Quantum Information Theory

PD Dr. Joseph M. Renes

Winter Semester 2012/2013

TECHNISCHE UNIVERSITÄT DARMSTADT

Problem Set #4 **07 March 2013**

Problem 4.1 Some common quantum channels

Find Kraus representations for the following qubit channels

- The dephasing channel: $\rho \to \rho' = \$(\rho) = (1-p)\rho + p\,\mathrm{diag}(\rho_{00}, \rho_{11})$ (the off-diagonal elements are annihiliated with probability p).

  Solution: The dephased output is the same as measuring the state in the standard basis: $\mathrm{diag}(\rho_{00}, \rho_{11}) = \sum_{j=0}^{1} P_j \rho P_j$ for $P_j = |j\rangle\langle j|$. Thus possible Kraus operators are $A_2 = \sqrt{1-p}\mathbb{1}$, $A_j = \sqrt{p} P_j$, $j = 0, 1$. But we can find a representation with fewer Kraus operators. Notice that $\sigma_z \rho \sigma_z = \begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{pmatrix}$. Thus $(\rho + \sigma_z \rho \sigma_z)/2 = \mathrm{diag}(\rho_{00}, \rho_{11})$ and $\rho' = \sum_{j=0}^{1} A_j \rho A_j^\dagger$ for $A_0 = \sqrt{1-p/2}\mathbb{1}$ and $A_1 = \sqrt{p/2}\sigma_z$.

- The depolarizing channel: $\rho \to \rho' = \$(\rho) = (1-p)\rho + p\mathbb{1}/2$.

  Solution: Since the action of conjugation by σ_z destroys off-diagonal elements in the basis in which σ_z is diagonal, we could try separately conjugating by each Pauli operator and see what happens. The result is that $\$(\rho) = \sum_{j=1}^{4} A_j \rho A_j^\dagger$ for $A_0 = \sqrt{1-3p/2}\mathbb{1}$, $A_k = \sqrt{p/2}\sigma_k$, $k = 1, 2, 3$.

- The amplitude damping (damplitude) channel, defined by the action $|00\rangle \to |00\rangle$, $|10\rangle \to \sqrt{1-p}|10\rangle + \sqrt{p}|01\rangle$

  Solution: From the unitary action we can read off the Kraus operators since $U|\psi\rangle|0\rangle = \sum_k A_k |\psi\rangle|k\rangle$. Therefore $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$.

Problem 4.2 Unital Channels

A superoperator $\$$ is unital if $\$(\mathbb{1}) = \mathbb{1}$, or in terms of Kraus operators, $\sum_k A_k A_k^\dagger = \mathbb{1}$. Show that the eigenvalues of the output of a unital superoperator majorize the eigenvalues of the input. (Recall that a vector p majorizes q if there exists a doubly stochastic matrix D such that $q = Dp$.) *Hint:* Express the input (output) as $U\Lambda U^\dagger$ ($V\Lambda' V^\dagger$) for U, V unitary and Λ, Λ' diagonal.

Solution: Let $B_k = V^\dagger A_k U$. Then it's easy to verify that $\Lambda' = \sum_k B_k \Lambda B_k^\dagger$ and that $\sum_k B_k B_k^\dagger = \sum_k B_k^\dagger B_k = \mathbb{1}$. Now consider the component form of each of these equations:

$$\begin{aligned}
(\Lambda')_\ell &= \sum_{k,n} (B_k)_{\ell n} (B_k^\dagger)_{n\ell} (\Lambda_n), \\
\delta_{\ell m} &= \sum_{k,n} (B_k)_{\ell n} (B_k^\dagger)_{nm}, \\
\delta_{\ell m} &= \sum_{k,n} (B_k^\dagger)_{\ell n} (B_k)_{nm}.
\end{aligned}$$

We only need one index for Λ and Λ' since they are diagonal. Defining $D_{\ell n} = \sum_k (B_k)_{\ell n} (B_k^\dagger)_{n\ell}$, we have $\Lambda' = D\Lambda$ (thinking of Λ and Λ' as vectors), and the two conditions on the B_k imply that D is doubly stochastic.

Problem 4.3 "All-or-Nothing" Violation of Local Realism

Consider the three qubit state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)_{123}$, the Greenberger-Horne-Zeilinger state.

a) Show that $|\mathrm{GHZ}\rangle$ is a simultaneous eigenstate of $X_1 Y_2 Y_3$, $Y_1 X_2 Y_3$, and $Y_1 Y_2 X_3$ with eigenvalue $+1$, where X and Y are the corresponding Pauli operators.

   Solution: Observe that the three operators commute, since X and Y anticommute. Since the state is invariant under permutations of the three systems, we only need to check that it is an eigenstate of the first operator, since the others are generated from it by permutation. Both X and Y flip bits in the standard basis, but Y adds an extra $-\mathrm{i}$ if the input is $|0\rangle$ and i if $|1\rangle$. Thus $XYY|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(-\mathrm{i})^2|111\rangle - (\mathrm{i}^2)|000\rangle = |\mathrm{GHZ}\rangle$.

b) Use the results of part (a) to argue by Einstein locality that each qubit has well-defined values of X and Y. For qubit j, denote these values by x_j and y_j. We say that these values are *elements of reality*. What would local realism, i.e. the assumption of realistic values that are undisturbed by measurements on other qubits, predict for the product of the outcomes of measurements of X on each qubit?

   Solution: Measuring Y on any two systems determines the X value on the third, so absent any "spooky action at a distance", the X value should be well-defined. Similarly, measurements of X and Y on any two determine the Y value of the third, so it should also be well-defined. For X measurements on each spin, the product $X_1X_2X_3 = 1$ since $X_1X_2X_3 = (X_1Y_2Y_3)(Y_1X_2Y_3)(Y_1Y_2X_3)$ (if X_j and Y_k all take the values ± 1.)

c) What does quantum mechanics predict for the product of the outcomes of measurements of X on each qubit?

   Solution: Measuring X on each system and taking the product is the same as measuring $X_1X_2X_3$. $|\mathrm{GHZ}\rangle$ is clearly an eigenstate of this operator with eigenvalue -1, so $X_1X_2X_3 = -1$.