Cent. Eur. J. Phys. • 8(6) • 2010 • 1001-1014
DOI: 10.2478/s11534-010-0018-8

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Central European Journal of **Physics**

Asymptotic evolution of random unitary operations

Research Article

Jaroslav Novotný[1 2]*, Gernot Alber[1], Igor Jex[2]

1 *Institut für Angewandte Physik, Technische Universität Darmstadt, Hochschulstraße 4a, D-64289 Darmstadt, Germany*

2 *Department of Physics, FJFI ČVUT v Praze, Břehová 7 Praha 1 - Staré Město, 115 19 Prague, Czech Republic*

Received 7 October 2009; accepted 15 February 2010

Abstract: We analyze the asymptotic dynamics of quantum systems resulting from large numbers of iterations of random unitary operations. Although, in general, these quantum operations cannot be diagonalized it is shown that their resulting asymptotic dynamics is described by a diagonalizable superoperator.We prove that this asymptotic dynamics takes place in a typically low dimensional attractor space which is independent of the probability distribution of the unitary operations applied. This vector space is spanned by all eigenvectors of the unitary operations involved which are associated with eigenvalues of unit modulus. Implications for possible asymptotic dynamics of iterated random unitary operations are presented and exemplified in an example involving random controlled-not operations acting on two qubits.

PACS (2008): 03.65.Ud,03.67.Bg,03.67.-a,03.65.Yz

Keywords: random unitary map • asymptotic evolution • iterations • attractor • open dynamics

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1. Introduction

In recent years the rapid advancement of quantum technology with its capabilities of controlling individual quantum systems has given rise to impressive developments in the areas of quantum information science and high-precision quantum metrology [1]. In particular, current experiments on large ensembles of interacting quantum systems open interesting perspectives to investigate in detail not only the transition from quantum to classical behavior but also to trace down those quantum phenomena or effects that still are observable on the mesoscopic or macroscopic scale. A paradigm of such large physical systems are interacting networks whose dynamics is currently investigated intensively in the classical domain [2]. Such networks are capable of simulating the behavior of real world systems like the internet or social dynamics [3]. Typically, in these systems a number of modes representing physical objects are coupled to each other by random interactions. A particularly interesting issue is to determine the dynamics of the system. In view of these current activities the natural question arises which characteristic properties govern the dynamics of such networks if each classical node is replaced by a quantum system and, correspondingly, the classical interactions by quantum operations.

In general, determining the time evolution of large quantum systems is difficult and analytic or closed-form solu-

*E-mail: novotny.jaroslav@seznam.cz

tions are possible in exceptional cases only. In particular, this applies to the dynamics of open quantum systems in which a large quantum system is in contact with an additional physical system. The influence of such an external system can be taken into account in various ways. In special cases it may be described by randomly applied unitary operations. Such a case is realized, for example, if the nodes of a large quantum network represent participants of a quantum communication network and if these nodes establish node-to-node communication in a random way by using quantum protocols which can be described by unitary transformations. A natural question arising in this context is what is the resulting quantum state of the network after a large number of such communication steps. More generally, such a quantum network involving random unitary transformations can characterize the dynamics of any interacting quantum system in which the interactions involved can be described by repeatedly applied random unitary transformations.

A natural approach to determine the dynamics of a quantum system involves diagonalization of the generator of the time evolution. This way the dynamics can be determined in a convenient way even in the asymptotic limit of arbitrarily long interaction times. The situation becomes significantly more complicated for open quantum systems because the relevant generators are often non-hermitian and not normal [4] so that they cannot be diagonalized. Nevertheless, in such cases it is still possible to use the Jordan canonical form (see Appendix A) of these operators for determining the dynamics for arbitrarily long interaction times. This leads to the highly nontrivial problem of handling generalized eigenvectors of the relevant generators which are in general not orthogonal.

Motivated by these aspects in this paper we address the problem of determining general properties of the asymptotic dynamics of quantum systems whose dynamics is governed by repeated applications of random unitary transformations. This large family of quantum stochastic dynamics is an example of so called quantum iterated functional systems which were proposed and studied in [5, 6]. A main goal of this paper is to demonstrate that the Jordan canonical form of the generators of random unitary transformations have rather unexpected and useful special properties which allow to obtain even closed-form expressions for the asymptotic quantum state resulting from a large number of iterations of random unitary transformations. It will be proved that there is always a vector subspace of so-called attractors on which the resulting superoperator governing the iterative time evolution of quantum states can be diagonalized and in which the asymptotic quantum dynamics takes place. As a main result a structure theorem is derived for this set of attractors which allows to determine them in a convenient way. Furthermore, it is shown how the asymptotic iterative dynamics of arbitrary quantum states can be written in terms of these attractors. Based on these findings we show that in general the asymptotic dynamics is non-monotonic. Finally, aspects of these general properties are exemplified by studying in detail the dynamics of two qubits which are coupled by randomly applied controlled-not operations. It should be mentioned that some of the results characterizing the asymptotic dynamics can also be obtained by a different approach which uses special properties of random unitary transformations in order to construct a convenient Ljapunov function [7].

This paper is structured as follows. In Sec. 2 we summarize basic properties of random unitary transformations which are useful for our subsequent discussion. In Sec. 3 we examine special properties of the Jordan canonical form of random unitary maps. The central statement of the paper, namely the structure theorem for attractors of random unitary operations, is derived in Sec. 4. Characteristic properties of attractors are investigated in Sec. 5. Sec. 6 is devoted to important implications resulting from the structure theorem. Finally, as an example the asymptotic dynamics of two qubits which are coupled by random controlled-not operations is discussed on the basis of our general results (Sec. 7).

2. Basic properties of random unitary operations

A random unitary operation (RUO) Φ is a completely positive trace-preserving map admitting a convex decomposition of the form [8]

$$\Phi(\rho) = \sum_{i=1}^{m} p_i U_i \rho U_i^{\dagger}. \qquad (1)$$

Thereby, U_i denotes a unitary operator acting on a Hilbert space $\mathscr{H}$ and this unitary operation is applied onto the quantum state ρ with probability $p_i > 0$ so that $\sum_{i=1}^{m} p_i = 1$. These latter probabilities take into account classical uncertainties in the realizations of the unitary quantum evolution involved. This uncertainty can be the result of an unknown error mechanism or of an unknown unitary evolution involving an additional ancillary system. In the following we are interested in the asymptotic dynamics resulting from many iterative applications of Φ. Starting with our quantum system in the initial state $\rho(0)$, the $(n+1)$-st step of this iteration procedure changes the state after the n-th iteration $\rho(n)$ to the state $\rho(n+1) = \Phi(\rho(n))$.

Our aim is to analyze the asymptotic behaviour of this iteration procedure. The random unitary map Φ of Eq. (1) belongs to the class of bistochastic or doubly stochastic maps [9–11] which leave the maximally mixed state invariant, *i.e.*

$$\Phi(I)=\sum_{i=1}^{m}p_iU_iU_i^{\dagger}=I, \tag{2}$$

and it acts on the Hilbert space $\mathcal{B}(\mathcal{H})$ of all linear operators defined on a d-dimensional Hilbert space $\mathcal{H}$. The dimension of the input and output system is the same. The Hilbert space $\mathcal{B}(\mathcal{H})$ is equipped with the Hilbert-Schmidt inner product $(A,B)_{HS}=\mathrm{Tr}(A^{\dagger}B)$ for all $A,B\in\mathcal{B}(H)$. With respect to this scalar product the adjoint operator of Φ is given by

$$\Phi^{\dagger}(A)=\sum_{i=1}^{m}p_iU_i^{\dagger}AU_i. \tag{3}$$

This can be shown directly or by using the matrix form of the map Φ (B6).

