Measuring the State of a Bosonic Two-Mode Quantum Field

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A measurement scheme is proposed to determine the state of a bosonic two-mode system completely. Without relying on the existence of a reference field, we reconstruct the quantum state from joint number measurement only, in case $[\rho, \hat{N}_{tot}] = 0$. Based on an analogy to angular momentum, we have obtained an explicit inversion procedure for the density matrix of the system and discuss its application. [S0031-9007(97)04719-4]

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Tomographic methods already had a well established history in many areas of applied classical physics, e.g., medical imaging, geological sciences, or signal processing, before it was recognized that the quantum mechanical state is also an “object” that can be viewed from different directions. Recording the information that is gained from a particular measurement from all possible view points of the object determines a quantum state uniquely.

For the measurement of the state of a magnetic dipole (a discrete degree of freedom) Newton and Young [1] introduced the concept of rotating quantum states into tomographic physics. More recently, it was applied in the context of cavity QED [2], as well. However, state rotation is also the center piece of phase space tomography [3,4], which is applicable to systems with a continuous degree of freedom.

The latter method has found many beautiful experimental realizations such as the pioneering measurement of the state of a single quantized cavity mode [5,6]. Successively, it has been applied to the study of molecular states [7], motional states of trapped ions [8–12], and atomic beams [13]. The method has even found its way back to classical diffractive optics [14] as it is based on the superposition principle and the uncertainty relation.

By extending the original ideas of state measurement to systems with more than 1 degree of freedom [15,16], or to systems that consist of identical particles, tomography provides a systematic tool to study the implications of state entanglement or aspects of quantum statistics. This is of particular importance as the controlled generation of entangled states, e.g., Schrödinger cat states or maximally entangled GHZ states [17,18], as well as Bose-Einstein condensation (BEC) of dilute alkali gases [19,20] are both currently research topics of central interest and experimentally feasible.

Thus, tomography can be used in a complementary manner to the proposed schemes of engineering quantum states [21] in the laboratory and to determine quantum correlations of noninteracting massive condensates, unambiguously.

The purpose of the present Letter is to present a novel state measurement scheme for an a priori unknown density matrix of a bosonic two-mode field,
For this purpose, we will consider four bosonic modes with the quantum description of angular momentum [24]. The Jordan-Schwinger analogy of harmonic oscillators of modes seems feasible by applying the required unitary A generalization of this procedure to an arbitrary number of interactions, that mix two modes of a massive condensate. Alternatively, it is possible to determine the quantum state. 

The measurement procedure is essentially based on the Jordan-Schwinger analogy of harmonic oscillators with the quantum description of angular momentum [24]. For this purpose, we will consider four bosonic modes \{a_0, a_1, b_0, b_1\} with \([a_i, a_j^+] = \delta_{ij}, \ [b_i, b_j^+] = \delta_{ij}\] that are related by a unitary transformation

\[
\begin{pmatrix} b_0 \\
 b_1 \\
\end{pmatrix} = \begin{pmatrix} C_{00} & C_{01} \\
 C_{10} & C_{11} \end{pmatrix} \begin{pmatrix} a_0 \\
 a_1 \end{pmatrix} = U_R^{\dagger} \begin{pmatrix} a_0 \\
 a_1 \end{pmatrix} U_R .
\]

This transformation could describe the time evolution of a dynamical system \([a_i = a_i(t_{\text{initial}}), b_i = a_i(t_{\text{final}})]\) or represent an input-output relationship between different mode sets \((a_i = a_i^{(\text{in})}, b_i = a_i^{(\text{out})})\). The constraint to preserve the commutation relations also fixes the number of excitations within the system, i.e., \([U_R, \hat{N}_{\text{tot}}] = 0\). Following Schwinger’s concept of angular momentum, one can define the rotation group by a set of bilinear combinations of creation and annihilation operators

\[
\hat{L}_1 = \frac{1}{2}(a_0^\dagger a_1 + a_1^\dagger a_0), \quad \hat{L}_2 = \frac{1}{2i}(a_0^\dagger a_1 - a_1^\dagger a_0),
\]

\[
\hat{L}_3 = \frac{1}{2}(a_0^\dagger a_0 - a_1^\dagger a_1), \quad [\hat{L}_i, \hat{L}_j] = i \epsilon_{ijk} \hat{L}_k.
\]

The analogy with angular momentum becomes complete if a square operator \(\hat{L}^2\) is defined according to

\[
\hat{L}^2 = \sum_{i=1}^{3} \hat{L}_i^2 = \hat{\ell}(\hat{\ell} + 1), \quad \hat{\ell} = \frac{1}{2} (\hat{n}_0 + \hat{n}_1),
\]

and a certain quantization axis \(e_3\) is chosen to select a projection operator \(\hat{m} = e_3 \cdot \hat{L} = \frac{1}{2}(\hat{n}_0 - \hat{n}_1)\). The eigenvectors \(|l, m\rangle_e\) of this complete set of commuting observables (CSCO) \(\{\hat{L}^2, \hat{m}\}\) are

\[
|l, m\rangle_e \equiv |n_0 = l + m\rangle_{a_0} \otimes |n_1 = l - m\rangle_{a_1} .
\]

