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Suppressing systematic control errors to high orders

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Dynamical decoupling is a powerful method for protecting quantum information against unwanted interactions with the help of open-loop control pulses. Realistic control pulses are not ideal and may introduce additional systematic errors. We introduce a class of self-stabilizing pulse sequences capable of suppressing such systematic control errors efficiently in qubit systems. Embedding already known decoupling sequences into these self-stabilizing sequences offers powerful means to achieve robustness against unwanted external perturbations and systematic control errors. As these self-stabilizing sequences are based on single-qubit operations, they offer interesting perspectives for future applications in quantum information processing.

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I. INTRODUCTION

Realistic systems implementing quantum information processing (QIP) are affected by various sources of errors. If no precautions are taken, the errors cumulate with time, destroying quantum information and thus preventing scalability. Merely increasing the experimental precision is not enough to achieve scalability, as there are fundamental limits on precision, so that it cannot be increased indefinitely. Nevertheless, it is possible to design the process itself in a way that it *protects* the information from the effects of error sources, without the need to increase experimental precision.

A powerful method to suppress errors is called *dynamical decoupling* (DD). The idea of DD is to perform a sequence of unitary operations on the system which effectively rotate the quantum state of the system in its state space. The action of an error caused by a Hamiltonian interaction depends on the current rotated frame, and if the applied rotations are chosen carefully, the error can be made to cancel itself up to a certain order. DD is a generalization of techniques used in the nuclear magnetic resonance community since the discovery of the Hahn spin echo in 1950 [1]; a general framework was formulated in [2,3]. Numerous results based on this framework followed, including experimental realizations. An efficient decoupling scheme for systems of pairwise interacting qubits was proposed in [4], which uses orthogonal arrays [5]. A similar strategy based on Hadamard matrices was described in [6]. Both approaches were shown to be equivalent in [7]. DD works perfectly only in the limit of infinitely fast control. For finite control frequencies, a residual error is present. This residual error can be further suppressed by higher-order decoupling schemes. Second-order decoupling can be achieved by using palindromic control sequences. However, finding efficient third- and higher-order schemes in the general (multiqubit) case is difficult. There is ongoing progress in higher-order decoupling of a single qubit (see, e.g., [8–11]). Additionally, randomized decoupling strategies have been proposed [12–14], which can offer certain advantages in the asymptotic time behavior of a protected system's fidelity compared to deterministic sequences. DD does not need any auxiliary qubits, and loosely speaking, it is applicable when the state of the neighborhood influencing the system does not change too fast. A comprehensive account on DD techniques is included in [15].

The above-mentioned DD techniques typically assume that the available control actions are perfect and instantaneous (bang-bang control), which is an assumption that cannot hold in practice. In a more realistic treatment, DD controls will take a certain time to execute, which introduces errors since the implementation of the controls does not generally commute with the acting system Hamiltonian. A method to suppress such errors to the first order, called Eulerian decoupling, is described in [16]. Another important source of errors is imperfections in the devices implementing the controls, leading to nonideal controls, which is the kind of error we are going to address.

In this paper, we introduce a class of single-qubit control sequences called self-stabilizing (SS) sequences that suppress systematic control errors to any desired order up to 11th order, conjecturing that arbitrarily high orders are achievable. This is a significant improvement over the first-order pulse error suppression achieved by single-qubit Eulerian decoupling [16] when systematic pulse error suppression is the criterion of comparison. The first-order SS sequence turns out to be identical to the one used in single-qubit Eulerian decoupling. Systematic control errors appear in scenarios such as NMR [17] or trapped electron or ion quantum computing, so our technique is applicable in realistic situations.

The paper is outlined as follows. In Sec. II we review DD basics. Then the construction of SS sequences is described in Sec. III. In Sec. IV we propose embedding other decoupling sequences into self-stabilizing sequences as a means to achieve robustness against systematic control errors. In Secs. V and VI we demonstrate the benefits of this approach in two important cases. Finally, in Sec. VII we recapitulate the presented results and outline future work directions. Throughout the article, we set $\hbar = 1$ and use dimensionless time.

II. DYNAMICAL DECOUPLING

Let us briefly review the foundations of dynamical decoupling with perfect controls, taking control imperfections into consideration later. We will formulate dynamical decoupling

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for qubit systems controlled by Pauli operations; decoupling of other systems with other operations as controls is analogous.

A. Decoupling scenario

We consider decoupling a target system S consisting of n qubits coupled to each other as well as to an environment or bath B. The Hilbert space of the environment is assumed to be of finite dimension. The intrinsic evolution of the joint system SB is assumed to be unitary and given by a time-independent Hamiltonian H. The environment B is understood as uncontrollable, while each target qubit can be controlled individually by applying local pulses at certain times. We approximate the pulses by instantaneous unitary operators and restrict these to Pauli operations. We will use $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ to denote the Pauli operations, with σ_0 denoting the identity operator.

