Quantum Information Theory

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Problem Set #1 **25 January 2013**

Problem 1.1 The Hadamard Gate

An important qubit transformation in quantum information theory is the Hadamard gate. In the basis of $\sigma_{\hat{z}}$, it takes the form

$$H = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}$$

That is to say, if $|0\rangle$ and $|1\rangle$ are the $\sigma_{\hat{z}}$ eigenstates, corresponding to eigenvalues $+1$ and -1, respectively, then

$$H = \tfrac{1}{\sqrt{2}} \left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right) \tag{2}$$

a) Show that H is unitary.

Solution: A matrix U is unitary when $U^\dagger U = \mathbb{I}$. In fact, $H^\dagger = H$, so we just need to verify that $H^2 = \mathbb{I}$, which is the case.

b) What are the eigenvalues and eigenvectors of H?

Solution: Since $H^2 = \mathbb{I}$, its eigenvalues must be ± 1. If both eigenvalues were equal, it would be proportional to the identity matrix. Thus, one eigenvalue is $+1$ and the other -1. By direct calculation we can find that the (normalized) eigenvectors are

$$|\lambda_\pm\rangle = \pm\frac{\sqrt{2 \pm \sqrt{2}}}{2}|0\rangle + \frac{1}{\sqrt{2(2 \pm \sqrt{2})}}|1\rangle \tag{3}$$

c) What form does H take in the basis of $\sigma_{\hat{x}}$? $\sigma_{\hat{y}}$?

Solution: The eigenbasis of $\sigma_{\hat{x}}$ is formed by the two states $|\hat{x}_\pm\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. From the form of H given in (2), it is clear that we can express H as

$$H = |\hat{x}_+\rangle\langle 0| + |\hat{x}_-\rangle\langle 1| \quad \text{or} \tag{4}$$

$$H = |0\rangle\langle \hat{x}_+| + |1\rangle\langle \hat{x}_-| \tag{5}$$

The latter form follows immediately from the first since $H^\dagger = H$. Finally, we can express the $\sigma_{\hat{z}}$ basis $|0/1\rangle$ in terms of the $\sigma_{\hat{x}}$ basis as $|0\rangle = \tfrac{1}{\sqrt{2}}(|\hat{x}_+\rangle + |\hat{x}_-\rangle)$ and $|1\rangle = \tfrac{1}{\sqrt{2}}(|\hat{x}_+\rangle - |\hat{x}_-\rangle)$. Thus, if we replace $|0\rangle$ and $|1\rangle$ by these expressions in the equation for H we find

$$H = |0\rangle\langle \hat{x}_+| + |1\rangle\langle \hat{x}_-| = \tfrac{1}{\sqrt{2}} \left(|\hat{x}_+\rangle\langle \hat{x}_+| + |\hat{x}_-\rangle\langle \hat{x}_+| + |\hat{x}_+\rangle\langle \hat{x}_-| - |\hat{x}_-\rangle\langle \hat{x}_-|\right). \tag{6}$$

Evidently, H has exactly the same representation in the $\sigma_{\hat{x}}$ basis! In retrospect, we should have anticipated this immediately once we noticed that H interchanges the $\sigma_{\hat{z}}$ and $\sigma_{\hat{x}}$ bases.

For $\sigma_{\hat{y}}$, we can proceed differently. What is the action of H on the $\sigma_{\hat{y}}$ eigenstates? These are $|\hat{y}_{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. Thus,

$$\begin{aligned} H|\hat{y}_{\pm}\rangle &= \tfrac{1}{\sqrt{2}}(H|0\rangle \pm iH|1\rangle) && (7)\\ &= \tfrac{1}{2}(|0\rangle + |1\rangle \pm i|0\rangle \mp i|1\rangle) && (8)\\ &= \left(\frac{1\pm i}{2}\right)|0\rangle + \left(\frac{1\mp i}{2}\right)|1\rangle && (9)\\ &= \tfrac{1}{\sqrt{2}}e^{i\pm\frac{\pi}{4}}\left(|0\rangle + \left(\frac{1\mp i}{1\pm i}\right)|1\rangle\right) && (10)\\ &= \tfrac{1}{\sqrt{2}}e^{i\pm\frac{\pi}{4}}(|0\rangle \mp i|1\rangle) && (11)\\ &= e^{i\pm\frac{\pi}{4}}|\hat{y}_{\mp}\rangle && (12) \end{aligned}$$

Therefore, the Hadamard operation just swaps the two states in the basis (note that if we used a different phase convention for defining the $\sigma_{\hat{y}}$ eigenstates, there would be extra phase factors in this equation). So, $H = \begin{pmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{pmatrix}$ in this basis.

d) Give a geometric interpretation of the action of H in terms of the Bloch sphere.

