Quantum Information Theory

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Problem Set #6 **14 March 2013**

Problem 6.1 Davies' Theorem

Consider an arbitrary CQ state $\sigma^{XB} = \sum_x p_x |x\rangle\langle x|^X \otimes \rho_x^B$ and imagine making a measurement $\mathscr{M}$ having elements E_y on B. By the Holevo bound, $I(X{:}Y) \leq I(X{:}B) = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$. Define the *accessible information* $I_{\text{acc}}(\sigma^{XB}) = \max_{\mathscr{M}} I(X:Y)$.

Show that the optimal measurement consists of rank-one elements and has no more than d^2 outcomes, where $d = \dim(B)$. Hint: the space of Hermitian operators on B is a vector space of size d^2.

Solution: Start with an arbitrary POVM $\mathscr{M} = \{E_y\}$ and consider the "fine-grained" version $\mathscr{M}' = \{\lambda_{y;z} P_{y;z}\}$, made by taking the (scaled-by-eigenvalue) eigenprojectors $\lambda_{y;z} P_{y;z}$ for $y = 1 \ldots \text{rank}(E_y)$ as separate POVM elements. We can think of the outcomes of the new POVM $\mathscr{M}'$ as defining a joint random variable YZ from which we can recover Y by marginalizing over Z since $p_{y|x} = \text{Tr}[E_y \rho_x] = \sum_z \lambda_{y:z} \text{Tr}[P_{y;z} \rho_x]$. It then follows that $I(X{:}Y) \leq I(X{:}YZ)$ because conditioning reduces entropy, $H(X|YZ) \leq H(X|Y)$. Thus, for any POVM $\mathscr{M}$ there exists a $\mathscr{M}'$ having rank-one elements and higher mutual information with X.
Now observe that $I(X{:}Y_{\mathscr{M}})_{\sigma^{XB}}$ is a convex function over the convex set of POVMs since mixing increases entropy: $H(pY_1 + (1-p)Y_2|X) \geq pH(Y_1|X) + (1-p)H(Y_2|X)$. But if we have a measurement $\mathscr{M}$ with $n > d^2$ elements E_y, there must exist a set of $q_y \neq 0$ such that $\sum_y q_y E_y$, which can be rescaled without loss of generality so that $-1 \leq q_y \leq 1$. Defining the measurements $\mathscr{M}_\pm = \{(1 \pm q_y)E_y\}$, which really are measurements since $(1 \pm q_y) \geq 1$ and $\sum_y (1 \pm q_y) E_y = \mathbb{1}$, we have $\mathscr{M} = \frac{1}{2}(\mathscr{M}_+ + \mathscr{M}_-)$.

Problem 6.2 Upper Bound on the Classical Capacity of a Quantum Channel

Suppose that Alice would like to send classical information to Bob over a quantum channel \$. For this purpose she uses $m < d$ quantum states σ_x^Q of dimension d encoding the message x with probability p_x. The channel turns the σ_x into $\rho_x^Q = \$(\sigma_x^Q)$. Bob makes a measurement which consists of at least m elements Λ_y, one for each message, plus one more to ensure that $\sum_y \Lambda_y = \mathbb{1}$. The average error probability is therefore $p = \frac{1}{m} \sum_{x=1}^m p_x \text{Tr}[(\mathbb{1} - \Lambda_x)\$(\sigma_x)]$.
Show that $p \geq (\log_2 m - S(X:Q)_\xi - 1)/\log_2 d$, where $\xi^{XQ} = \sum_{x=1}^m p_x |x\rangle\langle x|^X \otimes \rho_x^Q$.
Hint: use Fano's inequality.

Solution: By Fano's inequality we have

$$p \geq \frac{H(X|Y) - 1}{\log m} \geq \frac{H(X|Y) - 1}{\log d} = \frac{\log m - I(X{:}Y) - 1}{\log d}.$$

The Holevo bound is just $I(X{:}Y) \leq S(X{:}Q) = \chi$, so $p \geq \frac{\log m - \chi - 1}{\log d}$. If $\$ = \$^{\otimes n}$ and $m = 2^{nR}$, then in the limit of large n we have

$$p > \frac{R - \frac{1}{n}\chi(\{p_k, \$^{\otimes n}(\sigma_k)\})}{\log d},$$

where the σ_k are now states on the combined input space of dimension d^n. If the σ_k were product states $\xi_{k_1} \otimes \xi_{k_n}$ and the p_k was also a product distribution $q_{k_1} \cdots q_{k_n}$, then we could use subadditivity to conclude that communication at rates $R > \chi(\{q_k, \$(\xi_k)\})$ incurs a nonzero probability of error. However, for general inputs we don't know (and it's in fact not true) that the Holevo χ is additive, so we're stuck with the *regularized* expression $\frac{1}{n}\chi(\{p_k, \$^{\otimes n}(\sigma_k)\})$.

Problem 6.3 Quantum Data Processing Inequality

Consider two CPTP maps $\$_1$ and $\$_2$ acting on system Q. Call the initial state of Q ρ^Q, the output of the first map $\rho^{Q'} = \$(\rho^Q)$ and the output of the second map $\rho^{Q''} = \$_2 \circ \$_1(\rho^Q)$. Purifying the initial state with a system R and using the Stinespring dilations of the CPTP maps, we can regard this transformation as taking the pure state Ψ^{RQ} to $\Psi^{RQ'E_1}$ and then to $\Psi^{RQ''E_1E_2}$, where E_1 (E_2) is the environment of the first (second) map, so that E_1E_2 is the environment of the concatenated map $\$_2 \circ \$_1$. Now define the *coherent information* $I(A\rangle B) = -S(A|B)$. Show that

$$S(Q) \geq I(R\rangle Q') \geq I(R\rangle Q'').$$

Hint: use (strong) subadditivity.

Solution: The first inequality follows from subadditivity and the second from strong subadditivity. Observe that if C purifies AB, then $-S(A|B) = -S(AB) + S(B) = -S(C) + S(AC) = S(A|C)$, so $I(A\rangle B) = S(A|C)$. In the current context we have $I(R\rangle Q') = S(R|E_1) \leq S(R) = S(Q)$, where the last steps follow from the facts that system R is not involved in the transformation and system RQ is pure. For the second inequality we use strong subadditivity: $I(R\rangle Q'') = S(R|E_1E_2) \leq S(R|E_1) = I(R\rangle Q')$.