Equivalence of kinetic theories of Bose-Einstein condensation

J. Wachter, R. Walser, J. Cooper, and M. Holland
JILA, National Institute of Standards and Technology and University of Colorado, Boulder, Colorado 80309-0440
(Received 10 May 2001; published 12 October 2001)

We discuss the equivalence of two nonequilibrium kinetic theories that describe the evolution of a dilute, Bose-Einstein condensed atomic gas in a harmonic trap. The second-order kinetic equations of Walser *et al.* [Phys. Rev. A **63**, 013607 (2001)] reduce to the Gross-Pitaevskii equation and the quantum Boltzmann equation in the low- and high-temperature limits, respectively. These kinetic equations thus describe the system in equilibrium (finite temperature) as well as in nonequilibrium (real time). We have found this theory to be equivalent to the nonequilibrium Green's function approach originally proposed by Kadanoff and Baym and more recently applied to inhomogeneous trapped systems by Imamović-Tomasović and Griffin [in *Progress in Nonequilibrium Green's Functions*, edited by M. Bonitz (World Scientific, Singapore, 2000), p. 404].

DOI: 10.1103/PhysRevA.64.053612 PACS number(s): 03.75.Fi, 05.70.Ln

I. INTRODUCTION

Binary collisions are the essential mechanism for the formation of a Bose-Einstein condensate in an atomic gas. Moreover, many aspects of the system's dynamics require two-particle collisions, for example, sound propagation, the damping of elementary excitations, and the very mechanism that leads to the quantum phase transition—evaporative cooling. However, the conventional Hartree-Fock-Bogoliubov approach to generalize the Gross-Pitaevskii equation for dilute, trapped gases includes binary collisional interactions only as first-order energy shifts. Second-order kinetic theories that include collisional redistributions of excited atoms offer a more complete microscopic description of the gaseous system.

Why is a simplified kinetic description possible, when the evolution of the Bose-Einstein condensate might involve correlations between as many particles as the system contains? Would not binary collisions eventually entangle the quantum state of each atom in the system with that of every other atom? Fortunately, such complexity is not necessary to describe the measurable properties of a dilute, weakly interacting gas, because the duration of a collision, τ_0, is very short compared to the essentially interaction-free oscillation in the external potential between isolated collision events [1].

Because of this characteristic separation of time scales, correlations that arise during an individual collision decay rapidly before the next collision takes place. This rapid decay, in turn, implies the possibility of a Markov approximation, which assumes that only the current configuration of the system determines its future evolution. Furthermore, this decay of correlations allows us to parametrize the system's state by a reduced set of master variables, because we are interested in the system's time evolution only on the kinetic time scale, i.e., for times large compared to the duration of a collision τ_0. This reduced description with a set of master variables is possible, because for kinetic times the higher-order correlation functions can be expressed as functionals of these variables [2].

This set of master variables is common to both kinetic theories we will discuss: In the Kadanoff-Baym approach, abstract real-time Green's functions parametrize the condensating gas, whereas in the Walser *et al.* case [3], single-time density matrices, which contain the physical density and coherences of thermal atoms, as well as the mean field, represent the system. The equivalence of these two approaches is a general principle in nonequilibrium statistical mechanics [4,5]. However, it is not trivial to verify this fact in detail by explicitly connecting the complementary microscopic equations. Strictly speaking, we state equivalence after the Kadanoff-Baym theory has been restricted to single-time quantities using the Markov approximation.

We present the formulation of the quantum kinetic theory of dilute Bose-Einstein condensed gases in terms of nonequilibrium, real-time Green's functions and their Kadanoff-Baym equations of motion [6], which were generalized in Refs. [7,8] to include the condensate.

By transforming these equations to the single-particle energy basis and taking the single-time limit of the two-time Green's functions by means of the Markov approximation, we reproduce the equations of motion of the Walser *et al.* kinetic theory as presented in Ref. [3], thus providing an independent confirmation of these equations. Following Imamović-Tomasović and Griffin [9], we use the gapless Beliaev approximation for the self-energies in the Kadanoff-Baym equations, and thus prove the Walser *et al.* kinetic theory to be gapless as well.