In general, the RUO Φ is neither hermitian nor normal and consequently is not diagonalizable. Therefore, its resulting iterated dynamics has to be analyzed with the help of Jordan normal forms [4] (see Appendix A). It is a main goal of our subsequent discussion to prove that the Jordan normal forms of RUOs have interesting special properties which are particularly useful for the description of their asymptotic iterated dynamics. In particular, there exists a Jordan base in the Hilbert space $\mathcal{B}(\mathcal{H})$ in which the matrix of the map (1) has a block diagonal form (A1). Spectral properties of general quantum operations were studied in [12]. In the following we formulate several simple characteristic properties which are particularly useful for our subsequent considerations.

Proposition 2.1.

The random unitary map Φ defined by the relation (1) fulfills the following properties:

1) *The norm of the RUO Φ induced by the Hilbert-Schmidt norm of the Hilbert space $\mathcal{B}(\mathcal{H})$ equals unity.*

2) *If λ is an eigenvalue of the map Φ, then $|\lambda|\le 1$.*

3) *Let $X_\lambda\in\mathcal{B}(\mathcal{H})$ be a generalized eigenvector corresponding to the eigenvalue λ of the map Φ, then $\lambda=1$ or $\mathrm{Tr}\,X_\lambda=0$.*

Proof. (1-2) First we prove that the Hilbert-Schmidt norm is unitarily invariant. For this purpose consider an arbitrary operator $A\in\mathcal{B}(\mathcal{H})$ and two unitary operators $U,V\in\mathcal{B}(\mathcal{H})$. As a trace of matrix products is invariant under cyclic permutations we get

$$\begin{aligned}||UAV||_{HS}&=\left\{\mathrm{Tr}\left[(UAV)^{\dagger}(UAV)\right]\right\}^{\frac{1}{2}}\\&=\left\{\mathrm{Tr}(A^{\dagger}A)\right\}^{\frac{1}{2}}=||A||_{HS}.\end{aligned} \tag{4}$$

Therefore one can show that $||\Phi||=\sup_{||A||_{HS}\le 1}||\Phi(A)||_{HS}=1$. Let $A\in\mathcal{B}(\mathcal{H})$, then the Hilbert-Schmidt norm of the operator $\Phi(A)$ is bounded by

$$\begin{aligned}||\Phi(A)||_{HS}&=\left|\left|\sum_i p_iU_iAU_i^{\dagger}\right|\right|_{HS}\\&\le\sum_i p_i\left|\left|U_iAU_i^{\dagger}\right|\right|_{HS}=||A||_{HS}.\end{aligned} \tag{5}$$

Moreover, we have $||\Phi(I)||_{HS}=||I||_{HS}$. Hence, $||\Phi||=1$ and consequently $|\lambda|\le 1$.

(3) If X_λ is a generalized eigenvector corresponding to an eigenvalue λ of the map Φ, then there is a $n\in\mathbb{N}$ such that $(\Phi-\lambda I)^n(X_\lambda)=0$ because a simple calculation yields

$$\mathrm{Tr}\left\{(\Phi-\lambda I)^n(X_\lambda)\right\}=(1-\lambda)^n\,\mathrm{Tr}\,X_\lambda=0. \tag{6}$$

This equation can be fulfilled only if $\lambda=1$ or $\mathrm{Tr}\,X_\lambda=0$. ■

Thus, all Jordan blocks in the Jordan normal decomposition of the map Φ correspond to eigenvalues λ with $|\lambda|\le 1$. For our subsequent discussion let us introduce the following notation. Suppose that λ is an eigenvalue of the map Φ. We denote the corresponding eigen-subspace by $\mathrm{Ker}(\Phi-\lambda I)$, *i.e.*

$$\mathrm{Ker}(\Phi-\lambda I)=\{X\in\mathcal{B}(\mathcal{H})|\Phi(X)=\lambda X\}\,, \tag{7}$$

and the range of the map $\Phi-\lambda I$ by $\mathrm{Ran}(\Phi-\lambda I)$, *i.e.*

$$\mathrm{Ran}(\Phi-\lambda I)=\{X\in\mathcal{B}(\mathcal{H})|\exists Y\in\mathcal{B}(\mathcal{H}),X=\Phi(Y)-\lambda Y\}\,. \tag{8}$$

Furthermore, let us define $d_\lambda=\dim(\mathrm{Ker}(\Phi-\lambda I))$ and $\sigma_{|1|}$ as the set of all eigenvalues of the linear map Φ satisfying $|\lambda|=1$. Finally, the vector subspace spanned by all eigenstates corresponding to eigenvalues λ with $|\lambda|=1$ we call *the attractor space of the RUO* Φ and denote it by $\mathrm{Atr}(\Phi)$, *i.e.*

$$\mathrm{Atr}(\Phi)=\bigoplus_{\lambda\in\sigma_{|1|}}\mathrm{Ker}(\Phi-\lambda I). \tag{9}$$

We call elements of this subspace attractors of the dynamics because, as we will show later, the asymptotic iterated dynamics of the RUO is completely determined by these linear operators.

3. Jordan canonical form of random unitary operations

In this section we prove that all Jordan blocks corresponding to eigenvalues λ with $|\lambda| = 1$ are one-dimensional. In other words, generalized eigenvectors corresponding to eigenvalues $|\lambda| = 1$ are all eigenvectors. This statement is equivalent to the following theorem (for details see Appendix A).

Theorem 3.1.
Let $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a random unitary operation defined by (1) and λ its eigenvalue satisfying $|\lambda| = 1$, then we have

$$\mathrm{Ker}(\Phi - \lambda I) \cap \mathrm{Ran}(\Phi - \lambda I) = \{0\}. \tag{10}$$

Proof. We prove this theorem by contradiction. Suppose there is an operator $0 \neq A \in \mathcal{B}(\mathscr{H})$ and $A \in \mathrm{Ker}(\Phi-\lambda I) \cap \mathrm{Ran}(\Phi-\lambda I)$. This implies $\Phi(A) = \lambda A$ and there is an operator $0 \neq B \in \mathcal{B}(\mathscr{H})$ such that $\Phi(B) = \lambda B + A$. By induction one can conclude

$$\Phi^n(B) = \lambda^n B + n\lambda^{n-1} A \tag{11}$$

and consequently

$$\begin{aligned} n\,||A|| - ||B|| &\leq \left|\left|\lambda^n B + n\lambda^{n-1} A\right|\right| \\ &= ||\Phi^n(B)|| \leq ||\Phi||^n\, ||B|| = ||B||\,. \end{aligned} \tag{12}$$

Because the resulting inequality

$$||A|| \leq \frac{2}{n}\,||B|| \tag{13}$$

has to be fulfilled for arbitrary $n \in \mathbb{N}$ the only alternative left is that $A = 0$. ■

Let $Y_{j,k}$ ($j \in \hat{p}$, $k \in \{1, 2, \ldots, \dim(J_j)\}$) (compare with Appendix A) be the Jordan basis of the RUO Φ. J_j is a Jordan block corresponding to an eigenvalue λ_j with a basis formed by the generalized eigenvectors $Y_{j,k}$ ($k \in \{1, 2, \ldots, \dim(J_j)\}$). Let $\rho(0) \in \mathcal{B}(\mathscr{H})$ be an input density operator. We denote by $\beta_{j,k}^{(0)}$ the parameters of the unique decomposition of the density operator $\rho(0) \in \mathcal{B}(\mathscr{H})$ into this basis, *i.e.*

$$\rho(0) = \sum_{j=1}^{p} \sum_{k=1}^{\dim(J_j)} \beta_{j,k}^{(0)} Y_{j,k}\,. \tag{14}$$

Consider now the density operator $\rho(n) = \Phi^n(\rho(0))$ describing the physical system after n iterations and denote its decomposition coefficients (14) into the same basis by $\beta_{j,k}^{(n)}$. It is clear that the coefficients $\beta_{j,k}^{(n)}$ corresponding to eigenvectors of eigenvalues $\lambda_j \in \sigma_{|1|}$ evolve simply as

$$\beta_j^{(n)} = \lambda_j^n \beta_j^{(0)}. \tag{15}$$

(We omit the second index k intentionally because in this case all the Jordan blocks are one dimensional.)
Now we have to analyze the behavior of the remaining coefficients. It is governed by the following theorem which quantifies how the remaining coefficients $\beta_{j,k}^{(n)}$, corresponding to Jordan vectors $Y_{j,k}$ with $|\lambda_j| < 1$, evolve.