We suppose that the control pulses are applied in parallel, which is well defined as pulses applied to different qubits commute with each other. Parallel pulses will be specified by n-tuples of Pauli operation indices called operation multi-indices. The pulse corresponding to the multi-index α can be written as

$$P(\alpha) = \prod_{j=0}^{n-1} \sigma_{\alpha_j}^{(j)}. \tag{1}$$

Here and in the rest of the article, the upper index (j) denotes the target qubit. We index the n qubits starting from zero. For ease of exposition, we will occasionally refer to the multi-indices themselves as operations.

We assume the interpulse interval Δt to be constant, but this restriction is not essential. The evolution of the joint system SB is then

$$U(A,\Delta t H) = \prod_{\alpha \in A} \exp(-i\Delta t H)P(\alpha), \tag{2}$$

where A is a sequence of operation multi-indices, also called the (multiqubit) control sequence. Here and throughout the article, the product notation implies right-to-left ordering.

B. Decoupling conditions

The evolution given by a control sequence A can be separated into the ideal and nonideal parts

$$U(A,\Delta t H) = \exp[\Phi_B(A,\Delta t H) + \Phi_{\text{Error}}(A,\Delta t H)]U(A,0),$$

where $\Phi_B(A,\Delta t H)$ are all terms that affect solely the bath and $\Phi_{\text{Error}}(A,\Delta t H)$ are all the other terms. A control sequence A decouples S from the effects of H to the ω_Hth order if

$$\Phi_{\text{Error}}(A,\Delta t H) \in O(\Delta t^{\omega_H+1}). \tag{3}$$

The higher-order errors can, in theory, be made arbitrarily small by choosing a very small time distance Δt between two consecutive pulses. However, if the control pulses are not ideal, Δt cannot be made arbitrarily small without actually introducing errors, as the smaller the Δt is, the more pulses are performed per unit time, leading to accumulation of pulse errors.

III. SELF-STABILIZING SEQUENCES

Dynamical decoupling is based on open-loop control of the system. In the presence of operation nonidealities, the control introduces errors into the system, possibly negating the benefits of DD. Dealing with these errors in general is difficult, but the systematic component of such errors is a good candidate for suppression using DD-like mechanisms, by proper choice of the control sequence itself. Sequences that suppress their own systematic control errors will be called self-stabilizing sequences. This section focuses on the self-stabilization alone. Subsequent sections investigate how SS sequences can be used to add robustness against systematic control errors to other decoupling schemes.

We specify our control and control error model which we use to state formally the conditions that define self-stabilizing sequences of a particular order. At the end of this section we present a heuristic we used to find self-stabilizing sequences of up to 11th order.

Many decoupling schemes are based on sequences of Pauli operations, so we restrict ourselves to this kind of operations. In the following, we present our model for nonideal controls, which introduces a systematic error to the Pauli pulses. We assume that each Pauli operation except σ_0 is affected by a systematic single-qubit unitary error [18]. The errors of different Pauli operations are allowed to be different. The errors are systematic; that is, they are the same for different applications of the same operation. The ideal operation σ_k will be thus replaced as follows:

$$\sigma_k \rightarrow P_{1\text{q}}(k,q) = \exp[-iq_k]\sigma_k, \quad k \in \{0,1,2,3\}, \tag{4}$$

where

$$q \equiv (q_0,q_1,q_2,q_3) \in \left\{ \sum_{l=1}^{3} x_l\sigma_l : x_l \in \mathbb{R} \right\}^4 \tag{5}$$

is a quadruple holding the systematic errors of the individual Pauli operations, each representing a small rotation of the qubit's state in the Bloch sphere. As already said, q_0, the error of the identity operation, is assumed to be zero.

Let $d = (d_0,d_1,\ldots)$ be a finite sequence of Pauli operation indices, also called a single-qubit control sequence. The corresponding nonideal evolution is

$$U_{1\text{q}}(d,q) = \prod_{k\in d} P_{1\text{q}}(k,q) = \prod_{k\in d} \exp[-iq_k]\sigma_k. \tag{6}$$

This evolution is similar to the one described by Eq. (2), so it is tempting to simply use known decoupling sequences to suppress the systematic control error. However, this does not work as there is an important difference: The errors in (6) are not fixed, but they depend on the choice of the control sequence itself, necessitating new sequence construction schemes.

The evolution (6) can be separated into the ideal and nonideal parts,

$$U_{1\text{q}}(d,q) = \exp[\Phi_{1\text{q,Error}}(d,q)]U_{1\text{q}}(d,0).$$

The condition for ω_cth-order control error suppression is then as follows: For any realization of the systematic control errors q, the overall error satisfies

$$\Phi_{1\text{q,Error}}(d,\lambda_c q) \in O\left(\lambda_c^{\omega_c+1}\right), \tag{7}$$

TABLE I. Examples of self-stabilizing sequences. For nonideal operations, such sequences are associated with reduced overall error, provided the nonidealities are systematic in nature. Errors at different positions acquire different signs. An ω_cth-order self-stabilizing sequence uses these sign changes to achieve error cancellation of all errors up to ω_cth order.