II. NONEQUILIBRIUM GREEN'S FUNCTIONS

We begin the introduction to the Kadanoff-Baym description of the dilute Bose gas by defining its variables. Neglecting three-body interactions, the second-quantized many-body Hamiltonian $\hat{H}$ describing the atoms is

$$\hat{H}=\int d\mathbf{x}\int d\mathbf{y}\,\hat{a}^{\dagger}(\mathbf{x})\;\langle\mathbf{x}|\hat{H}^{(0)}|\mathbf{y}\rangle\;\hat{a}(\mathbf{y}) + \frac{1}{2}\int d\mathbf{x}\int d\mathbf{y}\;\hat{a}^{\dagger}(\mathbf{x})\hat{a}^{\dagger}(\mathbf{y})V_{\mathrm{bin}}(\mathbf{x}-\mathbf{y})\;\frac{\hat{a}(\mathbf{x})}{\hat{a}(\mathbf{y})}, \quad (1)$$

where $\hat{a}^{\dagger}(\mathbf{x})$ is the bosonic creation operator and $V_{\mathrm{bin}}(\mathbf{x}-\mathbf{y})$ the binary interaction potential. The single-particle Hamiltonian

$$\hat{H}^{(0)}=\frac{\hat{\mathbf{p}}^2}{2m}+V_{\mathrm{ext}}(\hat{\mathbf{x}}) \tag{2}$$

contains the kinetic energy of a boson with mass m and the external potential $V_{\mathrm{ext}}(\mathbf{x})$.

To represent the master variables in terms of nonequilibrium Green's functions, we first write the system's degrees of freedom in terms of spinor operators [10]

$$\hat{A}(1)=\begin{pmatrix}\hat{a}(1)\\ \hat{a}^{\dagger}(1)\end{pmatrix} \quad \text{and} \quad \hat{A}^{\dagger}(1)=(\hat{a}^{\dagger}(1) \quad \hat{a}(1)), \tag{3}$$

where we now follow Kadanoff-Baym and abbreviate $(1)\equiv(\mathbf{x}_1,t_1)$. The master variables are then contained in the following two-time propagators:

$$h(1,2)\equiv -i\langle\hat{A}(1)\rangle\langle\hat{A}^{\dagger}(2)\rangle, \tag{4}$$

$$g(1,2)\equiv -i\langle T\{\hat{A}(1)\hat{A}^{\dagger}(2)\}\rangle, \tag{5}$$

where $\langle\cdot\rangle$ denotes the grand-canonical average and $T\{\cdot\}$ the time ordering operator, which sorts its arguments in order of decreasing time. These two propagators are defined for real times by analytic continuation of the finite-temperature propagators for imaginary time, following [6] Chap. 8. We subtract the condensate propagator h from the full propagator g and thus define the Green's function for the fluctuations

$$\tilde{g}(1,2)\equiv g(1,2)-h(1,2). \tag{6}$$

The two time orderings of $\tilde{g}$,

$$\tilde{g}^{<}(1,2)\equiv\tilde{g}(1,2) \quad \text{for} \quad t_1<t_2 \tag{7}$$

and

$$\tilde{g}^{>}(1,2)\equiv\tilde{g}(1,2) \quad \text{for} \quad t_1>t_2, \tag{8}$$

define the generalized two-time fluctuation-density matrices. This can be seen by explicitly writing these two time orderings in terms of the fluctuating part $\hat{\tilde{a}}(1)$ of the field operators,

$$\hat{\tilde{a}}(1)\equiv\hat{a}(1)-\langle\hat{a}(1)\rangle\equiv\hat{a}(1)-\alpha(1), \tag{9}$$

as follows:

$$\tilde{g}^{<}(1,2)=\begin{pmatrix}\tilde{f}_{12} & \tilde{m}_{12}\\ \tilde{m}^{*}_{12} & (1+\tilde{f})^{*}_{12}\end{pmatrix}, \tag{10}$$

$$\tilde{g}^{>}(1,2)=\sigma_z+\tilde{g}^{<}(1,2), \tag{11}$$

where we defined the two-time normal ($\tilde{f}$) and anomalous ($\tilde{m}$) averages of the fluctuations in the position basis as

$$\tilde{f}_{12}=\langle\hat{\tilde{a}}^{\dagger}(2)\hat{\tilde{a}}(1)\rangle \quad \text{and} \quad \tilde{m}_{12}=\langle\hat{\tilde{a}}(2)\hat{\tilde{a}}(1)\rangle. \tag{12}$$

In the case $t_1=t_2$, the propagators in Eqs. (10) and (11) correspond to the dynamical quantities in the kinetic equations for the fluctuations given in Eqs. (24) and (25) of Ref. [3]; for $t_1=t_2$, the averages in Eq. (12) correspond to the density of thermal atoms around the condensate and correlations between these atoms. We have thus represented the condensate (h) and its fluctuations ($\tilde{g}^{<}$) and can now look for their corresponding evolution equations.