Theorem 3.2.
Let $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a quantum random unitary operation defined by (1) with its Jordan basis $Y_{j,k}$ ($j \in \hat{p}$, $k \in \{1, 2, \ldots, \dim(J_j)\}$) and $\rho(0) \in \mathcal{B}(\mathscr{H})$ be an input density operator. Furthermore, let $\beta_{j,k}^{(n)}$ be the decomposition coefficients of $\rho(n) = \Phi^n(\rho(0))$ into this Jordan basis, i.e.

$$\rho(n) = \sum_{j=1}^{p} \sum_{k=1}^{\dim(J_j)} \beta_{j,k}^{(n)} Y_{j,k}\,. \tag{16}$$

For any eigenvalue λ_s ($|\lambda_s| < 1$) of the map Φ with its corresponding Jordan block J_s and its Jordan chain $Y_{s,k}$ ($k \in \{1, 2, \ldots, \dim(J_s)\}$) the coefficients $\beta_{s,k}^{(n)}$ vanish in the limit of large n

$$\lim_{n\to+\infty} \beta_{s,k}^{(n)} \to 0, \qquad \text{for} \quad \forall k \in \{1, 2, \ldots, \dim(J_s)\}\,. \tag{17}$$

Proof. This theorem follows directly from the fact that the Jordan block $(J_s)^n$ of dimension $\dim(J_s)$ with

$$J_s = \begin{pmatrix} \lambda_s & 1 & & \\ & \lambda_s & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_s \end{pmatrix} \tag{18}$$

vanishes in the limit of large numbers of iterations n, *i.e.*

$$\lim_{n\to\infty} (J_s)^n = 0. \tag{19}$$

One can check that the entry $(J_s^n)_{ij}$ ($i \leq j \leq \dim J_s$) of the upper triangular matrix $(J_s)^n$ fulfills the inequality

$$|(J_s^n)_{ij}| = |\lambda_s|^{n-(j-i)} \binom{n}{n-(j-i)} \leq |\lambda|^{n-\dim J_s} n^{\dim J_s} \tag{20}$$

so that we obtain the relation

$$\lim_{n\to\infty}(J_s^n)_{ij} = 0, \qquad \forall i, j \in \{1, \ldots, \dim J_s\}. \tag{21}$$

■

In view of this theorem the asymptotic dynamics of the state $\rho(0)$ under iterations of the random unitary operation Φ is given completely in terms of its attractors. The remaining coefficients of the decomposition of the initial state $\rho(0)$ (16) become vanishingly small after sufficiently many iterations of the map. An interesting question which will be addressed in the following is how to determine the set of attractors.

4. Structure theorem for attractors

Let us now study the structure of the attractors, *i.e.* of all eigenspaces $\mathrm{Ker}(\Phi - \lambda I)$, with $\lambda \in \sigma_{|1|}$. In the case of random unitary operations the following powerful theorem can be proved which allows us to specify the space of attractors of the RUO Φ. In this context it should be also mentioned that for the more general case of arbitrary unital quantum operations interesting general results have been derived by Kribs [13, 14] recently.

Theorem 4.1.
Let $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a random unitary map Φ (1) and $\lambda \in \sigma_{|1|}$. Then the eigenspace $\mathrm{Ker}(\Phi - \lambda I)$ corresponding to this eigenvalue λ is equal to the set

$$D_\lambda := \{X \in \mathcal{B}(\mathscr{H}) | U_i X = \lambda X U_i \ \text{ for } \ i = 1, \ldots, m\}. \tag{22}$$

Proof. The map Φ is unital, that is $\Phi(I) = I$. Therefore, every $X \in D_\lambda$ fulfils $\Phi(X) = \lambda X$ and thus $D_\lambda \subset \mathrm{Ker}(\Phi - \lambda I)$. To prove the converse, let us consider $X \in \mathrm{Ker}(\Phi - \lambda I)$. If $X = 0$, then $X \in D_\lambda$. So let us assume that $X \neq 0$. Using the unitary invariance of the Hilbert-Schmidt norm we get

$$\begin{aligned} ||X|| = ||\lambda X|| &= \left|\left|\sum_{i=1}^m p_i U_i X U_i^\dagger\right|\right| \\ &\leq \sum_{i=1}^m p_i \left|\left|U_i X U_i^\dagger\right|\right| = ||X||. \end{aligned} \tag{23}$$

Therefore, the inequality (23) is in fact an equality and can be rewritten in the form

$$\left(\sum_{i=1}^m p_i v_i, \sum_{i=1}^m p_i v_i\right) = \left(\sum_{i=1}^m p_i (v_i, v_i)^{\frac{1}{2}}\right)^2 \tag{24}$$

with $v_i = U_i X U_i^\dagger$. Hence we get

$$\begin{aligned} \sum_{i<j} 2p_i p_j (v_i, v_i)^{\frac{1}{2}} (v_j, v_j)^{\frac{1}{2}} &= \sum_{i<j} p_i p_j \left[(v_i, v_j) + (v_j, v_i)\right] = \sum_{i<j} 2p_i p_j \mathrm{Re}(v_i, v_j) \leq \sum_{i<j} 2p_i p_j \left|\mathrm{Re}(v_i, v_j)\right| \\ &\leq \sum_{i<j} 2p_i p_j \left|(v_i, v_j)\right| \leq \sum_{i<j} 2p_i p_j (v_i, v_i)^{\frac{1}{2}} (v_j, v_j)^{\frac{1}{2}}. \end{aligned} \tag{25}$$

Because the left and right side of the relation (25) are the same, all inequalities are actually equalities. In particular, we have

$$\mathrm{Re}(v_i, v_j) = \left|(v_i, v_j)\right| = (v_i, v_i)^{\frac{1}{2}} (v_j, v_j)^{\frac{1}{2}} \neq 0 \qquad \text{for all } \ i, j \in \{1, \ldots, m\} \tag{26}$$

which can be fulfilled if and only if $v_j = \beta_{ij} v_i$ (for all i,j) with $\beta_{ij} > 0$. From the unitary invariance of the Hilbert-Schmidt norm

$$||X|| = \left|\left|U_i X U_i^\dagger\right|\right| = \beta_{ij} \left|\left|U_j X U_j^\dagger\right|\right| = ||X|| \tag{27}$$

we conclude that $\beta_{ij} = 1$ for all $i, j \in \{1, ..., m\}$ and hence

$$U_1 X U_1^\dagger = U_2 X U_2^\dagger = \ldots = U_m X U_m^\dagger. \tag{28}$$

Finally, using the equality $\Phi(X) = \lambda X$ we obtain $U_i X U_i^\dagger = \lambda X$, *i.e.* $X \in D_\lambda$. ■

As a consequence of this structure Theorem 4.1 the following corollary can be proved.

