Multiple qubits: Tensor Product Construction

The state of several qubits is described by a state in the tensor product of the underlying qubit spaces. This is the vector space formed by taking a basis from each of the constituent spaces and allowing arbitrary complex combinations. Formally, $V \otimes W = \text{span}\{\vec{v}_i \otimes \vec{w}_j\}_{i,j}$ for $\{\vec{v}_i\}$ a basis of V and $\{\vec{w}_j\}$ a basis of W.

For n qubits we can use the $\hat{z}$ basis for each, which gives elements

$$|0\rangle \otimes |0\rangle \cdots |0\rangle, \quad |0\rangle \otimes |0\rangle \cdots |1\rangle, \quad |0\rangle \otimes |0\rangle \cdots |1\rangle \otimes |0\rangle, \quad \text{etc.}$$

To save space we usually omit the $\otimes$ symbol, and just write $|0\rangle|0\rangle \cdots |0\rangle$ or even $|0 \cdots 0\rangle$.

Since there are 2^n basis states, it follows that $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n} \simeq \mathbb{C}^{2^n}$.

We will see in the next several lectures that if we only look at one part of a larger multipartite system, then it is no longer the case that states are described by rays, measurements by projectors, and dynamics by unitary operators. Nevertheless, the axioms will continue to hold on the larger multipartite system.

Bipartite States and Entanglement

Let's first focus on two systems A and B, of dimensions d_A and d_B. An arbitrary state can be written as $|\psi\rangle_{AB} = \sum_{j,k=1}^{d_A,d_B} \psi_{jk} |jk\rangle_{AB}$, and thinking of the coefficients ψ_{jk} as forming a $d_A \times d_B$ matrix, we can use the singular-value decomposition to form the Schmidt decomposition of $|\psi\rangle$. Recall that any matrix ψ_{jk} has an SVD:

$$\psi_{jk} = [U \Lambda V^\dagger]_{jk},$$

where U and V are unitary $d_A \times d_A$ and $d_B \times d_B$ matrices, and Λ is a diagonal $d_A \times d_B$ matrix whose entries, all positive, are called the singular values of ψ_{jk}.

Inserting this into the expression for $|\psi\rangle$ yields

$$|\psi\rangle = \sum_{jk} \psi_{jk} |jk\rangle = \sum_{jkl} U_{jl} \Lambda_{ll} (V^\dagger)_{lk} |j\rangle|k\rangle \qquad \text{let } \Lambda_{ll} = \lambda_l$$

$$= \sum_l \lambda_l \Big(\sum_j U_{jl}|j\rangle\Big) \otimes \Big(\sum_k V^*_{kl}|k\rangle\Big) = \sum_l \lambda_l \underbrace{U|l\rangle}_{|\xi_l\rangle} \otimes \underbrace{V^*|l\rangle}_{|\eta_l\rangle}$$

$$= \sum_l \lambda_l |\xi_l\rangle|\eta_l\rangle \qquad (\text{using } U_{jl} = \langle j|U|l\rangle)$$

Since U and V are unitary, the $\{|\xi_l\rangle\}$ and $\{|\eta_l\rangle\}$ are each orthonormal bases. Thus, it is possible to find a pair of bases for each subsystem such that the coefficients of the

global state are diagonal in this basis. Moreover, since the singular values are positive and the state must be normalized, the λ_ℓ actually form a probability distribution.
If there is only one nonzero λ_ℓ, then $|\psi\rangle = |\xi_\ell\rangle \otimes |\eta_\ell\rangle$, and it is said to be a product state, since each of the subsystems is described by its own state. If there is more than one nonzero λ_ℓ, the state is said to be entangled. The maximum number of nonzero Schmidt coefficients equals $d_m = \min(d_A, d_B)$ and when $\lambda_\ell = \frac{1}{\sqrt{d_m}}$, the state is said to be maximally entangled.

The possibility of entanglement, which is a consequence of the linear structure of quantum mechanics, is the reason quantum states cannot be copied. Repeating the derivation from the first lecture, we're looking for a unitary operator on two systems such that $U_{copy}|\psi\rangle|\text{blank}\rangle = |\psi\rangle|\psi\rangle$. $|\psi\rangle$ is an arbitrary state, meaning U_{copy} is supposed to be able to copy any state, and therefore its operation doesn't depend on the particulars of the input state. Now choose an arbitrary basis, say $|0\rangle$ and $|1\rangle$, for which it must be true that $U_{copy}|0\rangle|\text{blank}\rangle = |0\rangle|0\rangle$ and $U_{copy}|1\rangle|\text{blank}\rangle = |1\rangle|1\rangle$.

But if we then try to copy a superposition, we get an entangled state: $U_{copy}(a|0\rangle + b|1\rangle)|\text{blank}\rangle = a|00\rangle + b|11\rangle$.
It is already in Schmidt form, and when $a, b \neq 0$, the state is entangled according to the definition above.

Entanglement is at the heart of quantum information theory and is essentially responsible for all of the novel phenomena. Before moving on, let's consider two interesting and important examples involving two qubits.

- Superdense coding

The canonical maximally entangled state of two qubits is

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{AB}$$

Consider the action of one of the Pauli operators on system B, say σ_x:

$$|\Phi_x\rangle_{AB} = (\mathbb{1}_A \otimes (\sigma_x)_B)|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)_{AB}$$

Clearly this state is orthogonal to $|\Phi\rangle$. What about σ_z?

$$|\Phi_z\rangle_{AB} = (\mathbb{1}_A \otimes (\sigma_z)_B)|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)_{AB}$$

Also orthogonal to $|\Phi\rangle$, and to $|\Phi_x\rangle$. And σ_y:

$$|\Phi_y\rangle_{AB} = (\mathbb{1}_A \otimes (-i\sigma_y)_B)|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)_{AB}$$

orthogonal to all the others.