By choosing a rotated quantization axis, e.g., \(v = Re_3\), one obtains a different set of basis states \(|\ell, m\rangle_v\) that are labeled by the CSCO: \(\{\hat{L}^2, \hat{v} \cdot \hat{L}\}\). They induce an \(l\)-dimensional representation of the rotation group.
To be specific, we will adopt the Euler parametrization [25] for the unitary rotation operators $U_R(\alpha, \beta, \gamma)$ and the corresponding Wigner matrices $D^{(j)}_{m'm}(R) = e^{-i\alpha L_0} e^{-i\beta L_2} e^{-i\gamma L_3}$.

$$U(\alpha, \beta, \gamma) = e^{-i\alpha L_0} e^{-i\beta L_2} e^{-i\gamma L_3},$$

$$D^{(j)}_{m'm}(\alpha, \beta, \gamma) = e^{i(\beta,m' U(\alpha, \beta, \gamma))},$$

$$e^{-i\alpha L_0} e^{-i\beta L_2} e^{-i\gamma L_3}.$$ (9)

The key idea of the measurement scheme is based on a representation of the initial density matrix Eq. (1) with respect to the angular momentum states

$$\rho = \sum_{l, l'} \sum_{m=-l}^{l} |l, m\rangle \langle l, m| e_{l}^{(l)} e_{l'}^{(l')} \rho_{mm'}.$$ (11)

A temporal evolution or a unitary transformation between different modes in a beam splitter maps an initial (input) state $\psi$ onto a final (output) state $\varphi$.

$$\varphi(\alpha, \beta, \gamma) = U_R(\alpha, \beta, \gamma) \rho U_R^{\dagger}(\alpha, \beta, \gamma).$$ (12)

If the diagonal elements of Eq. (12) are taken with respect to the angular momentum basis

$$P_M(\beta, \gamma) = \langle L, M | \varphi(\alpha, \beta, \gamma) | L, M \rangle e_L,$$ (13)

one obtains an equation reminiscent of a discrete Fourier transform

$$P_M(\beta, \gamma) = \frac{2L}{\sqrt{2L}} \sum_{\nu=-L}^{L} e^{i\gamma \nu} X_{M}^{L\nu}(\beta),$$ (14)

$$X_{M}^{L\nu}(\beta) = \sum_{m=-L}^{L} d^{(L)}_{m' m}(\beta) d^{(L)}_{m' \nu + m}(\beta) \rho_{m'm' + m}.$$ (15)

A final phase shift $\alpha$ is not observable from a number measurement, thus vanishes identically.

In the following, we are going to show that Eq. (14) can be inverted with respect to the initial density matrix $\rho$, if all of the probabilities $\langle L, M, \nu| P_M(\beta, \gamma) \rangle \geq 0$, can be determined for $[\nu] \leq 2\nu$ different rotation angles $\gamma_k = 2\pi k/(4L + 1)$.

The inversion of Eq. (14) is based on two steps. First, by introducing a discrete Fourier transform of the probability $P_M(\beta, \gamma)$, we get

$$X_{M}^{L\mu}(\beta) = \frac{1}{4L + 1} \sum_{k=-2L}^{2L} e^{-i\gamma_k \mu} P_M(\beta, \gamma_k),$$

for $|\mu| \leq 2L$. So far, the inclination angle $\beta$ is an arbitrary constant.

Before inverting Eq. (15), it is worthwhile to note that an identity that is derived from the addition theorem for Wigner matrices as well as orthogonality relations for Clebsch-Gordan coefficients [1,25]

$$\delta_{mm'} = \sum_{j=0}^{2L} \sum_{M=-L}^{L} C_{m M M}^{j} C_{m M M}^{j} \times \frac{(-1)^{m-M}}{\delta_{m M M}(\beta)} \delta_{m M M}^{j}(\beta) \delta_{m M M}^{j}(\beta).$$ (16)

Finally, if this is applied to Eq. (15), one finds
measuring pairs of quantum numbers. The phase angles were equidistant, and numerically and notice a singular behavior for certain values of $\theta$. The authors reconstructed the density matrix for a constant particle number aware of a similar proposal [26]. The authors recon-
constructed the density matrix for a constant particle number.

We have simulated the outcome of an experiment numerically by evaluating joint count probabilities $P_{R(0,0,\gamma_k)}(n_0, n_1)$ [Eq. (13)] for various phase angles $\gamma_k$ from the given initial state $\rho(N = 7, \langle \hat{N}_{\text{tot}} \rangle = 3)$. This is shown in Fig. 4. With this set of data and by applying the inversion theorem Eq. (17), the initial density matrix can be recovered completely.

We have also tested the reconstruction procedure with respect to an initial density matrix that was chosen at random $\rho_{\text{rand}}(\langle \hat{N}_{\text{max}} \rangle = 8)$. The only constraints were $\text{Tr} [\rho_{\text{rand}}] = 1$, $\rho \geq 0$ (positive semidefinite) and $[\rho_{\text{rand}}, \hat{N}_{\text{tot}}] = 0$. Again, we obtained a faithful image of the original state.

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Note added.—After completing this work, we became aware of a similar proposal [26]. The authors reconstructed the density matrix for a constant particle number numerically and notice a singular behavior for certain values of $\theta$.

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[22] A generalization of the presented method to include case (i) is possible if a single mode quadrature and a number measurement on the other channel can be performed as well, i.e., $P_{R}(n_0, n_1) = \langle n_0, n_1 | U_R | n'_0, n'_1 \rangle$.