Quantum Information Theory

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Problem Set #2 **05 March 2013**

Problem 2.1 Indirect Measurement

Suppose a quantum system is prepared in one of two nonorthogonal states $|\varphi_1\rangle$ or $|\varphi_2\rangle$. We would like to make a measurement to determine which state was prepared, but do so without disturbing the state. To this end, we could consider making an indirect measurement in which we also prepare an auxiliary state $|\text{blank}\rangle$, apply a unitary U_{AB} which has the action

$$|\varphi_j\rangle_A|\text{blank}\rangle_B \to \mathrm{U_{AB}}|\varphi_\mathrm{j}\rangle_\mathrm{A}|\text{blank}\rangle_\mathrm{B} = |\varphi_\mathrm{j}\rangle_\mathrm{A}|\beta_\mathrm{j}\rangle_\mathrm{B},$$

and then measure system B in some way. This scheme evidently does not disturb the state of system A. What is the most we can learn about which state was prepared? What if the two states $|\varphi_j\rangle$ are orthogonal?

Solution: Since U is unitary, it preserves inner products, and therefore

$$\langle\varphi_1|\varphi_2\rangle = \langle\varphi_1|\varphi_2\rangle\langle\text{blank}|\text{blank}\rangle = \langle\varphi_1|\varphi_2\rangle\langle\beta_1|\beta_2\rangle \qquad \Rightarrow \quad \langle\beta_1|\beta_2\rangle = 1$$

This means the states $|\beta_j\rangle$ are identical, and are therefore completely indistinguishable. The implication only holds if $\langle\varphi_1|\varphi_2\rangle \neq 0$, i.e. except when the states are orthogonal, since in that case an extra factor $\langle\beta_1|\beta_2\rangle = 0$ will not violate the equality. Of course, orthogonal states can already be distinguished by direct, nondisturbing measurement of the associated projectors $P_j = |\varphi_j\rangle\langle\varphi_j|$.

Problem 2.2 Teleportation Redux

a) Show that for the entangled state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and any unitary operator U,

$$\left(U_A \otimes U_B^*\right)|\Phi\rangle_{AB} = |\Phi\rangle_{AB},$$

where $*$ denotes complex conjugation in the $|0\rangle, |1\rangle$ basis.

Solution:

$$\begin{aligned}
\left(U_A \otimes U_B^*\right)|\Phi\rangle_{AB} &= \frac{1}{\sqrt{2}}\sum_{jk\ell mt} U_{jk}U^*_{\ell m}\left(|j\rangle\langle k|_A \otimes |\ell\rangle\langle m|_B\right)|t,t\rangle_{AB} \\
&= \frac{1}{\sqrt{2}}\sum_{j\ell t} U_{jt}U^*_{\ell t}|j,\ell\rangle_{AB} = \frac{1}{\sqrt{2}}\sum_{j\ell}|j,\ell\rangle_{AB}\sum_t U_{jt}(U^\dagger)_{t\ell} \\
&= \frac{1}{\sqrt{2}}\sum_{j\ell}|j,\ell\rangle_{AB}\left[UU^\dagger\right]_{j\ell} = \frac{1}{\sqrt{2}}\sum_j |j,j\rangle_{AB} = |\Phi\rangle_{AB}
\end{aligned}$$

b) Show that for any state $|\psi\rangle$

$$ {}_A\langle\psi|\Phi\rangle_{AB} = \tfrac{1}{\sqrt{2}}|\psi^*\rangle_B. $$

Solution:

$$ {}_A\langle\psi|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\sum_{jk}\psi^*_{j\,A}\langle j|k\rangle_A|k\rangle_B = \frac{1}{\sqrt{2}}\sum_k \psi^*_k|k\rangle_B = \frac{1}{\sqrt{2}}|\psi^*\rangle_B \tag{1} $$

c) Use the results of (a) and (b) to give a derivation of the teleportation protocol without resorting to components.

Solution: Alice and Bob start out with the state $|\psi\rangle_{A'}|\Phi\rangle_{AB}$, where Alice holds systems A and A' and Bob B. When Alice measures $A'A$ in the Bell basis $|\Phi_j\rangle = \left(\mathbb{1}\otimes\sigma_j\right)|\Phi\rangle$, obtaining result j, the resulting state of $|\psi'_j\rangle_B$ Bob's system is

$$ \begin{aligned} |\psi'_j\rangle_B &= {}_{A'A}\langle\Phi_j|\left(|\psi\rangle_{A'}|\Phi\rangle_{AB}\right) = {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes(\sigma_j^\dagger)_A\otimes\mathbb{1}_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB} \\ &= {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes\mathbb{1}_A\otimes(\sigma_j^*)_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB} = \tfrac{1}{\sqrt{2}}{}_A\langle\psi^*|\left(\mathbb{1}_A\otimes(\sigma_j^*)_B\right)|\Phi\rangle_{AB} \\ &= \tfrac{1}{2}(\sigma_j^*)_B|\psi\rangle_B = \tfrac{1}{2}(\sigma_j)_B|\psi\rangle_B. \end{aligned} $$

(The last equality follows since σ_x and σ_z have real entries, $\sigma_y^* = -\sigma_y$, and we don't care about overall phases.) Alice then tells Bob the j she obtained in the measurement (which takes two bits of communication), and then he applies σ_j^T to get $|\psi\rangle$.

d) What happens if Alice and Bob use the state $(\mathbb{1}_A\otimes U_B)|\Phi\rangle_{AB}$ for teleportation? Or if Alice measures in the basis $U^\dagger_{A'}|\Phi_j\rangle_{A'A}$?