We've constructed a basis for $\mathbb{C}^2 \times \mathbb{C}^2$ comprised of maximally entangled states, all related by Pauli operators on system B alone.

Now return to the setup of our two separated parties, Alice and Bob. Alice would like to send a message to Bob, a message composed of two bits (sell stocks? buy gold?), but she's only got enough postage for either one classical bit or one quantum bit. Clearly one classical bit is insufficient. But quantum postage was even cheaper in the past, and Alice, predicting that it would go up, sent a qubit to Bob back when the rates were cheap.

How does that help her now? Suppose she originally prepared $|\Phi\rangle_{AB}$ and then sent system A using the cheap postage. Now she can apply one of the 3 Pauli operators, or do nothing, to B and send this qubit to Bob. This creates one of the 4 entangled basis states $|\Phi_j\rangle_{AB}$, and Bob can read out the message using the measurement $P_j = |\Phi_j\rangle\langle\Phi_j|$.

Notice that Alice managed to send 2 bits of information using just 1 qubit — when she sent the first one she hadn't yet made up her mind about selling stocks and buying gold. That's why this scheme is called superdense coding: one qubit is used to transfer 2 classical bits, though of course two qubits are ultimately involved (Bob needs 4 orthogonal projectors to read out the message).

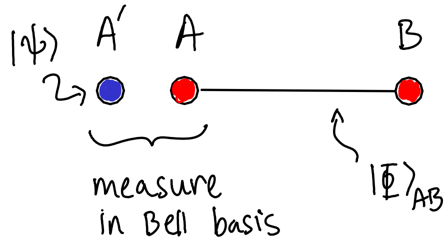
- Teleportation

Now imagine Alice and Bob are in the opposite situation: Instead of Alice wanting to send 2 classical bits and having only a quantum channel (plus preshared entanglement), she wants to send a qubit, but only has access to a classical channel. Can she somehow send the state to Bob using only a classical channel?

If that's all the resources they share, the answer is no. Alice could try to measure the qubit in some way, for instance to learn the values of the coefficients a and b in the expression $|\psi\rangle = a|0\rangle + b|1\rangle$ by building up statistics (since $Pr(0) = |a|^2$ and nevermind she also needs the relative phase between a and b), but she only has 1 copy of $|\psi\rangle$.

On the other hand, if Alice and Bob already share an entangled state, then it is possible to transfer $|\psi\rangle$ to Bob, and it only requires 2 bits! The "2 bits" might remind you of the 4 entangled states $|\Phi_j\rangle$ (called Bell states) used in superdense coding, and they play the same role as measurement in teleportation.

The protocol is very simple. Alice has a qubit prepared in $|\psi\rangle_{A'}$ as well as half of a maximally entangled state $|\Phi\rangle_{AB}$. She then measures her two systems in the Bell basis, producing a two-bit outcome.



What happens when the outcome corresponds to $|\Phi\rangle$?

$$
\begin{aligned}
{}_{A'A}\langle\Phi|\psi\rangle_{A'}|\Phi\rangle_{AB} &= {}_{A'A}\langle\Phi|\frac{1}{\sqrt{2}}\Big(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\Big)_{A'AB} \\
&= \frac{1}{2}\Big(\langle 00| + \langle 11|\Big)_{A'A}\Big(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\Big)_{A'AB} \\
&= \frac{1}{2}\Big(a|0\rangle + b|1\rangle\Big)_B = \frac{1}{2}|\psi\rangle_B
\end{aligned}
$$

The state is transferred to Bob! The norm^2 of the output tells us the probability, so the chance that Alice obtains result $|\Phi\rangle$ is $\frac{1}{4}$. And what about the other results?

To figure this out we can use a different method: write $|\psi\rangle_{A'}|\Phi\rangle_{AB}$ in the $|\Phi_j\rangle_{A'A}|k\rangle_B$ basis.

$$
\begin{aligned}
|\psi\rangle_{A'}|\Phi\rangle_{AB} &= \tfrac{1}{\sqrt{2}}\Big(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\Big)_{A'AB} \\
&= \tfrac{1}{2}\Big[a\Big(|\Phi\rangle|0\rangle + |\Phi_z\rangle|0\rangle + |\Phi_x\rangle|1\rangle + |\Phi_y\rangle|1\rangle\Big) \\
&\qquad + b\Big(|\Phi_x\rangle|0\rangle - |\Phi_y\rangle|0\rangle + |\Phi\rangle|1\rangle - |\Phi_z\rangle|1\rangle\Big)\Big] \\
&= \tfrac{1}{2}\Big[|\Phi\rangle\big(a|0\rangle + b|1\rangle\big) + |\Phi_x\rangle\big(a|1\rangle + b|0\rangle\big) \\
&\qquad + |\Phi_y\rangle\big(a|1\rangle - b|0\rangle\big) + |\Phi_z\rangle\big(a|0\rangle - b|1\rangle\big)\Big] \\
&= \tfrac{1}{2}\Big[|\Phi\rangle|\psi\rangle + |\Phi_x\rangle\sigma_x|\psi\rangle + |\Phi_y\rangle(-i\sigma_y)|\psi\rangle + |\Phi_z\rangle\sigma_z|\psi\rangle\Big]
\end{aligned}
$$

Notice how each term is of the form $|\Phi_j\rangle_{A'A}\,\sigma_j|\psi\rangle_B$, meaning that if Alice measures $A'A$ in the Bell basis and communicates the result to Bob, he can apply the corresponding Pauli operator to obtain the input state $|\psi\rangle$. Alice needs 2 bits to describe the outcome, and since each term has the same weight, the probability of every outcome is 1/4.