Corollary 4.1.
The random unitary operation Φ defined by (1) fulfills the following properties:

1) *If λ is an eigenvalue of the operation Φ fulfilling $|\lambda| = 1$, then*

$$\mathrm{Ker}(\Phi - \lambda I) \perp \mathrm{Ran}(\Phi - \lambda I). \tag{29}$$

2) *If λ_1,λ_2 are two different eigenvalues of the operation Φ fulfilling $|\lambda_1| = |\lambda_2| = 1$, then*

$$\mathrm{Ker}(\Phi - \lambda_1 I) \perp \mathrm{Ker}(\Phi - \lambda_2 I). \tag{30}$$

Proof. First, from the Theorem (4.1) follows that if $\Phi(X) = \lambda X$ and $|\lambda| = 1$, then $\Phi^\dagger(X) = \lambda^* X$ and $\Phi^\dagger(X^\dagger) = \lambda X^\dagger$. In order to show that the set $\mathrm{Ker}(\Phi-\lambda I)$ is orthogonal to the set $\mathrm{Ran}(\Phi-\lambda I)$ we have to prove that $(K, R) = 0$ is fulfilled for arbitrary elements $K \in \mathrm{Ker}(\Phi-\lambda I)$ and $R \in \mathrm{Ran}(\Phi - \lambda I)$. Therefore, there is an operator $Q \in \mathcal{B}(\mathscr{H})$ with $R = \sum_i p_i U_i Q U_i^\dagger - \lambda Q$. Hence, using Theorem 4.1 we have the orthogonality relation

$$(K, R) = \mathrm{Tr}\{K^\dagger R\} = \sum_i p_i \,\mathrm{Tr}\{K^\dagger U_i Q U_i^\dagger\} \\ - \lambda\,\mathrm{Tr}\{K^\dagger Q\} = \lambda \sum_i p_i \,\mathrm{Tr}\{K^\dagger Q\} - \lambda\,\mathrm{Tr}\{K^\dagger Q\} = 0. \tag{31}$$

The second property is a consequence of the identity

$$(X_1, X_2) = \frac{1}{\lambda_2}(X_1, \Phi(X_2)) = \frac{1}{\lambda_2}\left(\Phi^\dagger(X_1), X_2\right) \\ = \frac{\lambda_1}{\lambda_2}(X_1, X_2), \tag{32}$$

which is valid for $X_1 \in \mathrm{Ker}(\Phi-\lambda_1 I)$ and $X_2 \in \mathrm{Ker}(\Phi-\lambda_2 I)$ and for any mutually different non-zero eigenvalues λ_1 and λ_2. Therefore, the last equality can be satisfied only if $(X_1, X_2) = 0$. ■

This corollary together with theorem 3.1 has the following important consequence.

Theorem 4.2.
Let $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a quantum random unitary operation defined by (1) and $\rho(0) \in \mathcal{B}(\mathscr{H})$ be an input density operator, then the asymptotic iterative dynamics of the state $\rho(0)$ under the evolution map Φ is given by

$$\rho_\infty(n) = \sum_{\lambda\in\sigma_{|1|}, i=1}^{d_\lambda} \lambda^n \,\mathrm{Tr}\{\rho(0) X_{\lambda,i}^\dagger\} X_{\lambda,i} \tag{33}$$

and satisfies the relation

$$\lim_{n\to\infty} ||\rho(n) - \rho_\infty(n)|| = 0 \tag{34}$$

with $\rho(n) = \Phi^n(\rho(0))$ and with the complete set of orthonormal basis elements $X_{\lambda,i}$ ($i \in \{1, 2, \ldots, d_\lambda\}$) of the space $\mathrm{Ker}(\Phi - \lambda I)$.

Proof. In order to prove this theorem we have to show that the mutually orthogonal subspaces

$$\mathfrak{I}_0 = \bigoplus_{\lambda\in\sigma_{|1|}} \mathrm{Ker}(\Phi-\lambda I) \quad \text{and} \quad \mathfrak{I}_1 = \bigcap_{\lambda\in\sigma_{|1|}} \mathrm{Ran}(\Phi-\lambda I) \tag{35}$$

are invariant under the map Φ and that they satisfy the relation $\mathfrak{I}_0 \oplus \mathfrak{I}_1 = \mathcal{B}(\mathscr{H})$. The second claim is a direct consequence of corollary 4.1. The first claim follows from the fact that all subspaces $\mathrm{Ker}(\Phi - \lambda I)$ and $\mathrm{Ran}(\Phi - \lambda I)$ are invariant under the map Φ; that is, $\Phi(\mathrm{Ker}(\Phi - \lambda I)) \subset \mathrm{Ker}(\Phi - \lambda I)$ and $\Phi(\mathrm{Ran}(\Phi - \lambda I)) \subset \mathrm{Ran}(\Phi - \lambda I)$. Now we can choose some orthogonal basis vectors $X_{\lambda,i}$ in the subspaces $\mathrm{Ker}(\Phi - \lambda I)$ with $|\lambda| = 1$ and the Jordan basis $Y_{j,k}$ of the map Φ restricted to the subspace $\mathfrak{I}_1$. These vectors form a basis of the Hilbert space $\mathcal{B}(\mathscr{H})$. Now we consider a decomposition of $\rho(0)$ into these basis vectors. As was shown in Theorem 3.2, the part corresponding to the subspace $\mathfrak{I}_1$ vanishes for $n \to +\infty$ and the dynamics of the state $\rho(0)$ on the subspace $\mathfrak{I}_0$ is given by (15). ■

5. Basic properties of attractors

In this section we discuss some basic properties of RUOs which are useful for obtaining the complete set of attractors. A basic property arises straightforwardly from the Theorem 4.1.

Proposition 5.1. 1) *Let X_{λ_1} and X_{λ_2} be attractors of the RUO (1) corresponding to eigenvalue λ_1 and λ_2, respectively, then the product of these attractors $X_{\lambda_1}X_{\lambda_2}$ is either an attractor corresponding to eigenvalue $\lambda_1\lambda_2$ or it is the zero operator.*

2) *Let X_λ be an attractor of the RUO (1) corresponding to the eigenvalue λ, then $X_\lambda^\dagger$ is also an attractor of the RUO (1) corresponding to the eigenvalue λ^*.*

Proof. This proposition follows from the identities

$$U_i X_{\lambda_1} X_{\lambda_2} = \lambda_1 X_{\lambda_1} U_i X_{\lambda_2} = \lambda_1\lambda_2 X_{\lambda_1} X_{\lambda_2} U_i \tag{36}$$

and

$$U_i X_\lambda^\dagger = (X_\lambda U_i^\dagger)^\dagger = (\lambda U_i^\dagger X_\lambda)^\dagger = \lambda^* X_\lambda^\dagger U_i, \qquad (37)$$

which are valid for all $i \in \hat{m} := \{1, \ldots, m\}$. ■

Based on our preceding analysis a single step of the asymptotic dynamics is described by the superoperator

$$\Phi_{ass}(.) = \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda \operatorname{Tr}\left(X_{\lambda,i}^\dagger (.)\right) X_{\lambda,i}, \qquad (38)$$

which fulfils the property

$$\lim_{n\to\infty} ||\Phi^n - \Phi^n_{ass}|| = 0. \qquad (39)$$

The superoperator (38) is a unital quantum operation. In order to prove this statement, let us define the projector

$$\mathcal{P}(.) = \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \operatorname{Tr}\left(X_{\lambda,i}^\dagger (.)\right) X_{\lambda,i}, \qquad (40)$$

which projects all elements of the vector space $\mathcal{B}(\mathscr{H})$ onto the attractor space $\operatorname{Atr}(\Phi)$. The structure Theorem 4.1 and the orthogonality of all elements of the attractor space ensure that $[\Phi, \mathcal{P}] = 0$ and $[\Phi_{ass}, \mathcal{P}] = 0$. These commutation properties imply that for any integer n the action of the superoperator Φ^n_{ass} on an arbitrary operator $A \in \mathcal{B}(\mathscr{H})$ is given by

$$\Phi^n_{ass}(A) = \Phi^n_{ass}(\mathcal{P}(A)) = U^n_{i_0} \mathcal{P}(A) U^{\dagger n}_{i_0} \qquad (41)$$

for an arbitrary $i_0 \in \hat{m}$. Thus, for any integer n the action of the map Φ^n_{ass} on the Hilbert space $\mathcal{B}(\mathscr{H})$ is a sequence of a projection onto the attractor space and a unitary operation. As a consequence it is a completely positive map and in view of Eq. (39) it describes the dynamics of the iterated random unitary operation in the asymptotic limit of large numbers n of iterations.