Solution: In the first case we have

$$ \begin{aligned} |\psi'_j\rangle_B &= {}_{A'A}\langle\Phi_j|\left(|\psi\rangle_{A'}U_B|\Phi\rangle_{AB}\right) = {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes(\sigma_j^\dagger)_A\otimes U_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB} \\ &= {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes\mathbb{1}_A\otimes(U\sigma_j^*)_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB} = \tfrac{1}{2}(U\sigma_j^*)_B|\psi\rangle_B. \end{aligned} $$

After Bob receives Alice's message and applies σ_j^T they end up with the state $|\psi''_j\rangle = (\sigma_j^T U\sigma_j^*)|\psi\rangle$. For the second case

$$ |\psi'_j\rangle_B = {}_{A'A}\langle\Phi_j|U_{A'}\left(|\psi\rangle_{A'}U_B|\Phi\rangle_{AB}\right) = {}_{A'A}\langle\Phi_j|\left(|U\psi\rangle_{A'}U_B|\Phi\rangle_{AB}\right) = \tfrac{1}{2}(\sigma_j^*U)_B|\psi\rangle_B. $$

Now Bob's correction operation produces $|\psi''\rangle = U|\psi\rangle$. This is an important result, because it shows that it is possible to perform an arbitrary single-qubit operation solely by measuring an appropriately prepared state.

e) Instead of a single system state $|\psi\rangle_{A'}$, Alice has a bipartite state $|\psi\rangle_{A_1A_2}$. What happens if she performs the teleportation protocol on system A_2?

Solution: Work with the Schmidt decomposition: $|\psi\rangle_{A_1A_2} = \sum_k \sqrt{p_k}|\alpha_k\rangle_{A_1}|\beta_k\rangle_{A_2}$. Then following the same calculation above we get

$$\begin{aligned} |\psi'_j\rangle_{A_1B} &= {}_{A_2A}\langle\Phi_j| \left(|\psi\rangle_{A_1A_2}|\Phi\rangle_{AB}\right) = \sum_k \sqrt{p_k}\ {}_{A_2A}\langle\Phi_j|(|\alpha_k\rangle_{A_1}|\beta_k\rangle_{A_2}|\Phi\rangle_{AB}) \\ &= \sum_k \sqrt{p_k}|\alpha_k\rangle_{A_1}\ {}_{A_2A}\langle\Phi_j|(|\beta_k\rangle_{A_2}|\Phi\rangle_{AB}) = \tfrac{1}{2}\sum_k \sqrt{p_k}|\alpha_k\rangle_{A_1}(\sigma_j^*)_B|\beta_k\rangle_B \\ &= \tfrac{1}{2}(\sigma_j^*)_B|\psi\rangle_{A_1B}. \end{aligned}$$

Once again Bob can undo the σ_j^* on system B and thus teleportation can also faithfully transfer part of a larger, entangled system.

Problem 2.3 Remote Copy

Alice and Bob would like to create the state $|\Psi\rangle_{AB} = a|00\rangle_{AB} + b|11\rangle_{AB}$ from Alice's state $|\psi\rangle_A = a|0\rangle_A + b|1\rangle_A$, a "copy" in the quantum-mechanical sense. Additionally, they share the canonical entangled state $|\Phi\rangle$. Can they create the desired state by performing only local operations (measurements and unitary operators), provided Alice can only send one bit of classical information to Bob?

Solution: By the solution to part (e) of the previous problem, Alice could create the copied state herself using the CNOT gate $U_{\text{CNOT}}|j,k\rangle = |j, j \oplus k\rangle$ and then teleport half of it to Bob. However, this would take two bits of communication. Suppose Alice copies $|\psi\rangle_A$ to her half of the maximally entangled state $|\Phi\rangle_{A'B}$. This results in

$$\begin{aligned} U_{\text{CNOT}}^{AA'}|\psi\rangle_A|\Phi\rangle_{A'B} &= \tfrac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle\right)_{AA'B} \\ &= \tfrac{1}{\sqrt{2}}\left[(a|00\rangle + b|11\rangle)_{AB}|0\rangle_{A'} + (a|01\rangle + b|10\rangle)_{AB}|1\rangle_{A'}\right] \\ &= \tfrac{1}{\sqrt{2}}\left(|\Psi\rangle_{AB}|0\rangle_{A'} + (\sigma_x)_B|\Psi\rangle_{AB}|1\rangle_{A'}\right). \end{aligned}$$

As in teleportation, this creates the desired output state, up to the action of a Pauli operator on Bob's system which is indexed by an orthogonal state at Alice's end. By measuring system A' and telling Bob the result (using just one bit since there are only two outcomes) he can undo the Pauli operator to create $|\Psi\rangle_{AB}$.