Solution: All unitary operators on a qubit are rotations of the Bloch sphere by some angle about some axis. Since $H^2 = \mathbb{I}$, it must be a π rotation. Because the $\hat{y}$-axis is interchanged under H, the axis must lie somewhere in the $\hat{x}$-$\hat{z}$ plane. Finally, since H interchanges the $\sigma_{\hat{x}}$ and $\sigma_{\hat{z}}$ bases, it must be a rotation about the $\hat{m} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$ axis.

Problem 1.2 State Distinguishability

One way to understand the cryptographic abilities of quantum mechanics is from the fact that non-orthogonal states cannot be perfectly distinguished.

a) In the course of a quantum key distribution protocol, suppose that Alice randomly chooses one of the following two states and transmits it to Bob:

$$|\varphi_0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad \text{or} \qquad |\varphi_1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle). \tag{13}$$

Eve intercepts the qubit and performs a measurement to identify the state. The measurement consists of the orthogonal states $|\psi_0\rangle$ and $|\psi_1\rangle$, and Eve guesses the transmitted state was $|\varphi_0\rangle$ when she obtains the outcome $|\psi_0\rangle$, and so forth. What is the probability that Eve correctly guesses the state, averaged over Alice's choice of the state for a given measurement? What is the optimal measurement Eve should make, and what is the resulting optimal guessing probability?

Solution: The the probability of correctly guessing, averaged over Alice's choice of the state is

$$p_{\text{guess}} = \tfrac{1}{2}(|\langle\psi_0|\varphi_0\rangle|^2 + |\langle\psi_1|\varphi_1\rangle|^2) \tag{14}$$

To optimize the choice of measurement, suppose $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ for some $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. Then $|\psi_1\rangle = -\beta^*|0\rangle + \alpha^*|1\rangle$ is orthogonal as intended. Using this in (14) gives

$$p_{\text{guess}} = \frac{1}{2}\left(\left|\frac{\alpha^* + \beta^*}{\sqrt{2}}\right|^2 + \left|\frac{i\alpha - \beta}{\sqrt{2}}\right|^2\right) \tag{15}$$

$$= \tfrac{1}{2}(1 + 2\mathrm{Re}\left[\left(\tfrac{1-i}{2}\right)\alpha\beta^*\right]). \tag{16}$$

If we express α and β as $\alpha = ae^{i\theta}$ and $\beta = be^{i\eta}$ for real a, b, θ, η, then we get

$$p_{\text{guess}} = \tfrac{1}{2}(1 + 2ab\mathrm{Re}\left[\left(\tfrac{1-i}{2}\right)e^{i(\theta-\eta)}\right]). \tag{17}$$

To maximize, we ought to choose $a = b = \frac{1}{\sqrt{2}}$, and we may also set $\eta = 0$ since only the difference $\theta - \eta$ is relevant. Now we have

$$p_{\text{guess}} = \tfrac{1}{2}(1 + \mathrm{Re}\left[\left(\tfrac{1-i}{2}\right)e^{i\theta}\right]) \tag{18}$$

$$= \tfrac{1}{2}(1 + \tfrac{1}{\sqrt{2}}\mathrm{Re}\left[e^{-i\pi/4}e^{i\theta}\right]), \tag{19}$$

from which it is clear that the best thing to do is to set $\theta = \pi/4$ to get $p_{\text{guess}} = \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) \approx 85.4\%$. The basis states making up the measurement are $|\psi_0\rangle = \frac{1}{\sqrt{2}}(e^{i\pi/4}|0\rangle + |1\rangle)$ and $|\psi_1\rangle = \frac{1}{\sqrt{2}}(-|0\rangle + e^{-i\pi/4}|1\rangle)$.

b) Now suppose Alice randomly chooses between two states separated by an angle θ on the Bloch sphere. What is the measurement which optimizes the guessing probability? What is the resulting probability of correctly identifying the state?

