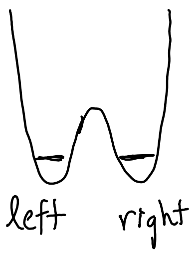
Qubits

Before proceeding with our generalization of the axioms of quantum mechanics, let's first look at the details of the simplest case — a two-level system, or <u>qubit</u>.

Denoting an orthormal basis by $|0\rangle, |1\rangle$, then a qubit is any system (or more precisely, degree of freedom) whose state vector $|\psi\rangle$ can be written as

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad \text{with } |a|^2 + |b|^2 = 1$$

Examples:
- spin-½ particle: $|0\rangle = |m=-\tfrac{1}{2}\rangle$, $|1\rangle = |m=\tfrac{1}{2}\rangle$
- photon polarization: $|0\rangle = |\text{horizontal}\rangle$, $|1\rangle = |\text{vertical}\rangle$
- "two-level atom": $|0\rangle = |\text{ground state}\rangle$, $|1\rangle = |\text{excited state}\rangle$
- position in a deep double-well potential:



$|0\rangle = |\text{left}\rangle \qquad |1\rangle = |\text{right}\rangle$

States

A useful parameterization of states comes from the spin-½ picture: $|\theta,\varphi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$, to which we can associate the point on a sphere described by (spherical) coordinates (θ,φ), or to the unit vector $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. This sphere of states is called the Bloch sphere.

The states $|\hat{n}\rangle$ and $|-\hat{n}\rangle$ are orthogonal: for $-\hat{n}$ we take $\theta \to \pi - \theta$ & $\varphi \to \varphi + \pi$, obtaining $|-\hat{n}\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\varphi}\cos\frac{\theta}{2}|1\rangle$.

The states along the six cardinal directions are

$$|\hat{z}\rangle = |0\rangle \qquad |\hat{x}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |\hat{y}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$
$$|-\hat{z}\rangle = |1\rangle \qquad |-\hat{x}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad |-\hat{y}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

which make up three orthonormal bases. The three sets are eigenstates of the Pauli spin operators

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1| \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Observables, Events, and Probability

These three, together with the identity $\mathbb{1}$, form a very convenient basis for operators on $\mathbb{C}^2$, which follows because we can construct the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, etc. When the coefficients are real, the set is a basis for Hermitian operators on $\mathbb{C}^2$:

$$A = a_0\mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z = a_0\mathbb{1} + \vec{a}\cdot\vec{\sigma} \qquad a_i \in \mathbb{R} \Rightarrow A = A^\dagger$$

It's easy to verify that $|\pm\hat{a}\rangle$ are eigenstates of $A = a_0\mathbb{1} + \vec{a}\cdot\vec{\sigma}$ with eigenvalues $\lambda = a_0 \pm |\vec{a}|$:

$$A|\hat{a}\rangle = a_0|\hat{a}\rangle + (\vec{a}\cdot\vec{\sigma})|\hat{a}\rangle \qquad (\vec{a}\cdot\vec{\sigma}) \to |\vec{a}|\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}$$

$$(\vec{a}\cdot\vec{\sigma})|\hat{a}\rangle \to \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ e^{i\varphi}(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \to |\hat{a}\rangle \quad \text{using the double-angle formulas.}$$

So $A|\vec{a}\rangle = (a_0 + |\vec{a}|)\,|\hat{a}\rangle$. Since $\langle -\hat{a}|\hat{a}\rangle = 0$, $|-\hat{a}\rangle$ must be the other eigenvector. The determinant gives us the product of the eigenvalues, and is

$$\mathrm{Det}\,A = (a_0+a_z)(a_0-a_z) - (a_x+ia_y)(a_x-ia_y) = a_0^2 - a_x^2 - a_y^2 - a_z^2 = a_0^2 - |\vec{a}|^2.$$

Thus the eigenvalue associated with $|-\hat{a}\rangle$ is $a_0 - |\vec{a}|$.