III. KADANOFF-BAYM EQUATIONS

The equations of motion for the nonequilibrium Green's functions h and $\tilde{g}^{<}$ are the Kadanoff-Baym equations; these equations are equivalent to the Dyson equation. In the second part of this section, we discuss the second-order Beliaev approximation for the self-energies that we use. For the condensed part of the atom cloud, which is parametrized by the propagator $h(1,2)$ defined in Eq. (4), we can write the Kadanoff-Baym equations as [7]

$$\int_{-\infty}^{\infty}d\bar{1}\{g_0^{-1}(1,\bar{1})-S_{\mathrm{HF}}(1,\bar{1})\}h(\bar{1},2) = \int_{-\infty}^{t_1}d\bar{1}\{S^{>}(1,\bar{1})-S^{<}(1,\bar{1})\}h(\bar{1},2) \tag{13}$$

and

$$\int_{-\infty}^{\infty}d\bar{1}\,h(1,\bar{1})\{g_0^{-1}(\bar{1},2)-S_{\mathrm{HF}}(\bar{1},2)\} = -\int_{-\infty}^{t_2}d\bar{1}\,h(1,\bar{1})\{S^{>}(\bar{1},2)-S^{<}(\bar{1},2)\}. \tag{14}$$

We write the corresponding equations for the fluctuations $\tilde{g}^{<}(1,2)$ and $\tilde{g}^{>}(1,2)$ [Eqs. (10) and (11)] around the condensate mean field as

$$\int_{-\infty}^{\infty}d\bar{1}\{g_0^{-1}(1,\bar{1})-\Sigma_{\mathrm{HF}}(1,\bar{1})\}\tilde{g}^{\gtrless}(\bar{1},2) = \int_{-\infty}^{t_1}d\bar{1}\{\Sigma^{>}(1,\bar{1})-\Sigma^{<}(1,\bar{1})\}\tilde{g}^{\gtrless}(\bar{1},2) - \int_{-\infty}^{t_2}d\bar{1}\Sigma^{\gtrless}(1,\bar{1})\{\tilde{g}^{>}(\bar{1},2)-\tilde{g}^{<}(\bar{1},2)\} \tag{15}$$

and

$$\int_{-\infty}^{\infty}d\bar{1}\tilde{g}^{\gtrless}(1,\bar{1})\{g_0^{-1}(\bar{1},2)-\Sigma_{\mathrm{HF}}(\bar{1},2)\} = \int_{-\infty}^{t_1}d\bar{1}\{\tilde{g}^{>}(1,\bar{1})-\tilde{g}^{<}(1,\bar{1})\}\Sigma^{\gtrless}(\bar{1},2) - \int_{-\infty}^{t_2}d\bar{1}\tilde{g}^{\gtrless}(1,\bar{1})\{\Sigma^{>}(\bar{1},2)-\Sigma^{<}(\bar{1},2)\}. \tag{16}$$

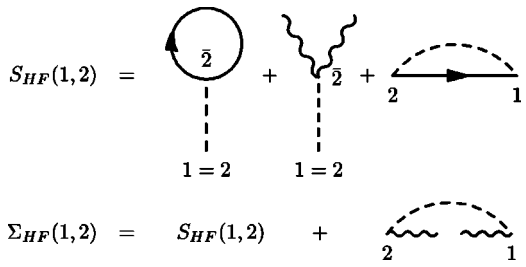

FIG. 1. The first-order Hartree-Fock self-energy diagrams. The solid lines depict the noncondensate propagator $\tilde{g}$, the wiggly lines the condensate propagator h, and the dashed lines the interaction potential v. The first two terms give the energy shifts due to both the mean field U_{f^c} and the normal fluctuations $U_{\tilde{f}}$. The third term in S_{HF} gives rise to a factor of 2 for $U_{\tilde{f}}$ and to $V_{\tilde{m}}$. The fourth term which only appears in Σ_{HF} causes the difference in the mean-field shifts that are experienced by the condensate and the fluctuations, respectively.