Order ω_r	Self-stabilizing sequence	Length
1	$(\sigma_1,\sigma_2,\sigma_1,\sigma_2,\sigma_2,\sigma_1,\sigma_2,\sigma_1)$	8
2	$(\sigma_1,\sigma_1,\sigma_1,\sigma_2,\sigma_1,\sigma_1,\sigma_2,\sigma_1,$ $\sigma_2,\sigma_2,\sigma_2,\sigma_1,\sigma_2,\sigma_2,\sigma_1,\sigma_2)$	16
3	$(\sigma_2,\sigma_2,\sigma_1,\sigma_2,\sigma_1,\sigma_2,\sigma_1,\sigma_1,$ $\sigma_2,\sigma_1,\sigma_2,\sigma_2,\sigma_1,\sigma_1,\sigma_2,\sigma_1,$ $\sigma_2,\sigma_2,\sigma_1,\sigma_2,\sigma_1,\sigma_1,\sigma_2,\sigma_1,$ $\sigma_2,\sigma_1,\sigma_2,\sigma_2,\sigma_1,\sigma_2,\sigma_1,\sigma_1)$	32

where λ_c is a scaling factor of the control errors q. A sequence d that satisfies this condition will be called an ω_cth-order self-stabilizing sequence. For $\omega_c \leqslant 3$, solutions can be obtained in a straightforward manner using an exhaustive automated search in the space of progressively longer sequences. For simplicity, we restrict the search to sequences that use only σ_1 and σ_2 operations. For higher orders, an exhaustive search is not tractable, but many solutions of order $\omega_c + 1$ can be obtained by searching the space of sequences formed by concatenation of two different solutions of order ω_c. When one solution is found, other solutions of the same order can be generated by cyclical permutations, sequence reversal, and swapping of σ_1 and σ_2. Iterating this procedure, we were able to obtain solutions of order $\omega_c = 11$. Higher orders were not attempted due to memory constraints of our hardware. We conjecture that SS sequences of arbitrary order exist. Some solutions for the first couple of orders are in Table I.

IV. ROBUST DECOUPLING IN THE PRESENCE OF SYSTEMATIC CONTROL ERRORS

We propose embedding of Pauli pulse-based decoupling sequences into self-stabilizing sequences as a means to achieve decoupling that is robust against systematic control errors. We consider decoupling of n qubits coupled to an environment, which is a scenario of considerable interest in quantum information processing. The notation is analogous to the one introduced in Sec. II A, but modified to take the control errors into consideration.

A. Decoupling scenario

We consider the scenario described in Sec. II A, but with nonideal Pauli pulses of the kind given by (4). The errors on different qubits are allowed to be different. Parallel application of pulses is still well defined as pulses applied to different qubits commute with each other even in the nonideal case.

The nonideal pulse corresponding to the multi-index α can be written as

$$P(\alpha,Q) = \prod_{j=0}^{n-1} P_{1q}(\alpha_j,Q_j)^{(j)} = \prod_{j=0}^{n-1}\{\exp[-i(Q_j)_{\alpha_j}]\sigma_{\alpha_j}\}^{(j)}.$$

Here and in the rest of the article, $(Q_j)_k$ specifies the control error associated with the pulse σ_k on the jth qubit [see (5)].

The evolution of the joint system SB in the presence of systematic control errors is then

$$U(A,Q,\Delta t H) = \prod_{\alpha\in A} \exp(-i\Delta t H) P(\alpha,Q),$$

where A is a control sequence.

B. Robust decoupling conditions

Robust decoupling is ordinary decoupling with the additional property that each qubit experiences self-stabilizing control. We will state the corresponding conditions formally.

The evolution given by a control sequence A can be separated into the ideal and nonideal parts,

$$U(A,Q,\Delta t H) \\ = \exp[\Phi_B(A,Q,\Delta t H) + \Phi_{\mathrm{Error}}(A,Q,\Delta t H)]U(A,0,0),$$

where $\Phi_B(A,Q,\Delta t H)$ are all terms that affect only the bath and $\Phi_{\mathrm{Error}}(A,Q,\Delta t H)$ are all the other terms. A control sequence A decouples S from the effects of H to the ω_Hth order if

$$\Phi_{\mathrm{Error}}(A,0,\Delta t H) \in O(\Delta t^{\omega_H+1}), \tag{8}$$

which is essentially the same as (3). In order for a control sequence A to be robust against control errors, the control applied to any given qubit has to fulfill the self-stabilization condition (7): For any realization of control errors Q,

$$\forall\, j \in \{0,\ldots,n-1\}: \\ [\Phi_{1q,\mathrm{Error}}((A_:)_j,\lambda_c Q_j) \in O(\lambda_c^{\omega_c+1})], \tag{9}$$

where $(A_:)_j = ((A_0)_j,(A_1)_j,\ldots)$, i.e., the single-qubit control sequence for the jth qubit.

C. Robust decoupling by embedding into self-stabilizing sequences

We will now construct a robust decoupling sequence by embedding a "regular" decoupling sequence into a self-stabilizing sequence.

Let D_H be a control sequence that decouples S to the order ω_H, i.e.,

$$\Phi_{\mathrm{Error}}(D_H,0,\Delta t H) \in O(\Delta t^{\omega_H+1}).$$

The first operation in D_H can be chosen arbitrarily without disturbing the decoupling condition, as H has not yet had any effect at the very beginning of the evolution. We will use this freedom to perform the embedding.

