Quantum Information Theory

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Problem Set #5 **13 March 2013**

Problem 5.1 Information and Description Length

Suppose we have a collection of 12 balls, all identical except that one is either lighter or heavier than the rest. At our disposal is a two-pan balance onto which we can place any number of balls in the left pan and the same number in the right pan. The balance registers which of the pans is heavier, or that they are equal. What is the fewest number of weighings needed to determine the odd ball and whether it is lighter or heavier? It might be useful to consider the following questions:

a) How much information is gained upon learning (i) the state of a flipped coin; (ii) the states of two flipped coins; (iii) the outcome of the roll of a four-sided die?

b) How much information is gained when the odd ball and its weight are identified?

c) How much information is gained on the first step if six balls are weighed against the other six? How much is gained by first weighing four against another four, leaving the rest aside?

d) For your prospective weighing strategy, draw a tree showing the possible outcomes of the chosen weighing and what weighing is to be performed next. At each node, how much information has been gained and how much remains to be gained?

Problem 5.2 Data Processing Inequality

Random variables X, Y, Z form a Markov chain $X \to Y \to Z$ if the conditional distribution of Z depends only on Y: $p(z|x,y) = p(z|y)$. The goal in this exercise is to prove the data processing inequality, $I(X:Y) \geq I(X:Z)$ for $X \to Y \to Z$.

a) First show the chain rule for mutual information: $I(X:YZ) = I(X:Z) + I(X:Y|Z)$, which holds for arbitrary X, Y, Z. The conditional mutual information is defined as

$$I(X:Y|Z) = \sum_z p(z) I(X:Y|Z=z) = \sum_z p(z) \sum_{x,y} p(x,y|z) \log \frac{p(x,y|z)}{p(x|z)p(y|z)}.$$

b) Next show that in a Markov chain $X \to Y \to Z$, X and Z are conditionally independent given Y; that is, $p(x,z|y) = p(x|y)p(z|y)$.

c) By expanding the mutual information $I(X:YZ)$ in two different ways, prove the data processing in equality.

Problem 5.3 Fano's Inequality

Given random variables X and Y, how well can we predict X given Y? Fano's inequality bounds the probability of error in terms of the conditional entropy $H(X|Y)$. The goal of this exercise is to prove the inequality

$$P_{\text{error}} \geq \frac{H(X|Y) - 1}{\log |X|}.$$

a) Representing the guess of X by the random variable $\widehat{X}$, which is some function, possibly random, of Y, show that $H(X|\widehat{X}) \geq H(X|Y)$.

b) Consider the indicator random variable E which is 1 if $\widehat{X} \neq X$ and zero otherwise. Using the chain rule we can express the conditional entropy $H(E, X|\widehat{X})$ in two ways:

$$H(E, X|\widehat{X}) = H(E|X, \widehat{X}) + H(X|\widehat{X}) = H(X|E, \widehat{X}) + H(E|\widehat{X}) \tag{1}$$

Calculate each of these four expressions and complete the proof of the Fano inequality. Hints: For $H(E|\widehat{X})$ use the fact that conditioning reduces entropy: $H(E|\widehat{X}) \leq H(E)$. For $H(X|E, \widehat{X})$ consider the cases $E = 0, 1$ individually.