It is instructive to analyze this property of complete positivity also from another perspective by using the concept of dynamical matrices (compare with Appendix B) [10]. In order to obtain the dynamical matrix of the asymptotic map Φ_{ass} we first calculate its matrix elements in an orthonormal basis, *i.e.*

$$(\Phi_{ass})^{m\mu}_{n\nu} = \langle m\mu|\Phi_{ass}|n\nu\rangle = \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda \operatorname{Tr}\left\{(|n\rangle\langle\nu|)\, X_{\lambda,i}^\dagger\right\} \operatorname{Tr}\left\{(|m\rangle\langle\mu|)^\dagger X_{\lambda,i}\right\} = \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda \left(X_{\lambda,i}\right)^*_{n\nu} \left(X_{\lambda,i}\right)_{m\mu}. \qquad (42)$$

The elements of the dynamical matrix $D_{\Phi_{ass}}$ are defined by

$$(D_{\Phi_{ass}})^{m\mu}_{n\nu} = (\Phi_{ass})^{mn}_{\mu\nu} = \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda \left(X_{\lambda,i}\right)^*_{\mu\nu} \left(X_{\lambda,i}\right)_{mn}, \qquad (43)$$

so that one obtains the relation

$$\begin{aligned} D_{\Phi_{ass}} &= \sum_{m,n,\mu,\nu} (D_{\Phi_{ass}})^{m\mu}_{n\nu} |m\mu\rangle\langle n\nu| = \sum_{m,n,\mu,\nu} \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda \left(X_{\lambda,i}\right)^*_{\mu\nu} \left(X_{\lambda,i}\right)_{mn} |m\mu\rangle\langle n\nu| \\ &= \sum_{m,n,\mu,\nu} \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda \left[\left(X_{\lambda,i}\right)_{mn} |m\rangle\langle n|\right] \otimes \left[\left(X_{\lambda,i}\right)^*_{\mu\nu} |\mu\rangle\langle\nu|\right] = \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda X_{\lambda,i} \otimes X^*_{\lambda,i}. \end{aligned} \qquad (44)$$

Using the identity (B5) one can rewrite the $d^2 \times d^2$ dynamical matrix as an operator acting on $d \times d$ matrices according to

$$D_{\Phi_{ass}}(A) = \sum_{\lambda \in \sigma_{|1|}, i=1}^{d_\lambda} \lambda X_{\lambda,i} A X^\dagger_{\lambda,i}, \qquad (45)$$

where A is an arbitrary $d \times d$ matrix. Expressions (44) and (45) describe the same dynamical matrix. The first relation describes it as a map acting on reshaped vectors of length d^2 and the second one as a map acting on $d \times d$ matrices (for details see Appendix B). Both expressions are useful to determine the properties of attractors.

According to Eq. (B9) the dynamical matrix (44) is always hermitian. Due to proposition 5.1 this property is fulfilled. Furthermore, the partial trace of the dynamical matrix (44) over each subsystem yields the identity operator. For a RUO of the form of Eq. (1) both properties lead to the condition

$$\sum_{i=1}^{d_1} \mathrm{Tr}\{X_{1,i}^{\dagger}\} X_{1,i} = I. \tag{46}$$

so that the dynamical matrix is positive. With the help of Eqs (45) and (B2) we find that this positivity is equivalent to the relation

$$\sum_{\lambda\in\sigma_{|1|},i=1}^{d_\lambda} \lambda \,\mathrm{Tr}\left\{A^{\dagger} X_{\lambda,i} A X_{\lambda,i}^{\dagger}\right\} \geq 0, \tag{47}$$

which has to be fulfilled for an arbitrary $d \times d$ matrix A. In view of Theorem 4.1 we can thus conclude that the map

$$\Phi_{ass}^{n}(.) = \sum_{\lambda\in\sigma_{|1|},i=1}^{d_\lambda} \lambda^n \,\mathrm{Tr}\{(.) X_{\lambda,i}^{\dagger}\} X_{\lambda,i}, \tag{48}$$

is a trace-preserving and completely positive unital map for an arbitrary $n \in \mathbb{Z}$.

6. Discussion and implications

Let us summarize and comment the results obtained so far for the asymptotic behaviour of a quantum system under a RUO.

First of all, the asymptotic iterative dynamics is determined completely by the attractor set $\mathrm{Atr}(\Phi)$ of a RUO. The Hilbert space $\mathcal{B}(\mathscr{H})$ can be decomposed as $\mathcal{B}(\mathscr{H}) = \mathrm{Atr}(\Phi) \oplus (\mathrm{Atr}(\Phi))^{\perp}$ with $\perp$ denoting the orthogonal complement with respect to $\mathcal{B}(\mathscr{H})$. Both mutually orthogonal subspaces, *i.e.* $\mathrm{Atr}(\Phi)$ and $\mathfrak{I}_1 = (\mathrm{Atr}(\Phi))^{\perp}$ are invariant under the RUO (1) and we proved that the component of any initial quantum state in the subspace $\mathfrak{I}_1$ vanishes after sufficiently large numbers of iterations. Furthermore, we proved that the vector space of attractors $\mathrm{Atr}(\Phi)$ is spanned by all elements X of the set $\mathcal{B}(\mathscr{H})$ which fulfil the generalized commutation relations $U_i X = \lambda X U_i$ for all unitary operators U_i of the decomposition (1) and for all eigenvalue λ with $|\lambda| = 1$.

The calculation of the asymptotic iterated dynamics of the random unitary map (1) can be divided into four steps:

- One determines the set $\sigma_{|1|}$. Usually, this step is highly nontrivial and depends significantly on the particular unitary Kraus operators involved. Any additional properties concerning the structure of the unitary operators U_i involved, for example, simplify this task considerably. In particular, the exploitation of symmetries may be useful in this respect.
- One identifies the set of attractors of the RUO $\mathrm{Atr}(\Phi)$. This step involves the calculation of all eigenspaces using the generalized commutation relations $\mathrm{Ker}(\Phi - \lambda I) = D_\lambda$ for all $\lambda \in \sigma_{|1|}$.
- One chooses an orthonormal basis $X_{\lambda,i}$ in each subspace $\mathrm{Ker}(\Phi - \lambda I)$ for $\lambda \in \sigma_{|1|}$.
- One calculates the asymptotic iterated dynamics according to the relation

$$\rho(n) = \Phi^n(\rho(0))(n \gg 1) = \sum_{\lambda\in\sigma_{|1|},i=1}^{d_\lambda} \lambda^n \,\mathrm{Tr}\{\rho(0) X_{\lambda,i}^{\dagger}\} X_{\lambda,i}, \tag{49}$$

which is valid asymptotically for $n \gg 1$.

These general features imply some important consequences. Firstly, the set of attractors $\mathrm{Atr}(\Phi)$ and its corresponding spectrum is independent of the nonzero probabilities p_i defining the convex decomposition of the RUO in Eq. (1). Thus, two RUOs with the same unitary operators in their convex decompositions (1) have the same attractors space $\mathrm{Atr}(\Phi)$. The nonzero probabilities p_i determine only how fast an input state converges to the asymptotic attractor space.