The overall ideal evolution of a particular qubit is the product of all the Pauli pulses applied to this qubit. The Pauli operations form a group up to a phase, so by the proper choice of the first operation in D_H, the overall ideal evolution on any qubit can be set to any desired Pauli operation. The whole pulse train can then be regarded as a higher-level implementation of any desired Pauli operation. In the nonideal case, the higher-level operations are nonideal too, but they fit the same error model as the low-level ones. The sequence D_H with the proper choice of the first operation therefore can be regarded as any desired nonideal parallel "pulse." A concatenation of such alterations of D_H can be made to mimic any sequence of parallel operations, including one where each single-qubit sequence is a self-stabilizing one (provided the single-qubit sequences are of the same length).

It is easy to show that if the higher-level sequence is self-stabilizing, so is the underlying sequence. Moreover, a concatenation of decoupling sequences is decoupling too. Therefore, the sequence created by concatenating altered copies of D_H as described above is both decoupling and self-stabilizing, thus achieving robust decoupling.

D. Mixed-term considerations

The set of single-qubit conditions (9) can also be written as

$$\Phi_{\text{Error}}(A,\lambda_c Q,0) \in O\left(\lambda_c^{\omega_c+1}\right). \tag{10}$$

The pair of conditions (8) and (10) is equivalent to the condition

$$\Phi_{\text{Error}}(A,\lambda_c Q,\Delta t H) \in O(\Delta t^{\omega_H+1}) + O\left(\lambda_c^{\omega_c+1}\right) + O(\Delta t \lambda_c). \tag{11}$$

The mixed term $O(\Delta t \lambda_c)$ is difficult to eliminate. Its impact on the overall performance of the decoupling scheme depends on several factors. The mixed term can be suppressed by reducing Δt. On the other hand, reducing Δt also increases the number of pulses per unit time, which actually amplifies the effect of this term. It turns out that for periodic decoupling, the effects roughly cancel each other, so further Δt reduction does not help performance. However, as shown in Secs. V and VI, decoupling schemes involving concatenation or randomization have the capacity to further reduce the impact of the mixed term, provided the self-stabilization order ω_c is high enough.

E. Example

We demonstrate the embedding process using specific inner and outer sequences. Let us consider a system of N qubits governed by a Hamiltonian with pairwise interactions; that is, any qubit can potentially be coupled to any other qubit, but there are no terms in the Hamiltonian involving more than two qubits. There exist decoupling sequences which can decouple these pairwise interactions; they were introduced in [4] along with a construction method based on orthogonal arrays [5]. For $N = 5$, one possible sequence to decouple these pairwise interactions consists of 16 control operations and looks like this:

$$D_H = \begin{pmatrix} \bullet & \\ \bullet & \\ \bullet & D'_H \\ \bullet & \\ \bullet & \end{pmatrix} = \begin{pmatrix} \bullet & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ \bullet & 1 & 0 & 1 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 2 \\ \bullet & 1 & 1 & 1 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 2 \\ \bullet & 2 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 2 & 0 \\ \bullet & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

Here, D_H is the sequence of multi-indices indicating the control pulses to apply. Each column of the matrix therefore corresponds to a single multi-index α and indicates a tensor product of Pauli operators as given by Eq. (1). We have freedom of choice when it comes to the first pulse, which is why the first column of D_H is intentionally left blank, as represented by the symbol $\bullet$. D'_H is D_H without this first arbitrary operation. The product of all the Pauli tensor-product operators represented by the columns in D'_H is (up to a global phase) the operator given by the multi-index

$$P_{D'_H} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}. \tag{12}$$

We would like to make this decoupling sequence robust against errors in the controls. For this purpose, we must embed the sequence into a self-stabilizing outer sequence. One possibility for the outer sequence is

$$D_r = \begin{pmatrix} 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 3 & 2 & 3 & 3 & 2 & 3 & 2 \\ 1 & 3 & 1 & 3 & 3 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \end{pmatrix}. \tag{13}$$

Each row in the matrix D_r represents a variant of the self-stabilizing sequence of order $\omega_r = 1$ given in Table I. Although we could use the same sequence in each row, using these variants will yield an additional beneficial property in the final sequence.

Multiplying each of the operators represented by the columns of D_r by the operator $P_{D'_H}$ yields the new sequence

$$F = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 \\ 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \end{pmatrix}. \tag{14}$$

The final combined sequence uses the columns in F as the first operation in the original sequence D_H, which is accordingly repeated eight times. The final sequence is thus given as

$$D_{H,r} = \begin{pmatrix} 1 & & 1 & & & 1 & & 2 & \\ 0 & & 1 & & & 1 & & 0 & \\ 0 & D'_H & 2 & D'_H & \cdots & 2 & D'_H & 0 & D'_H \\ 2 & & 1 & & & 1 & & 2 & \\ 1 & & 2 & & & 2 & & 1 & \end{pmatrix}. \tag{15}$$

This sequence is the result of embedding the original decoupling sequence in a self-stabilizing sequence. Like the original sequence, it decouples pairwise interactions between the qubits, but it is now robust against errors in the decoupling controls. Due to the particular choice of D_r, this sequence has the additional property of containing only two kinds of operations, σ_1 and σ_2. This property may be useful in experimental settings where only two Pauli operations are implemented.

