Problem Set #1 **25 January 2013**

Problem 1.1 The Hadamard Gate

An important qubit transformation in quantum information theory is the Hadamard gate. In the basis of $\sigma_{\hat{z}}$, it takes the form

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}$$

That is to say, if $|0\rangle$ and $|1\rangle$ are the $\sigma_{\hat{z}}$ eigenstates, corresponding to eigenvalues $+1$ and -1, respectively, then

$$H = \frac{1}{\sqrt{2}} \left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1| \right) \tag{2}$$

a) Show that H is unitary.

b) What are the eigenvalues and eigenvectors of H?

c) What form does H take in the basis of $\sigma_{\hat{x}}$? $\sigma_{\hat{y}}$?

d) Give a geometric interpretation of the action of H in terms of the Bloch sphere.

Problem 1.2 State Distinguishability

One way to understand the cryptographic abilities of quantum mechanics is from the fact that non-orthogonal states cannot be perfectly distinguished.

a) In the course of a quantum key distribution protocol, suppose that Alice randomly chooses one of the following two states and transmits it to Bob:

$$|\varphi_0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad \text{or} \qquad |\varphi_1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle). \tag{3}$$

Eve intercepts the qubit and performs a measurement to identify the state. The measurement consists of the orthogonal states $|\psi_0\rangle$ and $|\psi_1\rangle$, and Eve guesses the transmitted state was $|\varphi_0\rangle$ when she obtains the outcome $|\psi_0\rangle$, and so forth. What is the probability that Eve correctly guesses the state, averaged over Alice's choice of the state for a given measurement? What is the optimal measurement Eve should make, and what is the resulting optimal guessing probability?

b) Now suppose Alice randomly chooses between two states separated by an angle θ on the Bloch sphere. What is the measurement which optimizes the guessing probability? What is the resulting probability of correctly identifying the state?

Problem 1.3 Fidelity

a) Given a qubit prepared in a completely unknown state $|\psi\rangle$, what is the *fidelity* F of a random guess $|\varphi\rangle$, where $F(|\varphi\rangle, |\psi\rangle) = |\langle\varphi|\psi\rangle|^2$? The fidelity can be thought of as the probability that an input state (the guess) $|\varphi\rangle$ passes the "ψ" test, which is the measurement in the basis $|\psi\rangle, |\psi^\perp\rangle$.

b) In order to improve the guess, we might make a measurement of the qubit, say along the $\hat{z}$ axis. Given the result $k \in \{0, 1\}$, our guess is then the state $|k\rangle$. What is the average fidelity of the guess after the measurement, i.e. the probability of passing the "ψ" test?