In Eqs. (13) through (16), we use the definition of the matrix inverse of the interaction-free propagator g_0,

$$g_0^{-1}(1,2)=\left\{ i\sigma^z \frac{d}{dt_1}+\frac{\nabla_1^2}{2m}-V_{\mathrm{ext}}(1)+\mu \right\}\delta(1,2), \quad (17)$$

with the third Pauli matrix $\sigma^z=\mathrm{diag}(1,-1)$ and an energy shift μ, which removes mean-field oscillations. We define the δ function by $\delta(1,2)\equiv\delta(\mathbf{x}_1-\mathbf{x}_2)\,\delta(t_1-t_2)$ and integration $d\bar{1}$ as integration $dt_{\bar{1}}$ over time within the given time limits and $d\mathbf{x}_{\bar{1}}$ over all space. The approximations we choose for the Hartree-Fock self-energies for the condensate S_{HF} and for the fluctuations Σ_{HF} as well as the second-order collisional self-energies $S^<$ and $\Sigma^<$ will be discussed below.

Kadanoff and Baym derived these equations without including the condensate [6] and de Dominicis and Martin formulated a very general mathematical account [11,12]. The Green's function formalism traces back to Schwinger [13] and originally made use of the correspondence between the partition function and the time evolution operator in imaginary time ($e^{\beta H}=e^{iHt}$ for $t=-i\beta$). To get information about measurable quantities, the dynamic variables and equations of motion were extended to real times by analytic continuation (see [6] Chap. 8 and [14,15,5] for more details).

This nonequilibrium Green's function description was developed 40 years ago to eventually explain the behavior of superfluid helium [16]. Since this description involves a weak-coupling approximation but helium atoms are strongly interacting, the results at that time were disappointing and, for example, could not explain all predictions of the phenomenological Landau model. However, since the Green's function description holds for a dilute, weakly interacting gas, its application to Bose-Einstein condensation in this system is more appropriate.

To complete our exposition of the Kadanoff-Baym equations (13) through (16), we have to choose the Hartree-Fock and collisional self-energies. We draw the Hartree-Fock self-energy diagrams for both the condensate h and the thermal cloud $\tilde{g}^<$ in Fig. 1 and write them, respectively, as

$$S_{\mathrm{HF}}(1,2)=\frac{i}{2}\int d\bar{2}\, v(1,\bar{2})\mathrm{Tr}\{g(\bar{2},\bar{2})\}\,\delta(1,2)+iv(1,2)\tilde{g}(1,2) \quad (18)$$

and

$$\Sigma_{\mathrm{HF}}(1,2)=\frac{i}{2}\int d\bar{2}\, v(1,\bar{2})\mathrm{Tr}\{g(\bar{2},\bar{2})\}\,\delta(1,2) + iv(1,2)g(1,2), \quad (19)$$

with the local-time, binary interaction potential $v(1,2)=V_{\mathrm{bin}}(\mathbf{x}_1-\mathbf{x}_2)\,\delta(t_1-t_2)$ and the matrix trace Tr. When we evaluate the time ordered propagator g at equal times, we follow the convention $T\{\hat{a}(1)\hat{a}^\dagger(2)\}=\hat{a}^\dagger(2)\hat{a}(1)$.

For the second-order collisional self-energies $\Sigma^{>}_{<}$ we choose the gapless and conserving Beliaev approximation [9,17–19]. This means that, compared to Kane and Kadanoff [7], we include the exchange terms, which they deliberately excluded to obtain the simplest conserving approximation as proven in [20], and compared to Hohenberg and Martin [8], we include the terms containing no condensate contributions, which will give rise to the quantum Boltzmann terms for the fluctuations.

We depict the resulting self-energy diagrams in Fig. 2 and represent them mathematically as