Solution: The point of this exercise is to show that thinking in terms of the Bloch sphere is a lot more intuitive than just taking a brute force approach as we did in the solution of the previous exercise. Let $\hat{n}_0$ and $\hat{n}_1$ be the Bloch vectors of the two states. Call $\hat{m}$ the Bloch vector associated with one of the two basis vectors of the measurement, specifically the one which indicates that the state is $|\varphi_0\rangle$ (the other is associated with $-\hat{m}$). The guessing probability takes the form

$$p_{\text{guess}} = \tfrac{1}{2}(|\langle\psi_0|\varphi_0\rangle|^2 + |\langle\psi_1|\varphi_1\rangle|^2) \tag{20}$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}(1 + \hat{n}_0 \cdot \hat{m}) + \tfrac{1}{2}(1 - \hat{n}_1 \cdot \hat{m})\right) \tag{21}$$

$$= \tfrac{1}{4}\left(2 + \hat{m} \cdot (\hat{n}_0 - \hat{n}_1)\right) \tag{22}$$

The optimal $\hat{m}$ lies along $\hat{n}_0 - \hat{n}_1$ and has unit length, i.e.

$$\hat{m} = \frac{\hat{n}_0 - \hat{n}_1}{\sqrt{(\hat{n}_0 - \hat{n}_1) \cdot (\hat{n}_0 - \hat{n}_1)}} \tag{23}$$

$$= \frac{\hat{n}_0 - \hat{n}_1}{\sqrt{2 - 2\cos\theta}}. \tag{24}$$

Therefore,

$$p_{\text{guess}} = \tfrac{1}{4}\left(2 + \sqrt{2 - 2\cos\theta}\right) \tag{25}$$
$$= \tfrac{1}{2}\left(1 + \sqrt{\frac{1-\cos\theta}{2}}\right) \tag{26}$$
$$= \tfrac{1}{2}\left(1 + \sin\tfrac{\theta}{2}\right). \tag{27}$$

Finally, we should check that this gives sensible results. When $\theta = 0$, $p_{\text{guess}} = \frac{1}{2}$, as it should. On the other hand, the states $|\varphi_k\rangle$ are orthogonal for $\theta = \pi$, and indeed $p_{\text{guess}} = 1$ in this case. In the previous exercise we investigated the case $\theta = \frac{\pi}{2}$ and here we immediately find $p_{\text{guess}} = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$, as before.

Problem 1.3 Fidelity

a) Given a qubit prepared in a completely unknown state $|\psi\rangle$, what is the *fidelity* F of a random guess $|\varphi\rangle$, where $F(|\varphi\rangle, |\psi\rangle) = |\langle\varphi|\psi\rangle|^2$? The fidelity can be thought of as the probability that an input state (the guess) $|\varphi\rangle$ passes the "ψ" test, which is the measurement in the basis $|\psi\rangle, |\psi^\perp\rangle$.

Solution: First let's just try to guess the result. The unknown state $|\psi\rangle$ is somewhere on the Bloch sphere, and we might as well orient the sphere so that this direction is the $\hat{z}$ direction. The fidelity of $|\psi\rangle$ with any other state $|\varphi\rangle$ is given by

$$|\langle\psi|\varphi\rangle|^2 = \text{Tr}[P_\psi P_\varphi] = \tfrac{1}{2}(1 + \cos\theta), \tag{28}$$

where θ is the angle between the two states on the Bloch sphere. Any state in the $\hat{x}$-$\hat{y}$ plane has a fidelity of $\frac{1}{2}$, and since a random state is as likely to lie in the upper hemisphere as in the lower, i.e. $\theta = \frac{\pi}{2} + \alpha$ and $\theta = \frac{\pi}{2} - \alpha$ are equally-likely, the average fidelity ought to be $\frac{1}{2}$. A simple integration confirms this guess:

$$\langle F\rangle = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \, \frac{1}{2}(1 + \cos\theta) = \frac{1}{4}\int_0^{\pi} \mathrm{d}\theta \sin\theta = \frac{1}{2}. \tag{29}$$

b) In order to improve the guess, we might make a measurement of the qubit, say along the $\hat{z}$ axis. Given the result $k \in \{0, 1\}$, our guess is then the state $|k\rangle$. What is the average fidelity of the guess after the measurement, i.e. the probability of passing the "ψ" test?

Solution: Given the outcome $|k\rangle$, the fidelity is $F_k = |\langle k|\psi\rangle|^2$ and this occurs with probability $p_k = |\langle k|\psi\rangle|^2$, so averaging over the measurement outcome gives $F = \sum_k p_k F_k = \sum_k |\langle k|\psi\rangle|^4$. Now we average over $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle$:

$$\langle F\rangle = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \left(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}\right) = \frac{2}{3}. \tag{30}$$

Thus making the measurement increases the fidelity of the guess.