V. DECOUPLING A SINGLE QUBIT FROM ITS ENVIRONMENT BY NONIDEAL CONTROLS

We will now focus on decoupling a single qubit coupled to a bath modeled as several uncontrollable interacting qubits. It is shown that embedding the concatenated dynamical decoupling (CDD) sequence [8] into a self-stabilizing sequence of sufficiently high order dramatically reduces errors due to systematic control nonidealities when compared to CDD alone.

CDD applied to a single qubit is based on sequences generated by repeatedly embedding the sequence $\sigma_1,\sigma_2,\sigma_1,\sigma_2$ into itself. Each level of embedding attempts to reduce the residual error of the previous one. With ideal pulses, this approach works very well. With nonideal pulses, the errors of the last level always remain uncorrected, limiting the achievable fidelity. By using a self-stabilizing sequence for the last level of embedding, this limitation is overcome. These considerations are supported by simulations.

The simulation results are shown in Fig. 1. We simulated a single qubit coupled to a bath modeled as five qubits. This six-qubit system was subject to time-independent pairwise couplings with coupling coefficients chosen from a zero-centered normal distribution with $\sigma = 1$. Performance was measured by approximating the average fidelity F_{avg} by averaging over ten randomly chosen initial qubit states and ten randomly chosen initial bath states. The plot shows the infidelity $1 - F_{\text{avg}}$ vs sequence duration for a fixed Δt. With ideal pulses, CDD performs very well. For fixed Δt, there is an optimal CDD concatenation level, beyond which the error $1 - F_{\text{avg}}$ no longer decreases or even increases again. For a related analysis see [19].

With nonideal pulses, performance is limited by pulse errors. Embedding CDD into a first-order self-stabilizing sequence (SS1) significantly reduces the residual error. Remarkably, embedding CDD into a seventh-order SS is able to almost completely eliminate the effects of systematic pulse errors, approaching the performance of CDD with ideal pulses.

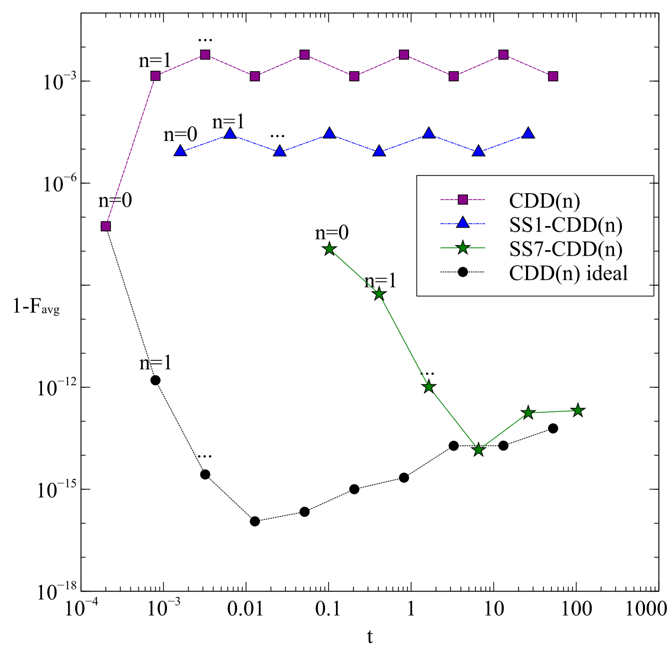

FIG. 1. (Color online) Performance of self-stabilizing CDD with a five-qubit bath, pairwise qubit coupling strength ~ 1, $\Delta t = 0.0002$, and $\|\lambda_c q\| \sim 0.1$. The performance is measured as $1 - F_{\text{avg}}$, the average infidelity. The parameter n denotes the concatenation level of CDD, with larger n resulting in longer sequences, which in turn take more time.

VI. SUPPRESSING QUBIT INTERACTIONS BY NONIDEAL CONTROLS

This section focuses on decoupling of arbitrary pairwise interactions in a closed multiqubit system. It is shown that orthogonal array (OA) based decoupling [4] embedded into a self-stabilizing sequence (using the method introduced in Sec. IV) reduces errors due to systematic control nonidealities when compared to the OA-based sequence alone. The improvement is less dramatic than in the single-qubit case, but further error reduction is shown to be possible using randomized decoupling, provided a high enough self-stabilization order is used.

Consider a closed system of n interacting qubits. Suppose that its state undergoes an undesirable evolution generated by some traceless [20] Hamiltonian H that *acts only on the system, contains only single- and two-qubit couplings*, and is *time independent.* In other words, H can be written as

$$H = \sum_{0\leqslant j<n} H_j + \sum_{0\leqslant j<k<n} H_{jk}, \tag{16}$$

where

$$H_j \in \left\{ \sum_{l=1}^{3} x_l \sigma_l^{(j)} : x_l \in \mathbb{R} \right\}, \tag{17}$$

$$H_{jk} \in \left\{ \sum_{l,m=1}^{3} X_{lm} \sigma_l^{(j)} \sigma_m^{(k)} : X_{lm} \in \mathbb{R} \right\}. \tag{18}$$

First-order decoupling ($\omega_H = 1$) of this kind of interaction is described in [4]. We will refer to it as orthogonal array decoupling (OAD). Second-order decoupling ($\omega_H = 2$) can be achieved by gluing the first-order scheme to its time reversal, which is known as symmetrical dynamical decoupling (SDD) [21]. Higher-order decoupling in this scenario is, in principle, also possible, but it involves much longer sequences.