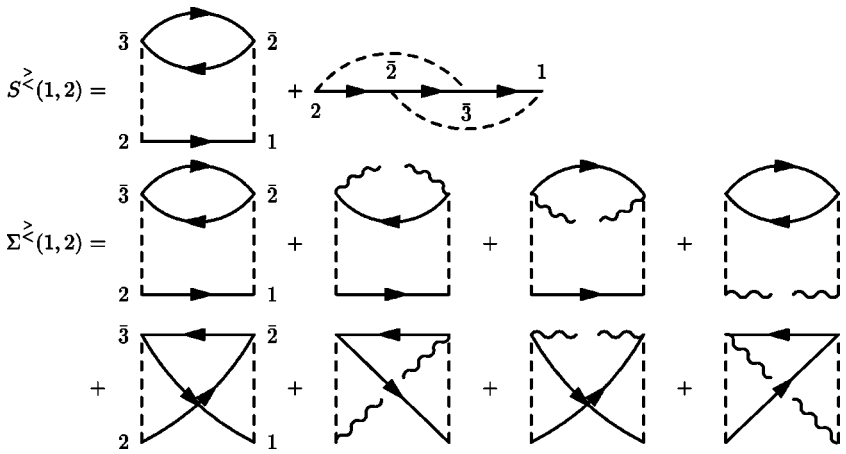

FIG. 2. The second-order collisional self-energies in the gapless Beliaev approximation. The solid lines depict the noncondensate propagator $\tilde{g}$, the wiggly lines the condensate propagator h, and the dashed lines the binary interaction potential v. The second diagram of S corresponds to the last four of Σ, when we replace each of the three fluctuation propagators by an open condensate one.

$$S^{\gtrless}(1,2) = -\frac{1}{2}\int d\bar{2}\int d\bar{3}\, v(1,\bar{2})v(2,\bar{3}) \times[\tilde{g}^{\gtrless}(1,2)\mathrm{Tr}\{\tilde{g}^{\lessgtr}(\bar{3},\bar{2})\tilde{g}^{\gtrless}(\bar{2},\bar{3})\} + 2\tilde{g}^{\gtrless}(1,\bar{3})\tilde{g}^{\lessgtr}(\bar{3},\bar{2})\tilde{g}^{\gtrless}(\bar{2},2)] \quad (20)$$

for the condensed part and

$$\Sigma^{\gtrless}(1,2) = -\frac{1}{2}\int d\bar{2}\int d\bar{3}\, v(1,\bar{2})v(2,\bar{3}) \times[\tilde{g}^{\gtrless}(1,2)\mathrm{Tr}\{g^{\lessgtr}(\bar{3},\bar{2})g^{\gtrless}(\bar{2},\bar{3}) - h(\bar{3},\bar{2})h(\bar{2},\bar{3})\} + h(1,2)\mathrm{Tr}\{\tilde{g}^{\lessgtr}(\bar{3},\bar{2})\tilde{g}^{\gtrless}(\bar{2},\bar{3})\} + 2\tilde{g}^{\gtrless}(1,\bar{3})\{g^{\lessgtr}(\bar{3},\bar{2})g^{\gtrless}(\bar{2},2) - h(\bar{3},\bar{2})h(\bar{2},2)\} + 2h(1,\bar{3})\{\tilde{g}^{\lessgtr}(\bar{3},\bar{2})\tilde{g}^{\gtrless}(\bar{2},2)\}] \quad (21)$$

for the fluctuations.

Instead of using lines for the matrix-valued propagators $\tilde{g}$ and h as in Fig. 2, one can also draw diagrams for the four elements of the matrix separately. The resulting diagrams for the first-order and second-order Beliaev terms can be seen in Figs. 15 and 17 of Ref. [21], where the interaction potential is replaced by a two-body T matrix.

IV. TRANSFORMATION TO THE ENERGY BASIS

We now demonstrate the key steps that connect the kinetic theory presented in the previous section to the work of Walser *et al.* presented in [3]: We rewrite the Kadanoff-Baym Eqs. (13) through (16) in the single-particle energy (SPE) basis and obtain the equations of motion for the master variables—the measurable quantities in our reduced description of the system—in this basis, exactly as given in the Walser *et al.* paper.

First, we define our master variables in the SPE basis $\{|1'\rangle\}_{1'} \equiv \{|\epsilon_{1'}\rangle\}_{\epsilon_{1'}}$ and determine their relation to their position basis counterparts, the Green's functions given in Eqs. (4), (10), and (11). The time-dependent, two-component mean-field state vector

$$\chi = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} \quad (22)$$

is defined in terms of $\alpha \equiv \alpha_{1'}|1'\rangle \equiv \Sigma_{1'}\langle \hat{a}_{1'}\rangle |1'\rangle$ and also contains the time-reversed mean field α^*. The time-dependent annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ transform as

$$\hat{a}(1) = \langle 1|1'\rangle \hat{a}_{1'} \quad \text{and} \quad \hat{a}^\dagger(1) = \langle 1'|1\rangle \hat{a}^\dagger_{1'}, \quad (23)$$

where $|1\rangle \equiv |\mathbf{x}_1\rangle$ are the position eigenstates. The fluctuating part of the master variables is contained in the single-time fluctuation-density matrix $\tilde{G}^<$, which we define as

$$\tilde{G}^< = \begin{pmatrix} \tilde{f} & \tilde{m} \\ \tilde{m}^* & (1+\tilde{f})^* \end{pmatrix}, \quad (24)$$

with the normal fluctuation density $\tilde{f} = \langle \hat{a}^\dagger_{2'}, \hat{a}_{1'}\rangle |1'\rangle \otimes \langle 2'|$ and the anomalous average $\tilde{m} = \langle \hat{a}_{2'}, \hat{a}_{1'}\rangle |1'\rangle \otimes |2'\rangle$ in the SPE basis.