Another simple consequence arises if the ensemble of random unitary operators defining the RUO Φ contains the identity operator I (apart from a global phase). Theorem 4.1 implies that the only possible eigenvalue of the map Φ is $\lambda = 1$. Hence from the set of attractors only fixed points can be formed and the resulting asymptotic dynamics is stationary. Moreover, assume that the unitary operators U_i are generators of a finite multiplicative group. As any group contains a unit element all possible eigenvalues of the RUO Φ fulfil the relation $\lambda^{n_\lambda} = 1$ for some integer $n_\lambda \in \mathbb{N}$. As a consequence the resulting asymptotic dynamics is periodic. Such a periodic asymptotic dynamics is also obtained obtained if the unitary operators U_i form an irreducible set of operators, *i.e.* they have no common nontrivial invariant subspace. This can be proven as follows. Consider an eigenvalue λ of the random unitary operation (1) with $|\lambda| = 1$ and its corresponding eigenvector $X_\lambda \neq 0$. Using Theorem 4.1 it can be checked that $U_i(\mathrm{Ker}(X_\lambda)) \subset \mathrm{Ker}(X_\lambda)$ and $U_i(\mathrm{Ran}(X_\lambda)) \subset \mathrm{Ran}(X_\lambda)$ is fulfilled for all $i \in \hat{m}$. Thus, $X_\lambda \neq 0$ is an invertible operator. Let $\alpha \neq 0$ be an eigenvalue of the operator X_λ and u_α its corresponding eigenvector. From the equation

$$X_\lambda U_i u_\alpha = \overline{\lambda} U_i X_\lambda u_\alpha = \alpha\overline{\lambda} U_i u_\alpha \quad (50)$$

follows that also $\alpha\overline{\lambda}$ is the eigenvalue of X_λ and $U_i u_\alpha$ are its corresponding eigenvectors. Therefore also $\alpha\overline{\lambda^2}, \alpha\overline{\lambda^3}, \ldots$ are eigenvalues of X_λ. Eigenvectors corresponding to different eigenvalues are linearly independent. Therefore there is $n \in \mathbb{N}$ such that $\lambda^n = 1$. Moreover, the direct sum of all eigensubspaces corresponding to eigenvalues $\overline{\lambda^j}\alpha$ ($j \in \{0, 1, \ldots, n-1\}$) is invariant under all unitary operators U_i and thus has to be equal to the whole Hilbert space $\mathscr{H}$, i.e.

$$\mathscr{H} = \bigoplus_{j=0}^{n-1} \mathrm{Ker}(X_\lambda - \overline{\lambda^j}\alpha I). \quad (51)$$

Therefore, X_λ is diagonalizable and can be written in the form

$$X_\lambda = \sum_{j=0}^{n-1} \overline{\lambda^j}\alpha (U_i)^j P (U_i^\dagger)^j, \quad (52)$$

where P is the projection on the eigensubspace corresponding to the eigenvalue α of X_λ and is determined by relations of the form

$$U_g^i P (U_g^i)^\dagger = U_h^i P (U_h^i)^\dagger, \qquad U_h^n P (U_h^n)^\dagger = P \neq 0. \quad (53)$$

Eq. (53) applies to an arbitrary pair of unitary operators U_g and U_h in the decomposition of the random unitary operation (1) and their arbitrary i-th power, $i \in \hat{n}$.
Assume now the opposite situation. Let P be a projection satisfying Eqs. (53) for some $n \in \mathbb{N}$ and consider $\lambda = \exp(i2\pi/n)$. Then one can make sure that the operator (52) fulfills the equations of Theorem 4.1 and thus it is an attractor of the RUO Φ corresponding to the eigenvalue λ. This brings us to the following interesting issue. Under which conditions does the attractor spectrum $\sigma_{|1|}$ of a RUO contain only the non-degenerate eigenvalue $\lambda = 1$ so that the resulting asymptotic dynamics always moves towards the maximally mixed state? We can formulate the answer to this question in the following corollary.

Corollary 6.1.
Let $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a random unitary map Φ (1). Then its attractor spectrum $\sigma_{|1|}$ contains only the non-degenerate eigenvalue $\lambda = 1$ if and only if the set of unitary operators U_i is irreducible and there is no projection P satisfying the set of Eqs. (53).

The second condition is necessary because irreducibility of the unitary operators U_i only ensures that the eigenvalue $\lambda = 1$ in the attractor spectrum $\sigma_{|1|}$ is not degenerate. This was also shown in [5]. To elucidate this fact let us show an example of a RUO whose attractor spectrum is given by $\sigma_{|1|} = \{1, \lambda, \overline{\lambda}\}$ with $\lambda = \exp(i2\pi/3)$. Based on our foregoing discussion we can construct the following set of irreducible unitary operators in a 6-dimensional Hilbert space with the orthonormal basis states $|j\rangle$ ($1 \leq j \leq 6$)

$$U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}, \quad U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}. \quad (54)$$

This set is chosen in such a way that $P = |1\rangle\langle 1| + |2\rangle\langle 2|$ is the only projection operator satisfying the set of Eqs. (53) with $n = 3$.
Finally, it can also be shown that all attractors of RUOs generated by an irreducible set of unitary operators are proportional to unitary operators. If we multiply the equations for attractors with their adjoint forms we obtain

$$U_i[X_\lambda X_\lambda^\dagger] = [X_\lambda X_\lambda^\dagger] U_i. \quad (55)$$

This inevitably leads us to the fact that

$$X_\lambda X_\lambda^\dagger = \alpha I, \quad (56)$$

with $\alpha > 0$.
The question remains what happens if the set of unitary operators U_i is not irreducible. It is shown in the following section that in special cases it may still be possible to decompose the Hilbert space into so-called minimal invariant subspaces for which the condition of irreducibility of unitary operators U_i still holds.

7. Asymptotic dynamics of a two-qubit CNOT-system

In this section we discuss the asymptotic dynamics of the RUO

$$\Phi(\rho) = p_1 C_1 \rho C_1 + (1 - p_1) C_2 \rho C_2, \quad (57)$$

which involves two controlled-not (CNOT) operations acting on two qubits. In the computational basis $\{|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\}$ of the two-qubit Hilbert space $\mathscr{H}_2$ the action of these CNOTs is defined by

$$C_1|i,j\rangle = |i, i\oplus j\rangle, \qquad C_2|i,j\rangle = |i\oplus j, j\rangle, \tag{58}$$

with $\oplus$ denoting addition modulo 2. This special RUO of Eq. (57) is a hermitian operator and therefore its only possible eigenvalue lying within $\sigma_{|1|}$ are 1 and −1. Let us first find a decomposition of the two-qubit Hilbert space $\mathscr{H}_2$ into subspaces $\mathscr{V}_x$, *i.e.*

$$\mathscr{H}_2 = \bigoplus_x \mathscr{V}_x, \tag{59}$$

within each of which the set of unitary operators C_1, C_2 acts irreducibly. Constructing such a decomposition is equivalent to constructing a decomposition of the finite multiplicative unitary group C generated by C_1 and C_2. The unitary group C is naturally a unitary representation I_C of itself. Therefore, the following considerations are immediate consequences of the standard theory of representations of finite groups [16].

The unitary group C contains six elements divided into three conjugated classes: $K_1 \equiv \{\text{identity element } I\}$, $K_2 \equiv \{C_1C_2, C_2C_1\}$, $K_3 \equiv \{C_1, C_2, C_1C_2C_1\}$. The characters of the representation I_C corresponding to these classes are $\chi_1 = 4$, $\chi_2 = 1$ and $\chi_3 = 2$. Thus, there are only three inequivalent irreducible representations of the group C, say D^μ ($\mu = \{1,2,3\}$), with dimensions n_μ satisfying the relation

$$\sum_{\mu=1}^{3} n_\mu^2 = 6. \tag{60}$$

Hence, there are two one-dimensional and one two-dimensional inequivalent irreducible representations of the group C. The reducible representation I_C can be expressed in terms of irreducible representations as

$$I_C(g) = \sum_{\mu=1}^{3} a_\mu D^\mu(g), \tag{61}$$

where a_μ are positive or zero integers and fulfil the relation

$$\sum_{\mu=1}^{3} a_\mu^2 = \frac{1}{g}\sum_{i=1}^{3} g_i = |\chi_i|^2 = 5 \tag{62}$$