We will evaluate sequences obtained by embedding the first-order decoupling sequence into self-stabilizing sequences of orders $\omega_c = 1,2,3$ in terms of performance and compare the results with the performance of first-order decoupling alone. The same can be done using the second-order decoupling sequence as the starting point, but it turns out that performance is typically not significantly improved by increasing the decoupling order ω_H. The reason is that the best performance is achieved by Δt so small that the terms $O(\lambda_c^{\omega_c+1})$ and $O(\Delta t \lambda_c)$ in the error expansion (11) are much larger than the term $O(\Delta t^{\omega_H+1})$ for $\omega_H \geqslant 1$ anyway. As we show in Sec. VI B, in order to improve performance, one has to apply randomization, reduce Δt, and increase the self-stabilization order ω_c instead.

To evaluate the performance of self-stabilized decoupling in comparison to plain decoupling, we use the entanglement fidelity, which serves as a good approximation of average fidelity. The ideal evolution is the identity operator, and the nonideal one is a unitary operator acting only on the system. In this case the entanglement fidelity can be expressed as

$$F_e(U) = \left\| \frac{\mathrm{tr}(U)}{d} \right\|^2. \tag{19}$$

A. Periodic decoupling

The evolution after N repetitions can be expressed using the single-cycle average Hamiltonian as

$$U = \exp(-iNT_c H_{\mathrm{avg}}), \tag{20}$$

where T_c is the cycle time. The global phase has no effect on measurement results, so we assume $\mathrm{tr}(H_{\mathrm{avg}}) = 0$. Substituting for U in (19) and expanding the exponential, we get

$$F_e(N) = 1 - (NT_c \|H_{\mathrm{avg}}\|_I)^2 + O(N^3),$$

where

$$\|A\|_I = \frac{\sqrt{\mathrm{tr}(AA^\dagger)}}{\sqrt{d}} = \frac{1}{\sqrt{d}}\|A\|_{HS}$$

is a dimension-invariant Hilbert-Schmidt norm. We see that the fidelity (at cycle boundaries) drops approximately quadratically with N and therefore also with time $T = NT_c$. The rapidity of the decay is given by $\|H_{\mathrm{avg}}\|_I$, which can be interpreted as the effective interaction strength, serving as a natural measure of performance for periodic decoupling [22]. It can be approximated by

$$\|H_{\mathrm{avg}}\|_I \simeq G_E = \frac{\sqrt{1-F_e(1)}}{T_c},$$

which is a good approximation for small values of $1 - F_e(1)$.

Comparing (20) with (11), we get

$$-iNT_c H_{\mathrm{avg}} = O(\Delta t^{\omega_H+1}) + O\left(\lambda_c^{\omega_c+1}\right) + O(\Delta t \lambda_c),$$

$$\|H_{\mathrm{avg}}\|_I = O(\Delta t^{\omega_H}) + \frac{1}{\Delta t} O\left(\lambda_c^{\omega_c+1}\right) + O(\lambda_c), \tag{21}$$