Second, we can recognize the fluctuation-density matrix $\tilde{G}^<$ as the single-time limit of its position-basis counterpart $\tilde{g}^<(1,2)$ in Eq. (10). The mean-field state vector χ, on the other hand, can be combined with its Hermitian conjugate into the matrix-$i\chi\chi^\dagger$, which corresponds to $h(1,2)$ in Eq. (4). We thus define $\chi(1) \equiv \langle \hat{A}(1)\rangle$. This allows us to explicitly connect the condensate mean-field state vectors $\chi(1)$ expressed in the position basis and $\chi(t_1)$ in the SPE basis as follows:

$$\chi(1) = \begin{pmatrix} \langle 1|1'\rangle & 0 \\ 0 & \langle 1'|1\rangle \end{pmatrix} \begin{pmatrix} \alpha_{1'}(t_1) \\ \alpha^*_{1'}(t_1) \end{pmatrix} \equiv T(1)\chi(t_1), \quad (25)$$

with a time-independent $2\times 2n$ transformation matrix $T(1) = T(\mathbf{x}_1)$. Because of the completeness of the position basis, we can also write

$$\chi(t_1) = \int d\mathbf{x}_1\, T^\dagger(1)\chi(1). \quad (26)$$

For the fluctuation density, we obtain similarly

$$i\,\tilde{g}^<(1,2)|_{t_1=t_2} = T(1)\tilde{G}^<(t_1)T^\dagger(2) \quad (27)$$

and

$$-i\,\tilde{G}^<(t_1) = \int d\mathbf{x}_1 \int d\mathbf{x}_2\, T^\dagger(1)\tilde{g}^<(1,2)T(2)|_{t_1=t_2}. \quad (28)$$

We can now use the transformation Eq. (27) to write the condensate's Hartree-Fock self-energy $S_{\mathrm{HF}}(1,\bar{1})$ in Eq. (18) as

$$T(1)\begin{pmatrix} U_{f^c} + 2U_{\tilde{f}} & V_{\tilde{m}} \\ V^\dagger_{\tilde{m}} & U^\dagger_{f^c} + 2U^\dagger_{\tilde{f}} \end{pmatrix} T^\dagger(\bar{1})\,\delta(t_1 - t_{\bar{1}}). \quad (29)$$

We here use the definitions of [3], where energy shifts due to both the mean field and the normal fluctuations are given by the matrices

$$U_f = 2\,\phi^{1'2'3'4'} f_{3'2'}|1'\rangle \otimes \langle 4'|, \quad (30)$$

whereas the first-order anomalous coupling strength is given by

$$V_{\tilde{m}} = 2\,\phi^{1'2'3'4'}\tilde{m}_{3'4'}|1'\rangle \otimes |2'\rangle. \quad (31)$$

The symmetrized two-body interaction matrix elements ϕ are here defined by

$$\phi^{1'2'3'4'}=\frac{1}{4}(\phi_u^{1'2'3'4'}+\phi_u^{1'2'4'3'}+\phi_u^{2'1'3'4'}+\phi_u^{2'1'4'3'}), \tag{32}$$

$$\phi_u^{1'2'3'4'}=\int d\mathbf{x}_1\int d\mathbf{x}_2\langle 1'|1\rangle\langle 2'|2\rangle\frac{V_{\rm bin}(\mathbf{x}_1-\mathbf{x}_2)}{2} \times\langle 1|3'\rangle\langle 2|4'\rangle. \tag{33}$$

