Quantum Information Theory

PD Dr. Joseph M. Renes

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TECHNISCHE UNIVERSITÄT DARMSTADT

Problem Set #6 **14 March 2013**

Problem 6.1 Davies' Theorem

Consider an arbitrary CQ state $\sigma^{XB} = \sum_x p_x |x\rangle\langle x|^X \otimes \rho_x^B$ and imagine making a measurement $\mathscr{M}$ having elements E_y on B. By the Holevo bound, $I(X{:}Y) \leq I(X{:}B) = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$. Define the *accessible information* $I_{\text{acc}}(\sigma^{XB}) = \max_{\mathscr{M}} I(X:Y)$.

Show that the optimal measurement consists of rank-one elements and has no more than d^2 outcomes, where $d = \dim(B)$. Hint: the space of Hermitian operators on B is a vector space of size d^2.

Problem 6.2 Upper Bound on the Classical Capacity of a Quantum Channel

Suppose that Alice would like to send classical information to Bob over a quantum channel \$. For this purpose she uses $m < d$ quantum states σ_x^Q of dimension d encoding the message x with probability p_x. The channel turns the σ_x into $\rho_x^Q = \$(\sigma_x^Q)$. Bob makes a measurement which consists of at least m elements Λ_y, one for each message, plus one more to ensure that $\sum_y \Lambda_y = \mathbb{1}$. The average error probability is therefore $p = \frac{1}{m}\sum_{x=1}^m p_x \text{Tr}[(\mathbb{1} - \Lambda_x)\$(\sigma_x)]$.
Show that $p \geq (\log_2 m - S(X:Q)_\xi - 1)/\log_2 d$, where $\xi^{XQ} = \sum_{x=1}^m p_x |x\rangle\langle x|^X \otimes \rho_x^Q$.
Hint: use Fano's inequality.

Problem 6.3 Quantum Data Processing Inequality

Consider two CPTP maps $\$_1$ and $\$_2$ acting on system Q. Call the initial state of Q ρ^Q, the output of the first map $\rho^{Q'} = \$(\rho^Q)$ and the output of the second map $\rho^{Q''} = \$_2 \circ \$_1(\rho^Q)$. Purifying the initial state with a system R and using the Stinespring dilations of the CPTP maps, we can regard this transformation as taking the pure state Ψ^{RQ} to $\Psi^{RQ'E_1}$ and then to $\Psi^{RQ''E_1E_2}$, where E_1 (E_2) is the environment of the first (second) map, so that E_1E_2 is the environment of the concatenated map $\$_2 \circ \$_1$. Now define the *coherent information* $I(A\rangle B) = -S(A|B)$. Show that

$$S(Q) \geq I(R\rangle Q') \geq I(R\rangle Q'').$$

Hint: use (strong) subadditivity.