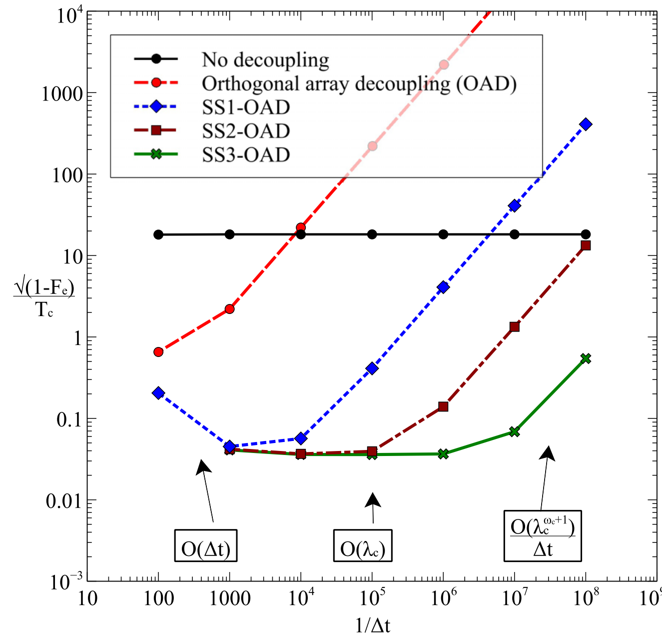

FIG. 2. (Color online) Comparison of *periodic* versions of orthogonal array decoupling (OAD), SS1-OAD, SS2-OAD, and SS3-OAD in terms of the *effective interaction strength* approximated as $G_E = \frac{\sqrt{1-F_e}}{T_c}$, which quantifies the rapidity of the *quadratic* fidelity decay associated with periodic decoupling. The quantity G_E was evaluated for $n = 9$ qubits, after one completed cycle of duration T_c, and is plotted as a function of the pulse rate $1/\Delta t$. The plots were calculated for a single random realization of the system Hamiltonian H and the nonideal controls. For each decoupling scheme, 25 control sequences were generated, and their performance was averaged. Other realizations of H and control nonidealities of similar magnitude led to very similar plots, so they are not shown. The control nonidealities were realized as noise added to control rotation vectors with standard deviation $\Delta v_i = 0.01$. The quantity G_E is a good performance measure only for not too large values of $1 - F_e$, so only points with $1 - F < 0.2$ are shown. The advantage of self-stabilization is obvious, but higher SS orders do not further improve performance in this case, they only widen the range of applicable values of Δt. The figure is consistent with (21).

where we used the fact that $T_c = \Delta t L_C$, with L_C being the number of pulses in a cycle.

Figure 2 shows the dependence of G_E on pulse rate $1/\Delta t$ for OAD with and without self-stabilization. With increasing pulse rate, G_E first decreases, indicating increasing decoupling efficiency, then reaches an optimum (possibly a flat one), and finally increases again, indicating accumulating pulse errors. Not all regimes are visible for all schemes. The optimal (i.e., minimal) value of G_E corresponds to the best achievable performance. It is clearly seen that the first-order self-stabilized (SS1) OAD significantly improves performance compared to the non-self-stabilizing OAD. Second-order SS does not further improve performance in the periodic decoupling case, as performance is now limited by the term $O(\lambda_c)$ in (21). However, this limitation can be overcome by applying randomization.

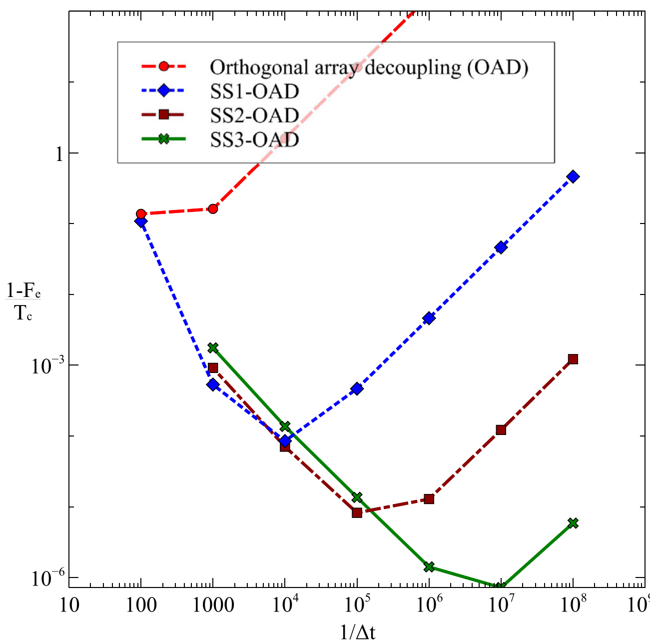

FIG. 3. (Color online) Comparison of *randomized* versions of OAD, SS1-OAD, SS2-OAD, and SS3-OAD, in terms of the*fidelity decay rate* approximated as $\frac{1-F}{T_c}$ after a single cycle, which quantifies the rapidity of the *linear* fidelity decay associated with *ideally* randomized decoupling. The performance is averaged over 25 sequence realizations. All parameters are the same as in Fig. 2. We do not have a proof that our randomization procedure achieves linear decay, but simulations (see Fig. 4) are consistent with linear fidelity decay for sequences up to 200 cycles long, justifying the use of $\frac{1-F}{T_c}$ after a single cycle as a practical performance predictor.

B. Randomized decoupling

In the periodic decoupling case, the average Hamiltonian H_{avg} is the same for all cycles, causing errors to accumulate coherently, leading to quadratic fidelity decay. There are many sequences that realize a particular decoupling scenario, and each is typically associated with a different H_{avg}. A well-known technique to combat the quadratic fidelity decay is to choose at random from these sequences for each decoupling cycle. In the case of perfect randomization, the correlations among H_{avg} of different cycles are destroyed, transforming the quadratic decay into a linear one on average [12,13]. We now present a randomization procedure for self-stabilized OAD and evaluate its performance.

Choosing randomly from the full set of SS OAD sequences may not be tractable. Instead, we first randomize the OAD sequence by shuffling the OA array used to generate it. Then we take a "master" SS sequence of the desired order and cyclically permute it a random number of positions. It is easy to show that this operation preserves the SS property. We prepare as many such randomized SS sequences as there are qubits. Finally, we embed the randomized OAD sequence into the randomized SS sequences. For each cycle, the randomization process is repeated. We do not know whether this process actually achieves perfectly linear fidelity decay on average, but simulations show that it is linear for practical purposes for up to several hundred cycles.