Like the first-order Hartree-Fock self-energies, we can rewrite the second-order self-energies in Eqs. (20) and (21) using the transformations in Eqs. (25) to (28) and (33). In particular, we now have to transform two potential factors, which makes the computation more complicated. Furthermore, the integrals over time to t_1 and t_2 in Eqs. (13) through (16) modify one of the binary potentials according to Eq. (65) of [22] to an approximately energy conserving two-particle matrix element

$$\phi_\eta^{1'2'3'4'}=\phi^{1'2'3'4'}\left\{\pi\delta_\eta(\Delta)+i\,\mathcal{P}_\eta\frac{1}{\Delta}\right\}, \tag{34}$$

with an energy difference Δ between the incoming and outgoing states of the collision event. This definition of the matrix elements ϕ_η introduces the Markov approximation into the Kadanoff-Baym equations. We obtain the second-order damping rates and energy shifts $\Upsilon^{\gtrless}$ for the condensate, corresponding to $S(1,2)$ in Eq. (20), and $\Gamma^{\gtrless}$ for the fluctuations, corresponding to $\Sigma(1,2)$ in Eq. (21); these are the collision integrals defined in [3]. The second-order terms appear in combinations $\Gamma^{<}\tilde{G}^{>}-\Gamma^{>}\tilde{G}^{<}$ that contain, for example, the Boltzmann collision terms

$$\begin{aligned}&\{\Gamma_{\tilde{f}\tilde{f}(1+\tilde{f})}(1+\tilde{f})-\Gamma_{(1+\tilde{f})(1+\tilde{f})\tilde{f}}\tilde{f}\}_{1'5'}\\&\quad=8\phi^{1'2'3'4'}\phi_\eta^{1''2''3''4''}\{\tilde{f}_{3'1''}\tilde{f}_{4'2''}(1+\tilde{f})_{4''2'}(1+\tilde{f})_{3''5'}\\&\qquad-(1+\tilde{f})_{3'1''}(1+\tilde{f})_{4'2''}\tilde{f}_{4''2'}\tilde{f}_{3''5'}\}\end{aligned} \tag{35}$$

and similar contributions involving the anomalous averages $\tilde{m}$ and $\tilde{m}^*$.

We can now exactly reproduce the coupled equations for the condensed fraction as well as the normal and anomalous fluctuations stated in Eqs. (10) and (26) of Ref. [3]. Considering the first column of the matrix Eq. (13) for the condensate at $t_1=t_2$, we obtain the generalized Gross-Pitaevskii equation

$$\frac{d}{dt}\chi=(-i\,\Pi+\Upsilon^{<}-\Upsilon^{>})\chi, \tag{36}$$

with the symplectic first-order propagator

$$\Pi=\begin{pmatrix}\Pi_{\mathcal{N}} & \Pi_{\mathcal{A}}\\ -\Pi_{\mathcal{A}}^* & -\Pi_{\mathcal{N}}^*\end{pmatrix}. \tag{37}$$

This propagator consists of the normal Hermitian Hamiltonian

$$\Pi_{\mathcal{N}}=\hat{H}^{(0)}+U_{f^c}+2U_{\tilde{f}}-\mu, \tag{38}$$

which contains the usual single-particle Hamiltonian $\hat{H}^{(0)}$ given in Eq. (2) and the mean-field and fluctuation shifts U_f given in Eq. (30); furthermore, the symmetric anomalous coupling

$$\Pi_{\mathcal{A}}=V_{\tilde{m}} \tag{39}$$

is defined in Eq. (31). The propagator Π contains the Hartree-Fock shifts, which are given in Eq. (29), and originally were contained in $S_{\rm HF}(1,2)$ [see Eq. (18)].

To obtain the equation of motion for the fluctuations, we subtract Eq. (15) from Eq. (16) and evaluate at $t_1=t_2$ to obtain

$$\frac{d}{dt}\tilde{G}^{<}=-i\,\Sigma\tilde{G}^{<}+\Gamma^{<}\tilde{G}^{>}-\Gamma^{>}\tilde{G}^{<}+\text{H.c.} \tag{40}$$

The reversible evolution of the fluctuations $\tilde{G}^{<}$ is governed by the Hartree-Fock-Bogoliubov self-energy operator

$$\Sigma=\begin{pmatrix}\Sigma_{\mathcal{N}} & \Sigma_{\mathcal{A}}\\ -\Sigma_{\mathcal{A}}^* & -\Sigma_{\mathcal{N}}^*\end{pmatrix}, \tag{41}$$

which in turn consists of the Hermitian Hamiltonian

$$\Sigma_{\mathcal{N}}=\hat{H}^{(0)}+2U_{f^c}+2U_{\tilde{f}}-\mu \tag{42}$$

and the symmetric anomalous coupling

$$\Sigma_{\mathcal{A}}=V_m. \tag{43}$$

The propagator Σ corresponds to $\Sigma_{\rm HF}(1,2)$ in Eq. (19). Its mean-field shift is twice as large as that of the condensate propagator Π, which is a well known property of first-order Hartree-Fock-Bogoliubov theories. Further details of this transformation can be found in [23].