with g and g_i denoting the number group elements and the number of elements of the conjugated class K_i, respectively. The only possibility to satisfy the dimensionality of the representation I_C and Eq. (62) is the solution: $a_1 = 1$ for the two-dimensional irreducible representation, $a_2 = 2$ for the one-dimensional irreducible representation, the second one-dimensional irreducible representation cannot be involved in the decomposition, *i.e.* $a_3 = 0$. Two one-dimensional representations contained in the irreducible decompositions (61) mean that there are just two common eigenvectors for the unitary group C and thus common eigenvectors of operators C_1 and C_2. ¿From the definition (58) it is clear that these eigenvectors are $e_1 = |00\rangle$ and $e_4 = (|01\rangle + |10\rangle + |11\rangle)/\sqrt{3}$. Subsequently, we know that the minimal invariant subspaces of operators C_1 and C_2 are: $V_1 = \text{span}(e_1)$, $V_2 = \text{span}\left(e_2 = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), e_3 = \frac{1}{\sqrt{6}}(|01\rangle + |10\rangle - 2|11\rangle)\right)$ and $V_3 = \text{span}(e_4)$. If we denote the restriction of the operator C_i to the subspace V_x as $C_i^{(x)}$, in the orthonormal basis system $\{e_i\}_{i=1}^4$ the operators C_1 and C_2 correspond to the matrices

$$C_1 = \begin{pmatrix} C_1^{(1)} & 0 & 0 \\ 0 & C_1^{(2)} & 0 \\ 0 & 0 & C_1^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} C_2^{(1)} & 0 & 0 \\ 0 & C_2^{(2)} & 0 \\ 0 & 0 & C_2^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{63}$$

Writing the general commutation relations (4.1) in the block structure form we obtain for $i \in \{1,2\}$

$$C_i^{(m)} X^{(mn)} = \lambda X^{(mn)} C_i^{(n)} \tag{64}$$

with the 1×1- matrices $X^{(11)}, X^{(13)}, X^{(31)}, X^{(33)}$, with the 2×2 matrix $X^{(22)}$, with the 1×2-matrices $X^{(12)}$ and $X^{(32)}$, and with the 2×1 matrices $X^{(21)}$ and $X^{(23)}$. Using Eqs. (64) one can check

$$\begin{aligned} C_i^{(n)}\left(\text{Ker}\left(X^{(mn)}\right)\right) &\subset \text{Ker}\left(X^{(mn)}\right), \\ C_i^{(m)}\left(\text{Ran}\left(X^{(mn)}\right)\right) &\subset \text{Ran}\left(X^{(mn)}\right) \end{aligned} \tag{65}$$

and thus $X^{(mn)}$ is either the zero operator or an invertible operator. Hence, $X^{(12)}, X^{(32)}, X^{(21)}, X^{(23)}$ are inevitably zero matrices.

Now, assume the case $\lambda = 1$. A simple evaluation of Eq. (64) leads to the relations $X^{(11)} = a$, $X^{(33)} = b$, $X^{(13)} = c$, and $X^{(31)} = d$ ($a, b, c, d \in \mathbb{C}$). The remaining matrix block $X^{(22)}$ has to commute with the irreducible set of 2×2 matrices $C_i^{(2)}$ ($i \in \{1,2\}$) and has to be equal to a multiple of the identity matrix $X^{(22)} = eI$ ($e \in \mathbb{C}$).

The eigenspace of the random unitary operation (57) corresponding to eigenvalue 1 is five-dimensional and the most general eigenvector reads

$$X_1 = \begin{pmatrix} a & 0 & 0 & c \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ d & 0 & 0 & b \end{pmatrix}. \tag{66}$$

The solution of Eq. (64) with $\lambda = -1$ yields $X^{(11)} = X^{(13)} = X^{(31)} = X^{(33)} = 0$. The last matrix block $X^{(22)}$ is determined by anticommutation relations with the irreducible set of operators $C_i^{(2)}$ $(i \in \{1, 2\})$, *i.e.*

$$C_i^{(2)} X^{(22)} = -X^{(22)} C_i^{(2)} X^{(22)}. \tag{67}$$

From the discussion in Sec. 6 and the Eq. (52) follows that $X^{(22)}$ is either the zero operator or

$$X^{(22)} = f\left(P - C_i^{(2)} P C_i^{(2)}\right), \qquad f \in \mathbb{C}, \tag{68}$$

with the projection P being determined by the equation

$$C_1^{(2)} C_2^{(2)} P = P C_1^{(2)} C_2^{(2)}. \tag{69}$$

Hence, the projection operator P is diagonal in the eigenbasis of the operator $C_1^{(2)} C_2^{(2)}$. Using Eq. (68) the most general form of the matrix block $X^{(22)}$ corresponding to eigenvalue -1 reads

$$X^{(22)} = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}, \qquad f \in \mathbb{C}. \tag{70}$$

Thus, the eigenspace of the random unitary operation (57) corresponding to eigenvalue -1 is one-dimensional and the general eigenvector reads

$$X_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & -f & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{71}$$

Therefore, in the computational basis the attractor space is spanned by the matrices

$$X_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{1,2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$X_{1,3} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{1,4} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{72}$$

$$X_{1,5} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{-1,1} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

In this notation the first index refers to the eigenvalues of the RUO (57) and the second index runs through the basis states of the corresponding eigenspaces.

Finally, consider the most general two-qubit input density matrix

$$\rho(0) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12}^* & a_{22} & a_{23} & a_{24} \\ a_{13}^* & a_{23}^* & a_{33} & a_{34} \\ a_{14}^* & a_{24}^* & a_{34}^* & a_{44} \end{pmatrix}.$$

Thus, theorem 4.2 implies that the asymptotic dynamics under the RUO (57) is periodic with period two and is determined by the relations

$$\lim_{n\to+\infty} \rho(2n) = \begin{pmatrix} a & c & c & c \\ c^* & b & d & d^* \\ c^* & d^* & b & d \\ c^* & d & d^* & b \end{pmatrix},$$
$$\lim_{n\to+\infty} \rho(2n+1) = \begin{pmatrix} a & c & c & c \\ c^* & b & d^* & d \\ c^* & d & b & d^* \\ c^* & d^* & d & b \end{pmatrix} \tag{73}$$

with $a = a_{11}$, $b = \frac{1}{3}(a_{22}+a_{33}+a_{44})$, $c = \frac{1}{3}(a_{12}+a_{13}+a_{14})$, $d = \frac{1}{3}(a_{23} + a_{34} + a_{24}^*)$. This two-qubit CNOT network is one of the simplest examples of a network allowing for oscillatory asymptotic dynamics.

8. Conclusions and outlook

We studied general properties of random unitary operations and presented several theorems allowing to determine the asymptotic long time dynamics. Thereby, a central result is the structure theorem which states that the asymptotic states are located completely inside the vector space spanned by a typically small set of attractors. The form of these asymptotic quantum states depends on this attractor space and on the choice of the initial state but is independent of the actual values of the probabilities with which the unitary transformations are applied. However, these probabilities affect the rate of the convergence towards the asymptotic quantum state.

It should be stressed that the asymptotic dynamics need not result in a stationary state. Thus, in contrast to thermalization the asymptotic dynamics might also be periodic as illustrated by the example of two qubits interacting by random C-NOT operations. Even an aperiodic non-stationary asymptotic dynamics is possible.

The obtained results rise several additional questions. First of all, it is not yet clear what determines the convergence rate of a quantum system towards its asymptotic dynamics. Numerical studies suggest that in many cases this convergence has an exponential character which depends on the probabilities with which the unitary operations are applied. Preliminary results also suggest that at least in the case of many-qubit networks involving controlled-not operations the topology of the network is related to the set of attractors.

Finally, it should also be mentioned that our results might have applications for quantum operations which involve an averaging procedure over a group, such as twirling operations. Our results might allow to choose efficiently the minimal set of unitary transforms leading to a particular asymptotic state. In addition, we expect that the theory presented might also contribute to other related problems concerning the determination of eigenvectors of random unitary maps [17] or their application in purification protocols [18].

