Quantum Operations

The final objects to generalize are the allowed transformations of quantum states and/or measurements. Continuing our quantum probability theory approach we will list a set of critera that quantum operations should fulfill and then see how what results is consistent with the axioms.

A couple of the criteria for any transformation S are immediate.

1. $\rho' = S(\rho) \geq 0$ for $\rho \geq 0$
2. $\mathrm{Tr}[\rho'] = 1$ for $\mathrm{Tr}[\rho] = 1$

The first condition ensures that S maps states to states while the second ensures that total probability is always equal to one.

We also want S to be linear so that it respects the ensemble description of density operators. If we don't require linearity the trouble is that the transformation of any particular ensemble member depends on the <u>other</u> members, even though only ostensibly only one element is actually realized. In other words, the dynamics of the true state depends on the nonexistent states! Since S is a linear mapping of operators to operators, we call it a <u>superoperator</u>.

Suprisingly, condition 1 runs afoul of the ensemble description in its coherent form, the ability to purify any state. Consider the transpose operator on a single qubit,

$$T : \rho = \sum_{jk} \rho_{jk} |j\rangle\langle k| \longrightarrow \rho^T = \sum_{jk} \rho_{kj} |j\rangle\langle k|.$$

Transposition is linear as well as trace- and positivity-preserving. What's the trouble? Consider transposing half of a maximally entangled state $|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_j |jj\rangle_{AB}$ ($|\Phi\rangle$ has a maximum number of Schmidt coefficients and the marginals are maximally-mixed: $\rho_A = \mathbb{1}_A/d$).

$$T_A \otimes \mathbb{1}_B \left(|\Phi\rangle\langle\Phi|_{AB} \right) = \frac{1}{d} \sum_{jk} |k\rangle\langle j| \otimes |j\rangle\langle k| = \rho'_{AB}.$$

A simple direct calculation shows that $d \cdot \rho'_{AB} |\psi\rangle_A |\varphi\rangle_B = |\varphi\rangle_A |\psi\rangle_B$, that is, it swaps the two systems. But antisymmetric combinations have eigenvalue -1, so $\rho'_{AB} \not\geq 0$.

$$d \cdot \rho'_{AB} \left(|\varphi\rangle|\psi\rangle - |\psi\rangle|\varphi\rangle \right) = |\psi\rangle|\varphi\rangle - |\varphi\rangle|\psi\rangle = -\left(|\varphi\rangle|\psi\rangle - |\psi\rangle|\varphi\rangle \right)$$

Since purifications of mixed states are always in principle possible, it is apparent that we must demand that quantum operations be <u>completely positive</u>: positive on ρ and all its purifications.

We have already encountered an example of completely positive superoperators in the section on generalized measurements. Performing a measurement described by operators A_k results in the ensemble $\{p_k = \mathrm{Tr}[A_k \rho A_k^\dagger], \frac{A_k \rho A_k^\dagger}{p_k}\}$.

Averaging over the outputs, i.e. forgetting which outcome occurred, leads to the average state $\mathcal{A}(\rho) = \sum_k A_k \rho A_k^\dagger$. $\mathcal{A}$ must be a completely positive operator because, as we saw, it can be thought of as a unitary operator U_{AB} defined by

$$U_{AB}|\psi\rangle_A|0\rangle_B = \sum_k A_k|\psi\rangle_A |k\rangle_B$$

followed by tracing out system B. Both of these are CPTP (completely positive, trace preserving) maps, so $\mathcal{A}$ is, too.

In fact, all CPTP maps are of this form, $\mathcal{A}(\rho) = \sum_k A_k \rho A_k^\dagger$ with $\sum_k A_k^\dagger A_k = \mathbb{1}$, a statement known as the Kraus representation theorem. We can easily prove this in finite dimensions by using the Choi matrix representation of a superoperator.

Given an arbitrary CPTP map $\$$ consider the result of applying it to half a maximally-entangled state $|\Phi\rangle_{AB} = \sum_{k=1}^{d} |k\rangle|k\rangle$. The state isn't properly normalized, but that won't matter here.

$$\$ \longrightarrow \Phi_\$^C = (\$ \otimes \mathbb{1})[\Phi] \in \mathbb{C}^d \otimes \mathbb{C}^d$$

Here $\Phi \equiv |\Phi\rangle\langle\Phi|$ and the superscript C stands for Choi.