It follows that the (measurement) projectors $P_{\hat{n}} = |\hat{n}\rangle\langle\hat{n}|$ are just

$$P_{\hat{n}} = \tfrac{1}{2}(\mathbb{1} + \hat{n}\cdot\vec{\sigma}),$$

and the probability of obtaining outcome $\hat{n}$ when measuring a system described by $|\hat{m}\rangle$ is

$$\begin{aligned}
\Pr(\hat{n}|\hat{m}) = \langle\hat{m}|P_{\hat{n}}|\hat{m}\rangle = \mathrm{Tr}[P_{\hat{n}}P_{\hat{m}}] &= \tfrac{1}{4}\mathrm{Tr}\left[(\mathbb{1}+\hat{n}\cdot\vec{\sigma})(\mathbb{1}+\hat{m}\cdot\vec{\sigma})\right] \\
&= \tfrac{1}{4}\mathrm{Tr}\left[\mathbb{1} + (\hat{n}+\hat{m})\cdot\vec{\sigma} + (\hat{n}\cdot\vec{\sigma})(\hat{m}\cdot\vec{\sigma})\right] \\
&= \tfrac{1}{4}\left(2 + \mathrm{Tr}\left[(\hat{n}\cdot\vec{\sigma})(\hat{m}\cdot\vec{\sigma})\right]\right) \\
&= \tfrac{1}{2}(1 + \hat{n}\cdot\hat{m})
\end{aligned}$$

In the last line we use the fact that the Pauli matrices anticommute and square to $\mathbb{1}$: $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbb{1}$.

Note that even though $|\hat{x}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$ looks sort of like a mixture of $|0\rangle$ and $|1\rangle$, and does give $Pr(\pm\hat{z}|\hat{x}) = \frac{1}{2}$, it isn't completely random because, of course, $Pr(\hat{x}|\hat{x}) = 1$. This what is meant by saying $|\hat{x}\rangle$ is a coherent combination of 0 and 1. Classically there's no such thing: if a classical bit has probability 1/2 to be in either state, then there's no other elementary event (like $P_{\hat{x}}$) for which the probability is 1. Quantum-mechanically we can also observe the relative phase between the two states. By measuring along $\hat{x}$, we are checking if the phase is ± 1, while a measurement along $\hat{y}$ checks for $\pm i$.

Another way to phrase this phenomena is to say that σ_x and σ_z are complementary observables. We just saw that an eigenstate of one observable gives a completely random result when measured using the eigenstates of the other. But the situation is even stranger than that. Since measurement also disturbs the state, alternately measuring in the two eigenbases can lead to a situation in which a once-certain outcome is made totally random. After a σ_z measurement of $|\hat{x}\rangle$ for instance, the state is either $|0\rangle$ or $|1\rangle$, and both of these only have probability 1/2 of returning $\hat{x}$ (as opposed to $-\hat{x}$) when measured in the σ_x eigenbasis. Needless to say, this is also impossible classically. Although measurement changes the distribution $\vec{p}$, it doesn't disturb the underlying states $\vec{s}_j$. So if some event $\vec{e}_k$ is certain (it contains the actual state $\vec{s}_j$), it remains certain no matter what measurements are done in the meantime.

Dynamics

The Pauli operators also satisfy the commutation relation $[\sigma_j, \sigma_k] = i\,\varepsilon_{jkl}\,\sigma_l$, where ε_{jkl} is the totally antisymmetric symbol. Thus the σ_j form a representation of the Lie algebra of SO(3), the group of rotations in three dimensions. In fact, this is the smallest nontrivial (irreducible) representation, and is associated with angular momentum $J = \frac{1}{2}\hbar$, hence the name spin-$1/2$.

By exponentiating linear combinations of the generators we therefore get unitary operators, which represent rotations in SO(3), but themselves live in SU(2).

The operator $U(\hat{n},\alpha) = e^{-i\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}} = \cos\frac{\alpha}{2}\,\mathbb{1} - i\sin\frac{\alpha}{2}\,\hat{n}\cdot\vec{\sigma}$ represents a rotation by α about the $\hat{n}$ axis. This follows from the fact that $U(\hat{n},\alpha)(\hat{m}\cdot\vec{\sigma})\,U^\dagger(\hat{n},\alpha) = [R(\hat{n},\alpha)\hat{m}]\cdot\vec{\sigma}$, where $R(\hat{n},\alpha)$ is the SO(3) rotation element.

Then $U(\hat{n},\alpha)|\hat{m}\rangle$ is an eigenstate of $[R(\hat{n},\alpha)\hat{m}]\cdot\vec{\sigma}$

$$\big([R(\hat{n},\alpha)\hat{m}]\cdot\vec{\sigma}\big)\,U(\hat{n},\alpha)|\hat{m}\rangle = U(\hat{n},\alpha)(\hat{m}\cdot\vec{\sigma})\,U^\dagger(\hat{n},\alpha)\,U(\hat{n},\alpha)|\hat{m}\rangle$$

$$= U(\hat{n},\alpha)(\hat{m}\cdot\vec{\sigma})|\hat{m}\rangle = U(\hat{n},\alpha)|\hat{m}\rangle.$$