Acknowledgements

Financial support by the Czech Ministry of Education (by MSM 6840770039 and MŠMT LC 06002), by the Alexander von Humboldt Foundation, by the DAAD, and by CASED is gratefully acknowledged.

Appendix A: Jordan canonical form

Let us recall the definition and properties of the Jordan canonical form of square matrices. Consider a complex square matrix $A = (A_{ij})_{i,j=1}^n$ of size $n \times n$ ($A \in \mathbb{C}^{n\times n}$). It is similar to a block diagonal matrix

$$A = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{pmatrix}, \tag{A1}$$

in which each Jordan block J_i ($i \in \hat{p} := \{1, ..., p\}$) is given by

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}. \tag{A2}$$

Thus, there is an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that $A = PJP^{-1}$ or equivalently there is a Jordan basis $x_{i,\alpha} \in \mathbb{C}^n$ ($i \in \hat{p}$, $\alpha \in \{1, 2, \ldots, \dim(J_i)\}$) in which the linear map corresponding to the matrix A has the diagonal form (A1). In general, this basis is non-orthogonal and the vectors $x_{i,\alpha}$ ($\alpha \in \{1, 2, \ldots, \dim(J_i)\}$) form the basis of the Jordan block J_i which corresponds to the eigenvalue λ_i of the matrix A. The geometric multiplicity of the eigenvalue λ_i is the number of Jordan blocks corresponding to λ_i and the sum of the sizes of all Jordan blocks corresponding to an eigenvalue λ_i is its algebraic multiplicity. Therefore, the matrix A is diagonalizable if and only if all Jordan blocks are one dimensional. In all other cases any Jordan block, say J_i, with dimension $s > 1$ gives rise to a Jordan chain. This means that there is a so-called lead vector or generator, say $x_{i,\dim(J_i)}$, which is a generalized eigenvector, *i.e.* $(A - \lambda_i I)^s x_{i,\dim(J_i)} = 0$. The vector $x_{i,1} = (A - \lambda_i I)^{s-1} x_{i,\dim(J_i)}$ is an eigenvector corresponding to the eigenvalue λ_i. In general, the vector $x_{i,j}$ is the image of the vector $x_{i,j+1}$ under the linear map $A - \lambda_i I$. In this sense all vectors $x_{i,\alpha}$ ($\alpha \in \{2, 3, \dots, \dim(J_i)\}$) are generalized eigenvectors of the matrix A.

Therefore, for every square matrix A there exists a basis consisting only of eigenvectors and generalized eigenvectors of the matrix A in which the matrix A can be put in Jordan normal form (A1).

Appendix B: Dynamical matrices

Let us summarize the concept of *dynamical matrices* which is useful to understand problems related to complete positivity of maps. We just recall its definition and present a short summary of characteristic properties needed in the main body of our text. Detailed proofs are given in Ref. [7], for example.

Assume that A is an operator acting on a d-dimensional Hilbert space $\mathscr{H}_d$. Hence A_{ij} ($i, j \in \hat{d} := \{1, ..., d\}$) are its matrix elements with respect to a given orthonormal basis. It is convenient to interpret a $d \times d$-matrix $(A)_{ij}$ as a vector $\boldsymbol{A} = (A_{m\mu}) \in \mathscr{H}_{d^2}$ of the length d^2

$$\boldsymbol{A} = (A_{11}, A_{12}, \dots, A_{1d}, A_{21}, A_{22}, \dots, A_{2d}, \dots, A_{d1}, A_{d2}, \dots, A_{dd}). \quad \text{(B1)}$$

One can check that two $d \times d$ matrices A and B fulfil

$$\langle A, B \rangle \equiv \mathrm{Tr}\{A^\dagger B\} = \boldsymbol{A}^* \boldsymbol{B} = \langle \boldsymbol{A}, \boldsymbol{B} \rangle. \quad \text{(B2)}$$

The vector $\boldsymbol{A}$ of the length d^2 may be linearly transformed into the vector $\boldsymbol{A}' = C\boldsymbol{A}$ by a matrix C of size $d^2 \times d^2$ whose matrix elements may be denoted by $C_{kk'}$ with $k, k' = 1, \dots, d^2$. In addition, it is also convenient to use a four index notation $C^{m\mu}_{n\nu}$ with respect to a two index notation of vectors (B1) with $m, n, \mu, \nu = 1, \dots, d^2$. The matrix C may represent an operator acting in a composite Hilbert space $\mathscr{H} = \mathscr{H}_d \otimes \mathscr{H}_d$. The tensor product of any two orthonormal basis systems in both factors provides a basis in $\mathscr{H}$ so that we obtain

$$C^{m\mu}_{n\nu} = \langle e_m \otimes f_\mu | C | e_n \otimes f_\nu \rangle \quad \text{(B3)}$$

with Latin indices referring to the first subsystem, $\mathscr{H}_A = \mathscr{H}_d$, and Greek indices to the second subsystem, $\mathscr{H}_B = \mathscr{H}_d$. The operation of partial trace over the second or first subsystem produces the $d \times d$ matrices $C^A \equiv \mathrm{Tr}_B\, C$ or $C^B \equiv \mathrm{Tr}_A\, C$, respectively, *i.e.*

$$C^A_{mn} = \sum_{\mu=1}^{d^2} C^{m\mu}_{n\mu}, \qquad \text{and} \quad C^B_{\mu\nu} = \sum_{m=1}^{d^2} C^{m\mu}_{m\nu}. \quad \text{(B4)}$$

If $C = A \otimes B$, then $C^{m\mu}_{n\nu} = A_{mn} B_{\mu\nu}$. The standard product of three matrices can be rewritten in the following useful form

$$ABC = \Phi \boldsymbol{B} \qquad \text{with} \qquad \Phi = A \otimes C^T. \quad \text{(B5)}$$

With the help of identity (B5) we can rewrite the RUO (1) in the form

$$\Phi = \sum_{i=1}^{m} p_i U_i \otimes U_i^*. \quad \text{(B6)}$$

Here, the RUO Φ is not understood as a map acting on the $d \times d$-dimensional matrix space but as a map acting on the vector space of the dimension d^2.

Let Φ be a completely positive trace-preserving map mapping an arbitrary $d \times d$ density matrix $\rho \in \mathcal{B}(\mathscr{H}_d)$ of a d-dimensional Hilbert space $\mathscr{H}_d$ on a density matrix $\rho' \in \mathcal{B}(\mathscr{H}_d)$, *i.e.*

$$\rho' = \Phi\rho \qquad \text{or} \qquad \rho'_{m\mu} = \sum_{n,\nu=1}^{d^2} \Phi^{m\mu}_{n\nu} \rho_{n\nu}. \quad \text{(B7)}$$

The meaning of complete positivity becomes rather transparent if we reshuffle Φ and define the *dynamical matrix* D_Φ

$$(D_\Phi)^{mn}_{\mu\nu} = \Phi^{m\mu}_{n\nu}. \quad \text{(B8)}$$

The dynamical matrix D_Φ uniquely determines the map Φ and has the following properties

$$\begin{array}{llcl} \text{(i)} & \rho' = (\rho')^\dagger & \Leftrightarrow & D_\Phi = D_\Phi^\dagger \\ \text{(ii)} & \mathrm{Tr}\,\rho' = 1 & \Leftrightarrow & \mathrm{Tr}_A\, D_\Phi = I \\ \text{(iii)} & \Phi(I) = I \quad \text{(unital)} & \Leftrightarrow & \mathrm{Tr}_B\, D_\Phi = I \\ \text{(iv)} & \Phi \quad \text{is CP map} & \Leftrightarrow & D_\Phi \quad \text{is positive.} \end{array} \quad \text{(B9)}$$

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