Problem Set #2 | # Quantum Information & Computation | **2009-05-11**

1. Decompositions of Density Matrices

Consider a mixed state ρ with two different pure state decompositions

$$\rho = \sum_{k=1}^{d} \lambda_k |k\rangle\langle k| = \sum_{\ell=1}^{n} p_\ell |\varphi_\ell\rangle\langle\varphi_\ell|,$$

the former being the eigendecomposition so that $\{|k\rangle\}$ is an orthonormal basis.

(a) Show that the probability vector $\vec{\lambda}$ *majorizes* the probability vector $\vec{p}$, which means that there exists a doubly stochastic matrix T_{jk} such that $\vec{p} = T\vec{\lambda}$. The defining property of doubly stochastic, or bistochastic, matrices is that $\sum_k T_{jk} = \sum_j T_{jk} = 1$.
Hint: Observe that for a unitary matrix U_{jk}, $T_{jk} = |U_{jk}|^2$ is doubly stochastic.

(b) The uniform probability vector $\vec{u} = (1/n, \ldots, 1/n)$ is invariant under the action of an $n \times n$ doubly stochastic matrix. Is there an ensemble decomposition of ρ such that $p_\ell = 1/n$ for all ℓ?
Hint: Try to show that $\vec{u}$ is majorized by any other probability distribution.

2. Generalized Measurement by Direct Sum

Consider a spin-1 subspace of atomic hyperfine levels, and suppose that in the basis corresponding to a magnetic field in the z-direction, only the levels corresponding to $m_z = -1$ and $m_z = 0$ angular momentum are populated (note: $\hbar = 1$). We can also formulate this situation in a basis corresponding to quantization of angular momentum about the x-axis. The transformation effecting this change of basis is given by

$$\begin{pmatrix} |m_x = 1\rangle \\ |m_x = 0\rangle \\ |m_x = -1\rangle \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} |m_z = 1\rangle \\ |m_z = 0\rangle \\ |m_z = -1\rangle \end{pmatrix}.$$

(a) Given a projection measurement onto the eigenstates in the x-basis, the resulting POVM elements in the populated two-dimensional subspace will have what rank?

(b) Calculate this POVM.

(c) Given a general measurement with d outcomes in d dimensions whose POVM elements are all rank one, is the measurement necessarily projective?

3. Neumark Extension

Construct POVMs and Neumark extensions corresponding to the following sets of Bloch vectors.

(a) The trine: $\{\hat{n}_k\}_{k=1}^{3}$ with

$$\hat{n}_1 = \hat{z} \qquad \hat{n}_2 = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z} \qquad \hat{n}_3 = -\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z}$$

(b) The tetrahedron: $\{\hat{n}_k\}_{k=1}^{4}$ with

$$\hat{n}_1 = \hat{z} \qquad \hat{n}_2 = \frac{\sqrt{8}}{3}\hat{x} - \frac{1}{3}\hat{z} \qquad \hat{n}_3 = -\frac{\sqrt{2}}{3}\hat{x} + \sqrt{\frac{2}{3}}\hat{y} - \frac{1}{3}\hat{z} \qquad \hat{n}_4 = -\frac{\sqrt{2}}{3}\hat{x} - \sqrt{\frac{2}{3}}\hat{y} - \frac{1}{3}\hat{z}$$

4. Generalized Measurement by Direct (Tensor) Product

Consider an apparatus whose purpose is to make an indirect measurement on a two-level system, A, by first coupling it to a three-level system, B, and then making a projective measurement on the latter. B is initially prepared in the state $|0\rangle$ and the two systems interact via the unitary U_{AB} as follows:

$$
\begin{aligned}
|0\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |0\rangle_A|2\rangle_B\right) \\
|1\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{6}}\left(2|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right)
\end{aligned}
$$

(a) Calculate the measurement operators acting on system A corresponding to a measurement on system B in the canonical basis $|0\rangle, |1\rangle, |2\rangle$.

(b) Calculate the corresponding POVM elements. What is their rank? Onto which states do they project?

(c) Suppose A is in the state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$. What is the state after a measurement, averaging over the measurement result?

5. Minimum-Error State Discrimination

Suppose that Alice sends Bob a signal which is either ρ_1 with probability p_1 or ρ_2 with probability p_2. Bob would like to know which one was sent, with the smallest possible error. His measurement consists of operators E_1 and E_2 such that $E_1 + E_2 = \mathbb{1}$. If outcome E_1 occurs he guesses that Alice sent ρ_1; if E_2, ρ_2.

(a) Shouldn't we consider the possibility that Bob's measurement has more than two outcomes?

(b) Show that the probability of error is given by

$$p_{\text{error}} = p_1 + \sum_i \lambda_i \langle e_i|E_1|e_i\rangle,$$

where $\{|e_i\rangle\}$ is the orthonormal basis of eigenstates of the operator $p_2\rho_2 - p_1\rho_1$, while λ_i are the corresponding eigenvalues.

(c) Find the nonnegative operator E_1 that minimizes p_{error}.

(d) Show that the corresponding error probability is

$$p^*_{\text{error}} = p_1 + \sum_{i:\lambda_i<0} \lambda_i = \frac{1}{2}(1 - ||p_2\rho_2 - p_1\rho_1||_{\text{Tr}}),$$

where $||A||_{\text{Tr}} = \sum_{\text{pos}} \lambda(A) - \sum_{\text{neg}} \lambda(A)$ is the trace-norm.

6. Unambiguous State Discrimination

Now Bob would like to decide between ρ_1 and ρ_2 as in the previous exercise, but wants to avoid the possibility of deciding incorrectly.

(a) Bob's measurement surely has outcomes E_1 and E_2 corresponding to ρ_1 and ρ_2, respectively. Assuming the two states ρ_j are pure, $\rho_j = |\varphi_j\rangle\langle\varphi_j|$ for some $|\varphi_j\rangle$, what is the general form of E_j such that $\text{prob}(E_j|\rho_k) = 0$ for $j \neq k$?

(b) Can these two elements alone make up a POVM? Is there generally an inconclusive result $E_?$?

(c) Assuming ρ_1 and ρ_2 are sent with equal probability, what is the optimal unambiguous measurement, i.e. the unambigous measurement with the smallest probability of an inconclusive result.

7. Broken Measurement

Alice and Bob share a state $|\Psi\rangle_{AB}$, and Bob would like to perform a measurement described by projectors P_j on his part of the system, but unfortunately his measurement apparatus is broken. He can still perform arbitrary unitary operations, however. Meanwhile, Alice's measurement apparatus is in good working order. Show that there exist projectors P'_j and unitaries U_j and V_j so that

$$|\Psi_j\rangle = (\mathbb{1} \otimes P_j)\,|\Psi\rangle = (U_j \otimes V_j)\left(P'_j \otimes \mathbb{1}\right)|\Psi\rangle.$$

(Note that the state is unnormalized, so that it implicitly encodes the probability of outcome j.) Thus Alice can assist Bob by performing a related measurement herself, after which they can locally correct the state.
Hint: Work in the Schmidt basis of $|\Psi\rangle$.