Aside on $S(A|B) > 0 \not\Rightarrow \rho_{AB}$ separable

The question of whether or not a state ρ_{AB} is entangled is the subject of much ongoing research which we won't have time to examine in any detail. However we can easily construct a counterexample to the statement $S(A|B) \geq 0 \Rightarrow \rho_{AB}$ separable by using the PPT criterion

Observe that for a separable state $\rho_{AB} = \sum_j p_j \sigma_j^A \otimes \xi_j^B$, we can apply the positive (but not completely positive) operation transpose to B and still have a valid state: $\rho_{AB}^{\Gamma} = \sum_j p_j \sigma_j^A \otimes (\xi_j^T)^B \geq 0$

This needn't be true for an entangled state (but could be), so this gives an entanglement criterion for bipartite states. In fact, we could use any positive but not completely positive operation and get an entanglement criterion by the same logic. Now consider the states

$$\rho_\alpha = \tfrac{2}{7}\Phi + \tfrac{\alpha}{7}\rho_+ + \tfrac{5-\alpha}{7}\rho_- \quad \text{on } \mathbb{C}_3 \otimes \mathbb{C}_3 \quad \text{with } |\Phi\rangle = \tfrac{1}{\sqrt{3}}\sum_{j=0}^{2} |j\rangle|j\rangle, \quad 2 \leq \alpha \leq 5,$$

$$\rho_+ = \tfrac{1}{3}\left(P_{01} + P_{12} + P_{20}\right) \qquad \rho_- = \tfrac{1}{3}\left(P_{10} + P_{21} + P_{02}\right) \qquad P_{jk} = |j\rangle\langle j| \otimes |k\rangle\langle k|$$

First let's compute $S(A|B)$. Since the three terms comprising ρ_α are disjoint, we have $S(AB) = \frac{2}{7} + \frac{\alpha}{7}S(\rho_+) + \frac{5-\alpha}{7}S(\rho_-) + H(\{\frac{2}{7}, \frac{\alpha}{7}, \frac{5-\alpha}{7}\}) = \frac{1}{7}(2 + 5\log 3) + H(\{\frac{2}{7}, \frac{\alpha}{7}, \frac{5-\alpha}{7}\})$

Since each term has $\rho_B \propto \mathbb{1}$, $S(B) = \log 3$, and

$$S(A|B) = \tfrac{1}{7}(2 - 2\log 3) + H(\{\tfrac{2}{7}, \tfrac{\alpha}{7}, \tfrac{5-\alpha}{7}\})$$

The entropy is lower bounded by $-\log p_{max}$, which in this case is $-\log\frac{5}{7}$, so

$S(A|B) \geq \frac{1}{7}(2 - 2\log 3) + \log 7 - \log 5 \geq 0$ for all permissible α.

What do we get if we partially transpose the state? In this case it's easiest just to write out the components of ρ_α as a matrix:

$$\rho_\alpha = \frac{1}{21}\left[\begin{array}{ccc:ccc:ccc}
2 & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 2 \\
\cdot & \alpha & & & \cdot & & & & \cdot \\
\cdot & & 5-\alpha & & \cdot & & & & \cdot \\
\hdashline
\cdot & & & 5-\alpha & \cdot & & & & \cdot \\
2 & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 2 \\
\cdot & & & & \cdot & \alpha & & & \cdot \\
\hdashline
\cdot & & & & \cdot & & \alpha & & \cdot \\
\cdot & & & & \cdot & & & 5-\alpha & \cdot \\
2 & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 2
\end{array}\right]$$

The partial transpose (on B) is found by transposing each of the 9 blocks in the matrix. It's not too hard to see that the resulting matrix is block diagonal with components

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix} \oplus \begin{pmatrix} \alpha & 2 \\ 2 & 5-\alpha \end{pmatrix} \times 3$$

Thus, $\rho_\alpha^\Gamma \geq 0$ whenever $\begin{pmatrix} \alpha & 2 \\ 2 & 5-\alpha \end{pmatrix} \geq 0$. Its eigenvalues are $\frac{1}{2}\left(5 \pm \sqrt{41 - 20\alpha + 4\alpha^2}\right)$ and therefore $\rho_\alpha^\Gamma < 0$ for $\alpha > 4$.

Altogether we have found that $\rho_{\alpha>4}$ is not separable, but nevertheless $S(A|B) > 0$