Assuming linear fidelity decay, one can evaluate the slope of this decay for different values of the decoupling frequency $1/\Delta t$ by extrapolating from the fidelity after one cycle. This dependence is shown in Fig. 3 for the self-stabilizing as well as ordinary decoupling. The plot shows that the higher the self-stabilization order is, the better the best achievable

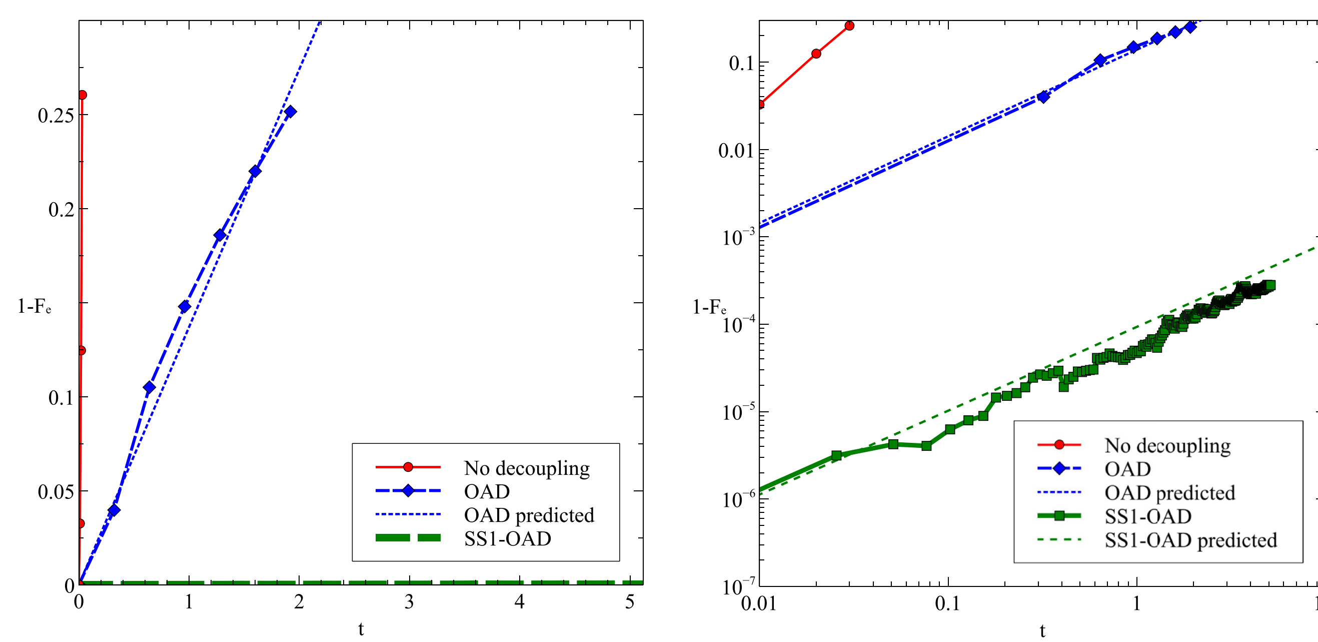

FIG. 4. (Color online) Time evolutions of the error probability $1-F$ for no decoupling, randomized OAD for $\Delta t = 0.01$, and randomized SS1-OAD for $\Delta t = 0.0001$. The values of Δt were chosen to maximize the performance of each scheme as predicted in Fig. 3. All parameters are the same as in Fig. 2. The evolution of error for SS1-OAD is in good agreement with the evolution predicted by extrapolation of fidelity after the first cycle, suggesting that the randomization procedure used for SS1-OAD is effective in keeping the fidelity decay linear for at least 200 cycles. The ability of self-stabilization to improve decoupling performance in the presence of systematic pulse errors is remarkable.

performance is. Figure 4 compares actual simulated fidelity decay for OAD and SS1-OAD. Both schemes were evaluated for their respectively optimal values of $1/\Delta t$ as determined by Fig. 3. We observe very good agreement with the linear decay predicted for an ideally randomized case. It is seen that randomization is very beneficial. Moreover, the performance benefits of self-stabilization are even more pronounced. In contrast to the periodic case, increasing the self-stabilization order further improves the results.

VII. CONCLUSIONS AND OUTLOOK

We have introduced a class of self-stabilizing pulse sequences capable of efficiently suppressing systematic control errors in qubit systems. A heuristic construction was presented that yields self-stabilizing sequences that suppress control errors up to 11th order. A method to embed already known decoupling sequences into self-stabilizing sequences was presented, and it was shown that such embedding offers powerful means to achieve robustness against unwanted external perturbations and systematic control errors. A randomized variant of the decoupling scheme was proposed and benefits of randomization were demonstrated by means of simulation.

We conclude that embedding decoupling sequences into self-stabilizing sequences is a very promising way to achieve robustness against systematic control errors. The method uses only single-qubit operations, opening interesting possibilities for qubit-based quantum information processing.

Based on the obtained and presented results, we see several directions we can pursue in the near future. It would be useful to investigate the performance of self-stabilizing sequences in the context of bounded strength controls and in the presence of more realistic control errors that include a stochastic component. It is worth looking in more detail into the structure and symmetries of self-stabilizing sequences of various orders, as this knowledge may help in constructing such sequences. Finally, it is a challenging question to prove whether self-stabilizing sequences of any order exist. In the case of an affirmative answer, their properties, in particular strict bounds on length and performance, should be studied in sufficient detail.

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