P}^n(B) - \frac{1}{n}\lambda^n B \tag{A.4}$$

for all numbers of iterations $n \geqslant 1$. As n increases, the second term on the right-hand side of (A.4) becomes arbitrarily small in comparison with the term on the left-hand side of (A.4) for $|\lambda| = 1$. Due to complete positivity of the map $\mathcal{P}$, this is also true for the first term on the right-hand side of (A.4). In order to demonstrate this, let us consider an orthonormal basis $\{|i\rangle\}$ of the Hilbert space $\mathscr{H}$, so that the linear operators $\{|i\rangle\langle j|\}$ form an orthonormal basis of the space $\mathsf{B}(\mathscr{H})$. This implies the relation

$$\mathcal{P}^n(B) = \sum_{i,j=1}^{N} B_{i,j}\mathcal{P}^n(|i\rangle\langle j|) \tag{A.5}$$

with $B_{i,j} = \langle i|B|j\rangle$. As the quantum operation $\mathcal{P}^n$ is completely positive and trace non-increasing, we can conclude from (A.1) that

$$0 \leqslant |[\langle l|P^n(|i\rangle\langle j|)]|k\rangle| \leqslant 1 \tag{A.6}$$

for all $n \geqslant 0$ and for all $1 \leqslant k, l \leqslant N$. So, in view of (A.5) and (A.6) also all matrix elements of the first term on the right-hand side of (A.4) tend to zero for $|\lambda| = 1$ as n tends to infinity. Thus, we finally arrive at the contradiction $A = 0$ to the initial assumption $A \neq 0$.

Appendix B. Proof of theorem 2.3

Any linear map $\mathcal{P} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ can be brought into the Jordan normal form by an appropriate basis transformation T, i.e.

$$\mathcal{P} = TJT^{-1} \tag{B.1}$$

with the (non-singular) linear operator $T \in \mathsf{B}(\mathscr{H})$ defining the basis transformation and with $J \in \mathsf{B}(\mathscr{H})$ denoting the Jordan normal form of $\mathcal{P}$. Thus, J is given by a direct sum of Jordan blocks J_k of dimensions d_k, i.e. $J = \sum_k \oplus J_k$. The matrix representation of any of these $d_k \times d_k$-dimensional Jordan blocks J_k is given by

$$(J_k)_{ij} = \begin{pmatrix} \lambda_k & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_k & 1 & 0 & \cdots & 0 & 0 \\ & & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_k \end{pmatrix}. \tag{B.2}$$

Consequently, also the iterated map $\mathcal{P}$ can be transformed to the Jordan normal form, i.e. $\mathcal{P}^n = T^{-1}J^nT$. It is straightforward to demonstrate [12] that the modulus of the (i, j)th matrix element of a Jordan block $(J_k)^n$ is given by

$$\left|(J_k)^n_{ij}\right| = \mid \lambda_k \mid^{n-(j-i)} \binom{n}{n-(j-i)} \leqslant \mid \lambda_k \mid^{n-d_k} n^{d_k}.$$

Therefore, in the limit of large numbers of iterations n the contributions of all Jordan blocks with eigenvalues $\mid \lambda_k \mid < 1$ become vanishingly small, so that in this limit only the Jordan blocks with eigenvalues $\mid \lambda_k \mid = 1$ contribute to the iterated map $\mathcal{P}^n$.

Let us now concentrate on these asymptotically contributing Jordan blocks J_k with eigenvalues $\mid \lambda_k \mid = 1$. From theorem 2.2, we can draw the conclusion that all these Jordan blocks are one dimensional, so that the map $\mathcal{P}^n$ can be diagonalized within the asymptotic subspace $\mathrm{Attr}(\mathcal{P})$ which is spanned by all eigenvectors $X_{\lambda,i}$, $i = 1 \ldots, d_\lambda$ with possibly d_λ-fold degenerate eigenvalues $\mid \lambda \mid = 1$. These eigenvectors $X_{\lambda,i}$ fulfil the relations $\mathcal{P}(X_{\lambda,i}) = \lambda X_{\lambda,i}$

for $i = 1 \ldots, d_\lambda$ and $\lambda \in \sigma_1 := \{\lambda \mid |\lambda| = 1\}$ and in general they are not orthogonal (with respect to the Hilbert–Schmidt scalar product). Consequently, in the limit of large numbers of iterations n, it is sufficient to expand any initial linear operator $X(0) \in \mathsf{B}(\mathscr{H})$ in terms of these eigenvectors, i.e.

$$X(0) = \sum_{\lambda \in \sigma_1} \sum_{i=1}^{d_\lambda} x_{\lambda,i} X_{\lambda,i} + Y(0) \tag{B.3}$$

with $Y(0)$ denoting the part of $X(0)$ which is contained in eigenspaces of eigenvalues $|\,\lambda\,| < 1$. The coefficients $x_{\lambda,i}$ are given by

$$x_{\lambda,i} = \mathrm{Tr}(X^{\lambda,i\dagger} X(0)) \tag{B.4}$$

with $X^{\lambda,i} \in \mathsf{B}(\mathscr{H})$ denoting the dual basis vector of $X_{\lambda,i}$ which fulfils the relations $\mathrm{Tr}(X^{\lambda,i} X_{\lambda',i'}) = \delta_{\lambda\lambda'}\delta_{ii'}$ for all generalized eigenvectors of a Jordan basis, i.e. $|\lambda|, |\lambda'| \leqslant 1$ and $i = 1 \ldots, d_\lambda$, $i' = 1 \ldots, d_{\lambda'}$. As $\mathcal{P}^n(Y(0))$ tends to zero in the limit $n \to \infty$, we finally obtain the result of theorem 2.3.

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