V. CONCLUSION

We independently rederive the kinetic equations of Walser *et al.* from the Kadanoff-Baym nonequilibrium Green's function formulation of kinetic theory, and recover identical factors in all second-order damping rates and energy shifts. This shows that for dilute, weakly interacting gases the Kadanoff-Baym nonequilibrium, real-time Green's function approach is microscopically equivalent to the density matrix approach used by Walser *et al.* [3]. The latter approach is more physical in two respects. First, its variables are measurable quantities: the mean field and the density and coherences of thermal atoms. Second, the variables' equations of motion reduce

to the Gross-Pitaevskii equation and the quantum Boltzmann equation in the low- and high-temperature limits, respectively.

Starting from the gapless Beliaev approximation for the collisional self-energy in the Kadanoff-Baym equations, we furthermore learn that the full second-order kinetic theory of Walser *et al.* is gapless itself [18,9]. This shows that the gap that appears in the first-order Hartree-Fock-Bogoliubov spectrum [24] is closed by the second-order energy shifts.

Furthermore, this work connects the kinetic theory of Walser *et al.* with work done by M. Imamović-Tomasović *et al.* [9,17,25], because they start from the same Kadanoff-Baym equations.

ACKNOWLEDGMENTS

J.W. acknowledges financial support by the National Science Foundation. R.W. gratefully acknowledges support from the Austrian Academy of Sciences through an APART grant. M.H. acknowledges support from the U.S. Department of Energy, Office of Basic Energy Sciences via the Chemical Sciences, Geosciences, and Biosciences Division.

[1] D. Zubarev, V. Morozov, and G. Röpke, *Statistical Mechanics of Nonequilibrium Processes* (Akademie Verlag GmbH, Berlin, 1997), Vol. 1.
[2] A. I. Akhiezer and S. V. Peletminskii, *Methods of Statistical Physics* (Pergamon Press, Oxford, 1981).
[3] R. Walser, J. Cooper, and M. Holland, Phys. Rev. A **63**, 013607 (2001).
[4] V. G. Morozov and G. Röpke, Ann. Phys. (N.Y.) **278**, 127 (1999).
[5] D. Zubarev, V. Morozov, and G. Röpke, *Statistical Mechanics of Nonequilibrium Processes* (Ref. [1]), Vol. 2.
[6] L. P. Kandanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).
[7] J. W. Kane and L. P. Kadanoff, J. Math. Phys. **6**, 1902 (1965).
[8] P. C. Hohenberg and P.C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965); [reprinted **281**, 636 (2000)].
[9] M. Imamović-Tomasović and A. Griffin, in *Progress in Nonequilibrium Green's Functions*, edited by M. Bonitz (World Scientific, Singapore, 2000), p. 404.
[10] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[11] C. de Dominicis and P. C. Martin, J. Math. Phys. **5**, 14 (1964).
[12] C. de Dominicis and P. C. Martin, J. Math. Phys. **5**, 31 (1964).
[13] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **37**, 452 (1951).
[14] G. Baym and N. D. Mermin, J. Math. Phys. **2**, 232 (1961).
[15] R. A. Craig, J. Math. Phys. **9**, 605 (1968).
[16] P. C. Martin, J. Math. Phys. **4**, 208 (1961).
[17] M. Imamović-Tomasović and A. Griffin, J. Low Temp. Phys. **122**, 617 (2001).
[18] S. T. Beliaev, Sov. Phys. JETP **34**, 289 (1958).
[19] S. T. Beliaev, Sov. Phys. JETP **34**, 299 (1958).
[20] G. Baym and L. P. Kadanoff, Phys. Rev. **124**, 287 (1961).
[21] H. Shi and A. Griffin, Phys. Rep. **304**, 1 (1998).
[22] R. Walser, J. Williams, J. Cooper, and M. Holland, Phys. Rev. A **59**, 3878 (1999).
[23] J. Wachter, MS, University of Colorado at Boulder, 2000.
[24] A. Griffin, Phys. Rev. B **53**, 9341 (1996).
[25] M. Imamović-Tomosović and A. Griffin, Phys. Rev. A **60**, 494 (1999).