This map is an isomorphism because the components of $\Phi^c_\$$ are the same as those of \$ itself. Suppose we define $\$_{jk;lm}$ by

$$\$(|j\rangle\langle k|) = \sum_{lm} \$_{jk;lm} |l\rangle\langle m| \iff \$_{jk;lm} = \langle l|\$(|j\rangle\langle k|)|m\rangle.$$

Then $\Phi^c_\$$ can be expressed as

$$\Phi^c_\$ = (\$\otimes\mathbb{1})\Big[\sum_{jk} |j\rangle\langle k|\otimes|j\rangle\langle k|\Big] = \sum_{jk} \$(|j\rangle\langle k|)\otimes|j\rangle\langle k| = \sum_{jklm} \$_{jk;lm}|l\rangle\langle m|\otimes|j\rangle\langle k|$$

and $\$(|j\rangle\langle k|)$ can be recovered from $\Phi^c_\$$ by

$$\$(|j\rangle\langle k|) = \langle j|\Phi^c_\$|k\rangle$$

In fact we can recover any $\$(|\varphi\rangle\langle\varphi|)$ from $\Phi^c_\$$ by using $|\varphi^*\rangle$, the complex conjugate of $|\varphi\rangle$ in the basis $\{|k\rangle\}$.

$$\langle\varphi^*|\Phi^c_\$|\varphi^*\rangle = \sum_{jk} \$(|j\rangle\langle k|)\langle\varphi^*|j\rangle\langle k|\varphi^*\rangle = \sum_{jk}\varphi_j\varphi_k^*\,\$(|j\rangle\langle k|) = \$(|\varphi\rangle\langle\varphi|)$$

Now the Kraus rep. theorem is easy to prove. Since \$ is a CPTP map $\Phi^c_\$$ must be positive. It therefore has an eigendecomposition with positive eigenvalues: $\Phi^c_\$ = \sum_l \lambda_l|\lambda_l\rangle\langle\lambda_l|$. Folding the eigenvalue into the normalization we can write $\Phi^c_\$ = \sum_l |\tilde\lambda_l\rangle\langle\tilde\lambda_l|$. Extracting $\$(|j\rangle\langle k|)$ we obtain

$$\$(|j\rangle\langle k|) = \sum_l {}_B\langle j|\tilde\lambda_l\rangle\langle\tilde\lambda_l|_{AB}\,|k\rangle_B = \sum_l {}_B\langle j^*|\tilde\lambda_l\rangle\langle\tilde\lambda_l|_{AB}\,|k^*\rangle_B$$

Defining the map $A_l : |\varphi\rangle \to {}_B\langle\varphi^*|\tilde\lambda_l\rangle_{AB}$, which is linear since

$$A_l\Big(\sum_k \varphi_k|k\rangle\Big) = A_l|\varphi\rangle = \langle\varphi^*|\tilde\lambda\rangle = \sum_k\varphi_k\langle k|\tilde\lambda_l\rangle = \sum_k \varphi_k A_l|k\rangle,$$

the superoperator action is given by

$$\$(|j\rangle\langle k|) = \sum_l A_l|j\rangle\langle k|A_l^\dagger$$

By linearity of $\$$ and A_ℓ the action on an arbitrary operator is $\$(\rho) = \sum_\ell A_\ell \rho A_\ell^\dagger$. And since $\$$ is trace preserving,

$$\mathrm{Tr}[\$(\rho)] = \sum_\ell \mathrm{Tr}[A_\ell^\dagger A_\ell \rho] = 1 \Rightarrow \sum_\ell A_\ell^\dagger A_\ell = \mathbb{1}.$$

Thus we have established the Kraus representation theorem. In this context the A_ℓ's are often called Kraus operators.

There are two important corollaries to this theorem, both following from the Choi matrix $\Phi_\$^C$. First, since $\Phi_\$^C$ is a $d^2 \times d^2$ matrix, it has at most d^2 eigenvectors, and therefore a CPTP map $\$$ on operators on $\mathbb{C}^d$ always has an Kraus decomposition with d^2 Kraus operators. Secondly, in the construction of the Kraus operators we are free to use any decomposition of $\Phi_\$^C$ into a convex combination of pure states — it doesn't have to be the eigendecomposition. The result would be another set of Kraus operators A_ℓ', in general with more elements. But by the HJW theorem we know that the pure states in the decomposition are unitarily-related, and so this holds for the Kraus operators as well:

$$|\tilde{\lambda}_\ell'\rangle = \sum_m U_{\ell m} |\tilde{\lambda}_m\rangle \Rightarrow \langle \varphi^* | \tilde{\lambda}_\ell' \rangle = \sum_m U_{\ell m} \langle \varphi^* | \tilde{\lambda}_m \rangle$$

$$\Rightarrow A_\ell' = \sum_m U_{\ell m} A_m, \text{ for } U_{\ell m} \text{ a unitary matrix.}$$